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Quintic B-Spline Technique for Numerical Treatment of Third Order Singular Perturbed Delay Differential Equation

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Abstract
In this paper, a class of third order singularly perturbed delay differential equation with large delay is considered for numerical treatment. The considered equation has discontinuous convection-diffusion coefficient and source term. A quintic trigonometric B-spline collocation technique is used for numerical simulation of the considered singularly perturbed delay differential equation by dividing the domain into the uniform mesh. Further, uniform convergence of the solution is discussed by using the concept of Hall error estimation and the method is found to be of first-order convergent. The existence of the solution is also established. Computation work is carried out to validate the theoretical results.

Keywords- Quintic trigonometric spline, Error estimate, Perturbed equation, Delay.

1. Introduction
The frequent emerge of singular perturbed delay differential equations (SPDDE) in every field of science and technology has triggered the researchers for numerical treatment of these equations. These equations involve two sensitive parameters: perturbation ($\boldsymbol{\varepsilon}$) and retarded (or delay)($\boldsymbol{\delta}$) parameter. The study of these differential equations is a stiff job for the researchers due to startlingadapt of the solution at boundary as $\boldsymbol{\varepsilon} \rightarrow \mathbf{0}$, and such variation in solution at the boundary is well known as boundary layer. The other factor which provokes the mathematicians for analysis of SPDDE is recurrent emerge of these equations in real life problems. Ample mathematical models in science and technology result in SPDDE. Reader can refer (Stein, 1965; Stein, 1967; Longtin and Milton, 1988; Nelson and Perelson, 2002; Rihan, 2013; Wilkie and Hahnfeldt, 2013) for some of applications of SPDDE in biosciences.

The delay differential equations are also the result of mathematical models of the real-life problems in various fields. One such model exists in study of the development of the bearing of the population of a system of organisms. This system can be well explained by its mathematical modeling with delay as the prominent insight of the dynamic conduct of the system. This time delay appears in a polluted ecosystem, and is one of the important factors affecting the expansion of the biological system. Therefore, delay differential equations are the result of mathematical modeling explaining organic structure. The disparity and bifurcation in the system emerge due to the time delay. One of the common bifurcations (the division of system into two parts) is hop bifurcation which refers to the division of the system at a point where the firmness turns around and periodic solution arises as a particular variable changes its values. One such mathematical model of the singular biological system is proposed by Zhang et al. (2016) by considering the delay parameter into account.

The model is framed by considering τ as the time delay of the transformation of immature organisms into mature organisms, u(t) and v(t) taken as the densities of the immature and mature organisms, $E(t)$ as capture capability of mature creatures at time t, p_1 as a unit price, c is a unit cost, and m is economic profit. $p_1 v(t)E(t)$ represents the total revenue, and $cE(t)$ denotes the total cost. The delay model of the biological system given by:

$$\boldsymbol{\dot{u}(t) = pv(t) - qu(t-\tau) - d_1 u(t) - \phi_1 u_1(t) u(t)}$$
$$\boldsymbol{\dot{v}(t) = qu(t-\tau) - d_2 v(t) - \beta v^2(t) - E(t)v(t) - \phi_2 u_1(t) v(t)}$$
and $\boldsymbol{\dot{u_1}(t) = \theta - hu_1(t), 0 = E(t)(p_1 v(t) - c) - m}$.

A range of numerical methods have been proposed in recent years for numerical treatment of these differential equations. Some of the reported work can be cited from (Andargie and Reddy, 2013; Nicaise and Xenophontos, 2013; Swamy et al., 2015; Cimen, 2017). With the incitement from the reported work, this paper is an extension of numerical simulation of SPDDE.

This paper is organized as follows. In the next section (Section 2) the considered SPDDE is stated followed by the existence of the solution of the problem stated in Section 3. In Section 4, the method applied is interpreted with convergence of the scheme explored in Section 5. Section 6, involves the estimated solution for some numerical examples with conclusion in Section 7.

2. Problem Statement

The problem is to find solution $y \epsilon C^1(\bar{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^*)$ so that it satisfies the following equation:

$$\begin{aligned} Ly(x) &= -\varepsilon y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) + d(x)y'(x-1) \\ &= f(x), x\epsilon\Omega^* \end{aligned} \tag{1}$$

with boundary conditions

$$y(x) = \phi(x), x\epsilon[-1,0], \; y'(2) = \gamma \tag{2}$$

where ε is very-very small positive number and $a(x)$, $f(x)$ are discontinuous functions as shown below:

$$a(x) = \begin{cases} a_1(x), x\epsilon[0,1] \\ a_2(x), x\epsilon(1,2] \end{cases} \text{ and } f(x) = \begin{cases} f_1(x), x\epsilon[0,1] \\ f_2(x), x\epsilon(1,2] \end{cases}$$

with the conditions $a_1(1-) \neq a_2(1+), f_1(1-) \neq f_2(1+)$, $a_i(x) > \alpha_i(x) > \alpha + 2 > 3$, $b(x) \geq \beta_0 \geq 0, \gamma_0 \leq c(x) \leq \gamma \leq 0, \eta_0 \leq d(x) \leq 0$ such that $2\alpha+4\gamma_0+5\eta_0>0$ (Subburayan and Mahendran, 2018).

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Here, the functions $a(x)$ and $f(x)$ are sufficiently smooth and bounded on Ω^*, $b(x)$, $c(x)$ and $d(x)$ are sufficiently smooth functions on $\overline{\Omega}$, where $\Omega^* = \Omega^+ \cup \Omega^-, \Omega^- = (0,1), \Omega^+ = (1,2)$ and $\Omega = (0,2)$.

Now, to handle the delay parameter Taylor's series up to second order accuracy is used as

$$y'(x-1) = y'(x) - y''(x).$$

By using Taylor's series expansion in equation (1), we get the equation in form:

$$P(x)y'''(x) + Q(x)y''(x) + R(x)y'(x) + S(x)y(x) = f(x) \tag{3}$$

where $P(x) = -\varepsilon$, $Q(x) = a(x) - d(x), R(x) = b(x)+d(x)$ and $S(x) = c(x)$.

3. Existence of Solution

Theorem: The equation (1)-(2) has solution $\bar{y} = (y_1, y_2)$ where $y_1 \epsilon C^0(\overline{\Omega}) \cap C^1(\Omega \cup \{2\})$ and $y_2 \epsilon C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$.

Proof: Proof of this theorem is by construction.

Let $\overline{y_L} = (y_{LA}, y_{LB})$ and $\overline{y_R} = (y_{RA}, y_{RB})$ be the particular solution of the following problems:

$$\begin{cases} y'_{LA}(x) - y_{LB}(x) = 0, x \epsilon \Omega^- \\ -\varepsilon y'''_{LB}(x) + a(x)y''_{LB}(x) + b(x)y'_{LB}(x) + c(x)y_{LA}(x) + d(x)y'_{LA}(x-1) = f(x), x \epsilon \Omega^- \\ y_{LA}(x) = \phi(x), y_{LB} = \phi'(x), x \epsilon [-1,0] \end{cases}$$

and

$$\begin{cases} y'_{RA}(x) - y_{RB}(x) = 0, x \epsilon \Omega^+ \\ -\varepsilon y'''_{RB}(x) + a(x)y''_{RB}(x) + b(x)y'_{RB}(x) + c(x)y_{RA}(x) + d(x)y'_{RA}(x-1) = f(x), x \epsilon \Omega^+ \end{cases}.$$

Now we consider the function $\bar{y} = (y_1, y_2)$ as:

$$y_1(x) = \begin{cases} y_{LA}(x) + A\emptyset_{1A}(x), x \epsilon \Omega^- \\ y_{RA}(x) + \emptyset_{2A}(x)[y_2(2) - y_{RA}(2)] + B\emptyset_{3A}(x), x \epsilon \Omega^+ \end{cases}$$

$$y_2(x) = \begin{cases} y_{LB}(x) + A\emptyset_{1B}(x), x \epsilon \Omega^- \\ y_{RB}(x) + \emptyset_{2B}(x)[y_2(2) - y_{RB}(2)] + B\emptyset_{3B}(x), x \epsilon \Omega^+ \end{cases}$$

where the functions $\overline{\emptyset_1} = (\emptyset_{1A}, \emptyset_{1B}), \overline{\emptyset_2} = (\emptyset_{2A}, \emptyset_{2B})$ and $\overline{\emptyset_3} = (\emptyset_{3A}, \emptyset_{3B})$ satisfies the following boundary value problems respectively:

$$\begin{cases} \emptyset'_{1A}(x) - \emptyset_{1B}(x) = 0, x \epsilon \Omega \cup \{2\} \\ -\varepsilon \emptyset'''_{1B}(x) + a(x)\emptyset''_{1B}(x) + b(x)\emptyset'_{1B}(x) + c(x)\emptyset_{1A}(x) + d(x)\emptyset'_{1A}(x-1) = f(x), x \epsilon \Omega \\ \emptyset_{1A}(x) = 0 \; for x \epsilon [-1,0], \emptyset_{1B}(x) = 0 \; for x \epsilon [-1,0] \; and \; \emptyset_{1B}(2) = 1 \end{cases}$$

$$\begin{cases} \emptyset_{2A}'(x) - \emptyset_{2B}(x) = 0, x\epsilon\Omega\cup\{2\} \\ -\varepsilon\emptyset_{2B}'''(x) + a(x)\emptyset_{2B}''(x) + b(x)\emptyset_{2B}'(x) + c(x)\emptyset_{2A}(x) + d(x)\emptyset_{2A}'(x-1) = f(x), x\epsilon\Omega \\ \emptyset_{2A}(x) = 0 \; for x\epsilon[-1,0], \emptyset_{2B}(x) = 0 \; for x\epsilon[-1,0] \; and \emptyset_{2B}(2) = 1 \end{cases}$$

$$\begin{cases} \emptyset_{3A}'(x) - \emptyset_{3B}(x) = 0, x\epsilon\Omega\cup\{2\} \\ -\varepsilon\emptyset_{3B}'''(x) + a(x)\emptyset_{3B}''(x) + b(x)\emptyset_{3B}'(x) + c(x)\emptyset_{3A}(x) + d(x)\emptyset_{3A}'(x-1) = f(x), x\epsilon\Omega \\ \emptyset_{3A}(x) = 0 \; for x\epsilon[-1,0], \emptyset_{3B}(x) = 0 \; for x\epsilon[-1,0] \; and \emptyset_{3B}(2) = 0. \end{cases}$$

It is apparent that $\bar{\boldsymbol{y}}$ satisfy (1)-(2) and the constants A and B can be obtained by procedure given by Subburayan and Mahendran (2018) to show existence of the solution.

4. Quintic Trigonometric B-Spline Collocation Method

The first order trigonometric function is given by

$$TS_{1,i}(x) = \begin{cases} 1, for \; x_i \le x < x_{i+1} \\ 0, \quad otherwise \end{cases}$$

and the τ^{th} ordered trigonometric i[th] function is given by

$$TS_{\tau,i}(x) = \sin\left(\frac{x - x_i}{2}\right) B_{\tau-1,i}(x) + sin\left(\frac{x_{i+\tau} - x}{2}\right) B_{\tau-1,i+1}(x)$$

where $B_{\tau-1,\gamma}(x) = \begin{cases} \dfrac{T_{\tau-1,\gamma}(x)}{sin\left(\frac{x_{\gamma+\tau+1}-x_\gamma}{2}\right)} & , x_\gamma < x_{\gamma+\tau-1} \\ 0 & , x_\gamma = x_{\gamma+\tau-1} \end{cases}$ for $\gamma = i$or $\gamma = i+1$.

The quintic trigonometric b-spline collocation method is used for the numerical simulation of the considered SPDDE. The domain $0 \le x \le 2$ is partitioned into a uniform mesh with each sub intervals of length $1/N$ with resultant partition $\{0{=}x_0 < x_1 < x_2 < \cdots \ldots . < x_N = 2\}$ where N is the total number of partition points and $\{T_{5,-2}(x), T_{5,-1}(x), \ldots, T_{5,N+2}(x)\}$ form the basis for function defined over the interval $[x_0, x_N]$.The approximated solution is considered as:

$$y(x) = \sum_{i=-2}^{N+2} \alpha_i T_{5,i}(x) \tag{4}$$

where $T_{5,i}(x)$'s are the trigonometric B-spline basis function of fifth order as described by Zakaria et al. (2017) and α_i's are unknown coefficients to be determined . At the nodal points x_i, the function y(x) and its derivatives are defined as:

$y(x_i) = k_1\alpha_{i-2} + k_2\alpha_{i-1} + k_3\alpha_i + k_2\alpha_{i+1} + k_1\alpha_{i+2}$, $y'(x_i) = k_4\alpha_{i-2} + k_5\alpha_{i-1} + k_6\alpha_{i+1} + k_7\alpha_{i+2}$

$y''(x_i) = k_8\alpha_{i-2} + k_9\alpha_{i-1} + k_{10}\alpha_i + k_9\alpha_{i+1} + k_8\alpha_{i+2}$, $y'''(x_i) = k_{11}\alpha_{i-2} + k_{12}\alpha_{i-1} + k_{13}\alpha_{i+1} + k_{14}\alpha_{i+2}$

where the constants $k_i$for quintic Trigonometric B-spline are as follows:

$k_1 = \dfrac{sin(h/4)^4}{\sin h \sin\left(\frac{3h}{2}\right)\sin(2h)\sin\left(\frac{5h}{2}\right)}$, $k_2 = \dfrac{5+8cosh}{4\cosh\cos\left(\frac{h}{2}\right)(1+2cosh)(1+2cosh+2cos2h)}$,

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$k_3 = \frac{6+6\cosh \text{sech} \sec(h/2)^2}{4+8\cos h+8cos2h}$, $k_4 = \frac{-5sin(h/2)^2}{4\sin(3h/2)\sin(2h)\sin(5h/2)}$, $k_5 = \frac{-5(1+4cosh)cosec(h/2)\sec(h)}{8(1+2cosh)(1+2cosh+2cos2h)}$,

$k_6 = \frac{5(1+4cosh)cosec(h/2)\sec(h)}{8(1+2cosh)(1+2cosh+2cos2h)}$, $k_7 = \frac{5sin(h/2)^2}{4\sin(3h/2)\sin(2h)\sin(5h/2)}$,

$k_8 = \frac{5(3+5cosh)}{16cosh(1+2cosh)(1+2cosh+2cos2h)sin(h)^2}$,

$k_9 = \frac{5(3+5cosh)}{32\cosh\cos(h/2)(1+2cosh)(1+2cosh+2cos2h)sin(h/2)^2}$, $k_{10} = \frac{5(2+5cosh+2cos2h)}{8cosh(1+2cosh+2cos2h)sin(h)^2}$,

$k_{11} = \frac{-5(-1+25cosh)}{128\cosh\cos(h/2)(1+2cosh)(1+2cosh+2cos2h)sin(h/2)^3}$,

$k_{12} = \frac{-5(1-27cosh+2cos2h)}{64cosh(1+2cosh)(1+2cosh+2cos2h)sin(h/2)^3}$,

$k_{13} = \frac{5(1-27cosh+2cos2h)}{64cosh(1+2cosh)(1+2cosh+2cos2h)sin(h/2)^3}$,

$k_{14} = \frac{5(-1+25cosh)}{128\cosh\cos(h/2)(1+2cosh)(1+2cosh+2cos2h)sin(h/2)^3}$.

Now, to apply the collocation technique collocation points are selected in such a way that they concur with the nodal points. By substituting the values of $y_i, y'_{i,} y_i'' and y_i'''$ at the nodal points in equation (3) we get a system of *N+1* linear equations in *N+5* unspecified variables as:

$$W_1{}^i\alpha_{i-2} + W_2{}^i\alpha_{i-1} + W_3{}^i\alpha_i + W_4{}^i\alpha_{i+1} + W_5{}^i\alpha_{i+2} = f_i, \quad 0 \le i \le N \tag{5}$$

where $W_1{}^i = k_{11}P(x) + k_8Q(x) + k_4R(x) + k_1S(x)$,

$W_2{}^i = k_{12}P(x) + k_9Q(x) + k_5R(x) + k_2S(x)$, $W_3{}^i = k_{10}Q(x) + k_3S(x)$

$W_4{}^i = k_{13}P(x) + k_9Q(x) + k_6R(x) + k_2S(x)$, $W_5{}^i = k_{14}P(x) + k_8Q(x) + k_7R(x) + k_1S(x)$.

To calculate the four unknown variables α_{-1}, $\alpha_{-2}, \alpha_{N+1}$ and α_{N+2}, one can use the two given boundary conditions and two assumed conditions as following:

(i) $y_0 = \emptyset$ (ii) $y'(2) = \gamma$ and (iii) $y''(0) = 0$, (iv) $y''(2) = 0$.

By solving the system of equations $y_0 = \emptyset$ and $y''(0) = 0$, we obtained values of α_{-2} and α_{-1} as:

$\alpha_{-2} = \frac{Lk_2 - Rk_9}{k_8k_2 - k_1k_9} = A_1(say)$ and $\alpha_{-1} = \frac{Rk_8 - Lk_1}{k_8k_2 - k_1k_9} = A_2$

where, L=$-k_{10}\alpha_0 - k_9\alpha_1 - k_8\alpha_2$ and R=$\emptyset - k_3\alpha_0 - k_2\alpha_1 - k_1\alpha_2$.

Similarly, by solving the system of equations $y'(2) = \gamma$ and $y''(2) = 0$, we obtained values of α_{N+2} and α_{N+1} as:

$\alpha_{N+1} = \frac{L_1k_7 - R_1k_8}{k_9k_7 - k_8k_6} = A_3$ and $\alpha_{N+2} = \frac{R_1k_9 - L_1k_6}{k_9k_7 - k_8k_6} = A_4$

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where, $L_1 = -k_8\alpha_{N-2} - k_9\alpha_{N-1} - k_{10}\alpha_N$ and $R_1 = \gamma - k_4\alpha_{N-2} - k_5\alpha_{N-1}$.

Substituting these values in equation (5) for i=0, 1, N-1 and i=N, we get following equations:

$$\alpha_0 {W_3}^0 + \alpha_1 {W_4}^0 + \alpha_2 {W_5}^0 = f_0 - A_1 {W_1}^0 - A_2 {W_2}^0 \tag{6}$$

$$\alpha_0 {W_2}^1 + \alpha_1 {W_3}^1 + \alpha_2 {W_4}^1 + \alpha_3 {W_5}^1 = f_1 - A_2 {W_1}^1 \tag{7}$$

$$\alpha_{N-3} {W_1}^{N-1} + \alpha_{N-2} {W_2}^{N-1} + \alpha_{N-1} {W_3}^{N-1} + \alpha_N {W_4}^{N-1} = f_{N-1} - A_3 {W_5}^{N-1} \tag{8}$$

and

$$\alpha_{N-2} {W_1}^N + \alpha_{N-1} {W_2}^N + \alpha_N {W_3}^N = f_N - A_3 {W_4}^N - A_4 {W_5}^N \tag{9}$$

Now, by considering above equations (6) to (9) and other equations from equation (5) for i=2, 3 ,.., N-2, a system of order N+1 is obtained with N+1 variables as A α =B where $\alpha = \alpha_0, \alpha_1, \alpha_2, \dots\dots\dots, \alpha_N$. A is a penta-diagonal matrix given by

$$A=\begin{bmatrix} W_3^0 & W_4^0 & W_5^0 & \cdots & \cdots & \cdots & \cdots & 0 \\ W_2^1 & W_3^1 & W_4^1 & W_5^1 & \cdots & \cdots & \cdots & 0 \\ W_1^2 & W_2^2 & W_3^2 & W_4^2 & W_5^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & W_1^i & W_2^i & W_3^i & W_4^i & W_5^i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & W_1^{N-2} & W_2^{N-2} & W_3^{N-2} & \vdots W_4^{N-2} & W_5^{N-2} \\ 0 & \cdots & \cdots & \cdots & W_1^{N-1} & W_2^{N-1} & W_3^{N-1} & W_4^{N-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & W_1^N & W_2^N & W_3^N \end{bmatrix}$$

with coefficient matrix B=
$[f(x_0) - A_1W_1^0 - A_2W_2^0, f(x_1) - A_2W_1^1, f(x_2), \dots, f(x_{N-2}), f(x_{N-1}) - A_3W_5^{N-1}, f(x_N) - A_3W_4^N - A_4W_5^N]^t$.

5. Anatomy of Convergence

This section is preserved for the exploration of the convergence of the quantic trigonometric B-spline collocation technique. We have assumed C as a non-specific positive constant independent of δ, ε and N, which may capture different values at different points and h=1/N and N=2^m to ensure at least one point in boundary layer.

Lemma 5.1 (Kumar and Kadalbajoo, 2012) If the functions $a(x)$, $b(x)$, $c(x)$, $d(x)$ and $f(x)$ are sufficiently smooth and are independent of ε, then the solution y of (1) –(2) satisfies

$$\left|y^{(k)}(x)\right| \le C\left(1 + \varepsilon^{-k} e^{-\frac{\alpha x}{\varepsilon}}\right), k = 0, 1, 2 \dots$$

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For proof of this lemma, one can follow procedure similar to as given in Kellogg and Tsan (1978).

Lemma 5.2 Hall error estimation: If $f(x) \in C^2[0,1]$ and $y(x) \in C^4[0,1]$, then $\|D^j(y-Y)\| \le \lambda_j \|y^4\| h^{4-j}, j = 0,1,2, \ldots$ where λ_j are the constants (Hall, 1968).

Lemma 5.3 If A is diagonally dominant by rows and $\alpha = \min_i(|a_{i,i}| - \sum_{i\neq j}|a_{i,j}|)$. Then $\|A^{-1}\|_\infty < 1/\alpha$ (Varah, 1975).

Theorem 5.1 Let S(x) be the approximation obtained by collocation method to the solution y(x) of boundary value problem (1)-(2). If $f \epsilon C^2[0,1]$, then the error estimate is given by
$\sup_\varepsilon max_i |y(x_i) - S(x_i)| \le CN^{-1} ln^3 N$, where $0 \le i \le N$ and $0 < \varepsilon \le 1$.

Proof: Consider $Y(x)$ be the unique spline interpolate to the solution y(x) of SPDDE given in (1)-(2) given by (4) and the estimated error is given by $|y(x) - S(x)|$.

Now by using Hall error estimation as defined in Lemma 5.2 we get the following estimation:

$$|Ly(x_i) - LY(x_i)| = |-\varepsilon||y'''(x_i) - Y'''(x_i)| + |a(x) - d(x)||y''(x_i) - Y''(x_i)| \\ +|b(x) + d(x)||y'(x_i) - Y'(x_i)| + |c(x)||y(x_i) - Y(x_i)| \\ \le (c_\varepsilon \lambda_3 h + (\|a(x)\| + \|b(x)\|)\lambda_2 h^2 + (\|b(x)\| + \|d(x)\|)\lambda_1 h^3 + \|c(x)\|\lambda_0 h^4)\|y^4\|.$$

Using Lemma 5.1, we get

$$|Ly(x_i) - LY(x_i)| \le (c_\varepsilon \lambda_3 h + (\|a(x)\| + \|b(x)\|)\lambda_2 h^2 + (\|b(x)\| + \|d(x)\|)\lambda_1 h^3 \\ +\|c(x)\|\lambda_0 h^4)\, C\left(1 + \varepsilon^{-k} e^{-\alpha x/\varepsilon}\right) \tag{10}$$

where $\varepsilon^{-1} \le C \ln N$. Thus, $|Ly(x_i) - LY(x_i)| \le CN^{-1} ln^3 N$.

Therefore, we have $|Ly(x_i) - LY(x_i)| = |f(x_i) - LY(x_i)| \le CN^{-1} ln^3 N$ (11)

Now consider the boundary value problem as $LY(x) = \bar{f}(x_i)$ with conditions $Y(x_0) = \phi(0), Y(x_N) = \gamma$.

A$\bar{\alpha}$=$\bar{B}$ is a linear system of equations obtained from the above problem, which follows that

$$A(\alpha - \bar{\alpha}) = B - \bar{B} \tag{12}$$
where $B - \bar{B} = [f(x_0) - \bar{f}(x_0), f(x_1) - \bar{f}(x_1), \ldots, f(x_N) - \bar{f}(x_N)]^t$

By using (11), $\|B - \bar{B}\| \le CN^{-1} ln^3 N$ (13)

The matrix A is strictly diagonal dominant for sufficiently small values of h and

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$$|a_{i,i}| - (|a_{i,i-1}| + |a_{i,i+1}|) = \begin{cases} F(x_0), for\ first\ row \\ L(x_N), for\ last\ row \\ M(x_i), otherwise \end{cases}$$

where $F(x_0) = -(k_{13} + k_{14})P(x_0) + (k_{10} - k_9 - k_8)Q(x_0) - (k_6 + k_7)R(x_0) + (k_3 - k_2 - k_1)S(x_0)$,

$L(x_N) = -(k_{11} + k_{12})P(x_N) + (k_{10} - k_9 - k_8)Q(x_N) + (k_5 - k_4)R(x_N) + (k_6 - k_2 - k_1)S(x_N)$,

and $M(x_i) = -\varepsilon A' + (a(x_i) - d(x_i))A'' - (b(x_i) + d(x_i))A''' + c(x_i)A''''$,

where $A' = k_{11} + k_{10} + k_{13} + k_{14}, A'' = k_{10} - 2k_8 - 2k_9, A''' = k_4 + k_5 + k_6 + k_7, A'''' = k_3 - 2k_1 - 2k_2$.

It is apparent, that A is strictly diagonally dominant. So, by using above and Lemma 5.3, it is concluded that

$$\|A^{-1}\| \leq C \tag{14}$$

Now combining (12), (13), and (14), $|\alpha - \bar{\alpha}| \leq CN^{-1}ln^3N, 0 \leq i \leq N$.

Similarly, estimating $|\alpha_i - \bar{\alpha_i}|$ from the boundary and assumed conditions as defined in Section 4, we get,

$$max|\alpha_i - \bar{\alpha_i}| \leq CN^{-1}ln^3N,\ for - 2 \leq i \leq N + 2 \tag{15}$$

Now by using Lemma 5.1 and (15) to estimate$|S(x) - Y(x)| = \sum_{i=-2}^{N+2}(\alpha_i - \bar{\alpha_i})T_{5,i}(x)$, we get $|S(x) - Y(x)| \leq CN^{-1}ln^3N$, which leads to result of theorem with triangleinequality.

sup(ε) max(i) $|y(x_i) - S(x_i)| \leq CN^{-1}ln^3N$, where $0 \leq i \leq N$ and $0 < \varepsilon \leq 1$.

Hence the theorem is proved.

6. Numerical Examples

To validate the proposed scheme two examples are considered for the numerical simulation. As the exact solution of the equation is not known hence the double mesh principle is used to calculate the maximum absolute error and $D^N = max|y_i^N - y_{2i}^{2N}|$ where $1 \leq i \leq N$.

Example 1: $-\varepsilon y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x)d(x)1/2y'(x-1) = 0,$

$$y(x) = 1 + x, x\epsilon[-1,0], y'(2) = 2$$

where $a_1 = 16, a_2 = 10, b(x) = 0, c(x) = -1, d(x) = -1, f_1 = 1, f_2 = -1$.

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The maximum absolute error obtained for example 1 at different domain partitions is presented in Table 1 and Figure 1 depicts the solution of Example 1 for *N*=128.

Table 1. Maximum absolute error of Example 1

Epsilon	N=16	N=32	N=64	N=128	N=256	N=512	N=1024
$\varepsilon = 2^{-6}$	1.29E+00	1.08E+00	1.41E-01	6.13E-03	2.69E-04	2.46E-02	1.93E-01
$\varepsilon = 2^{-7}$	1.10E+00	1.04E+00	1.18E-01	2.00E-03	1.38E-06	8.69E-09	7.16E-08
$\varepsilon = 2^{-8}$	1.00E+00	1.01E+00	1.04E-01	7.20E-04	8.63E-07	1.29E-11	7.84E-18
$\varepsilon = 2^{-9}$	9.52E-01	9.98E-01	9.69E-02	5.81E-04	1.41E-07	1.22E-13	2.29E-23
$\varepsilon = 2^{-10}$	9.26E-01	9.90E-01	9.31E-02	6.46E-04	1.24E-07	4.94E-15	1.81E-27
$\varepsilon = 2^{-11}$	9.13E-01	9.86E-01	9.12E-02	6.71E-04	1.19E-07	1.95E-15	1.05E-30
$\varepsilon = 2^{-12}$	9.06E-01	9.84E-01	9.03E-02	6.82E-04	1.15E-07	1.10E-15	1.16E-30
$\varepsilon = 2^{-13}$	9.03E-01	9.83E-01	8.98E-02	6.87E-04	1.12E-07	7.29E-16	2.74E-31
$\varepsilon = 2^{-14}$	9.01E-01	9.83E-01	8.96E-02	6.89E-04	1.11E-07	5.61E-16	1.82E-31
$\varepsilon = 2^{-15}$	9.00E-01	9.83E-01	8.94E-02	6.90E-04	1.10E-07	4.82E-16	1.38E-31
$\varepsilon = 2^{-16}$	9.00E-01	9.83E-01	8.94E-02	6.91E-04	1.10E-07	4.44E-16	1.17E-31
$\varepsilon = 2^{-17}$	9.00E-01	9.82E-01	8.93E-02	6.91E-04	1.10E-07	4.25E-16	1.07E-31
$\varepsilon = 2^{-18}$	9.00E-01	9.82E-01	8.93E-02	6.91E-04	1.09E-07	4.15E-16	1.02E-31
$\varepsilon = 2^{-19}$	9.00E-01	9.82E-01	8.93E-02	6.91E-04	1.09E-07	4.11E-16	9.97E-32
$\varepsilon = 2^{-20}$	9.00E-01	9.82E-01	8.93E-02	6.91E-04	1.09E-07	4.08E-16	9.85E-32
$\varepsilon = 2^{-21}$	9.00E-01	9.82E-01	8.93E-02	6.91E-04	1.09E-07	4.07E-16	9.79E-32
$\varepsilon = 2^{-22}$	9.00E-01	9.82E-01	8.93E-02	6.91E-04	1.09E-07	4.07E-16	9.76E-32
$\varepsilon = 2^{-23}$	9.00E-01	9.82E-01	8.93E-02	6.91E-04	1.09E-07	4.06E-16	9.75E-32

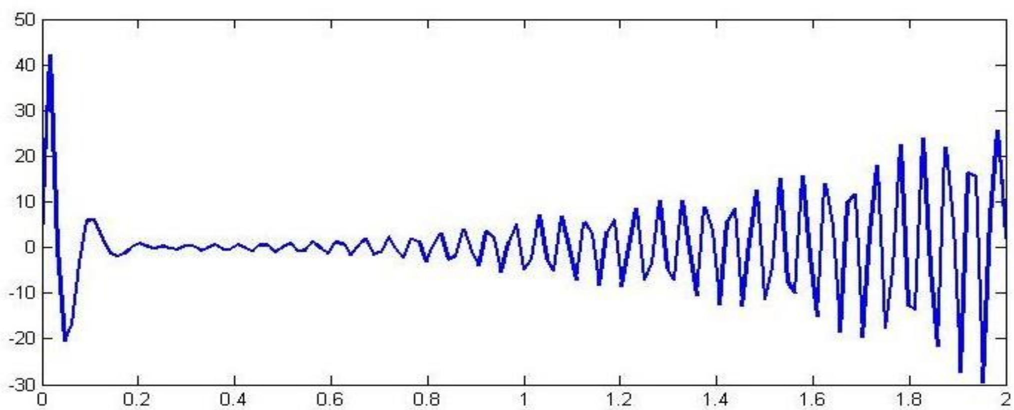

Figure 1. Graph of solution of Example 1 for *N*=128 and epsilon=0.25

Example 2: $-\varepsilon y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x)d(x)1/2y'(x-1) = 0$

$$y(x) = 1 + x, x\epsilon[-1,0], y'(2) = 2$$

where $a_1 = 10 + \exp(x^2)$, $a_2 = 10 + \exp(-\mathrm{x})$, $b(x) = 3, c(x) = -1, d(x) = -1, f_1 = 1, f_2 = -1$.

Table 2, demonstrates the maximum absolute error obtained for example 2 at different values of *N*. Figure 2 shows the solution of Example 2 for *N*=128.

Table 2. Maximum absolute error of Example 2

Epsilon	N=16	N=32	N=64	N=128	N=256	N=512	N=1024
$\varepsilon = 2^{-6}$	3.37E+00	7.46E-01	7.90E-02	6.84E-03	2.04E-04	1.67E-02	2.48E-01
$\varepsilon = 2^{-7}$	3.13E+00	5.87E-01	8.32E-02	2.23E-03	5.57E-06	5.61E-09	9.54E-06
$\varepsilon = 2^{-8}$	3.01E+00	5.16E-01	8.74E-02	9.33E-04	7.23E-07	1.02E-11	4.45E-18
$\varepsilon = 2^{-9}$	2.95E+00	4.83E-01	8.79E-02	4.20E-04	7.41E-08	1.49E-13	2.19E-23
$\varepsilon = 2^{-10}$	2.92E+00	4.67E-01	8.78E-02	2.06E-04	4.57E-08	1.67E-15	1.34E-27
$\varepsilon = 2^{-11}$	2.91E+00	4.59E-01	8.77E-02	1.91E-04	6.42E-08	2.49E-15	1.68E-29
$\varepsilon = 2^{-12}$	2.90E+00	4.55E-01	8.76E-02	1.97E-04	7.28E-08	1.85E-15	3.69E-31
$\varepsilon = 2^{-13}$	2.90E+00	4.53E-01	8.76E-02	1.98E-04	7.57E-08	1.42E-15	2.65E-31
$\varepsilon = 2^{-14}$	2.90E+00	4.52E-01	8.75E-02	1.99E-04	7.69E-08	1.21E-15	2.34E-31
$\varepsilon = 2^{-15}$	2.89E+00	4.52E-01	8.75E-02	2.00E-04	7.74E-08	1.10E-15	1.96E-31
$\varepsilon = 2^{-16}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.76E-08	1.05E-15	1.74E-31
$\varepsilon = 2^{-17}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.78E-08	1.02E-15	1.63E-31
$\varepsilon = 2^{-18}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.78E-08	1.01E-15	1.57E-31
$\varepsilon = 2^{-19}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.78E-08	1.00E-15	1.54E-31
$\varepsilon = 2^{-20}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.79E-08	1.00E-15	1.53E-31
$\varepsilon = 2^{-21}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.79E-08	1.00E-15	1.52E-31
$\varepsilon = 2^{-22}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.79E-08	9.99E-16	1.52E-31
$\varepsilon = 2^{-23}$	2.89E+00	4.51E-01	8.75E-02	2.00E-04	7.79E-08	9.99E-16	1.52E-31

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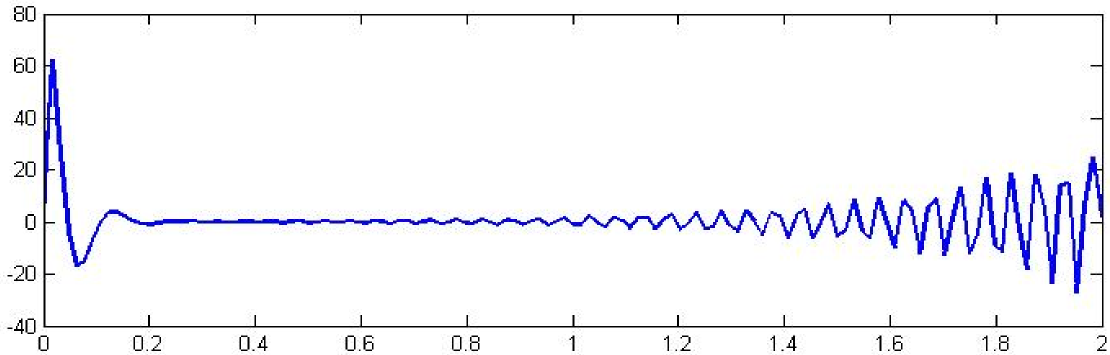

Figure 2. Graph of solution of Example 2 for N=128 and epsilon=0.25

7. Conclusion

A class of third order SPDDE for ordinary delay differential equations with large delay is considered with discontinuous convection-diffusion coefficient and source term. In section 3 existence of the solution of the problem is shown. With the uniform partition of the domain, the solution is approximated by quintic trigonometric B-spline basis by collocation technique. Discussion of convergence is carried out by the Hall theorem and the order of convergence of the method presented in this paper is of the almost first order. It can be concluded from the numerical results that maximum absolute error declines as N increases. Hence the presented scheme is efficient to simulate the considered SPDDE and other related types of differential equations.

Conflict of Interest

The authors confirm that this article contents have no conflict of interest.

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