Discrimination between Gamma and Rayleigh Models

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Abstract
The two popular life testing models are considered to verify whether one can be an alternative to other. The motivation for this study is as follows. It is well known that the cumulative distribution function of Rayleigh distribution can be analytically inverted whereas it is not so with Gamma distribution. Generally analytical inversion of cumulative distribution function would be advantageous in the study of problems of inference. “Whether this advantage can be explored in assessing the discrimination or otherwise of the two models” is studied in this paper.

Keywords: Rayleigh distribution, Gamma distribution, likelihood ratio criterion.

1. Introduction
The cumulative distribution function and probability density function of Gamma distribution with shape parameter 2 and Rayleigh distribution (Weibull with shape parameter 2) are given by Eqs. (1-4):

\[ F_0(x) = 1 - e^{-x/\sigma}(1 + \frac{x}{\sigma}); x \geq 0, \sigma > 0. \]  
(1)

and

\[ f_0(x) = \frac{x}{\sigma^2} e^{-x/\sigma}; x \geq 0, \sigma > 0. \]  
(2)

\[ F_1(x) = 1 - e^{-x^2/(2\sigma^2)}; 0 \leq x < \infty, \sigma > 0. \]  
(3)

and

\[ f_1(x) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}; 0 \leq x < \infty, \sigma > 0. \]  
(4)

Here \( \sigma \) is called the scale parameter in the above models. We see in most of the text books that the frequency curves of the two distributions look alike. The cumulative distribution function of Rayleigh distribution is analytically invertible whereas that of Gamma distribution is an incomplete Gamma function extensively tabulated. This advantage of Rayleigh distribution is the motivation to study whether it can be an alternative to Gamma with shape parameter 2. Some of the authors working in this direction are of Gupta and Kundu (2003) for the Weibull and the

The likelihood ratio criterion is narrated in Section-2. The adopted simulation study results and conclusions are given in Section-3 and Section-4.

2. Likelihood Ratio Criterion

In our study Gamma 2 is taken as null population and Rayleigh is taken as an alternative population.

i.e., $H_0$: A given sample $(x_1, x_2, ..., x_n)$ belongs to Gamma 2 model.

$H_1$: The sample $(x_1, x_2, ..., x_n)$ belongs to Rayleigh model.

The MLE of $\sigma$ of Gamma 2 is known as (Eq. (5))

$$ \frac{\partial \log L_0}{\partial \sigma} = 0 \Rightarrow \sigma = \frac{x}{2} $$

(Eq. 5)

Similarly MLE of $\sigma$ of Rayleigh is known to be (Eq. (6))

$$ \frac{\partial \log L_1}{\partial \sigma} = 0 \Rightarrow \sigma = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}} $$

(Eq. 6)

The LR statistic is defined as $\lambda = \ln \left( \frac{L_1}{L_0} \right)$ evaluated at the respective MLEs of $\sigma$.

i.e.,

$$ \lambda = 2n \ln(\bar{x}) - n \ln\left(\sum_{i=1}^{n} x_i^2\right) + n \ln(n) + n - n \ln(2) . $$
If $\lambda$ is negative then the null hypothesis that the sample is from Gamma is accepted. In other words for a sample that is generated from Gamma, a negative value of $\lambda$ yields correct decision otherwise it will give wrong decision.

Therefore $\lambda$ can be used as a criterion for testing the above $H_0$ vs. $H_1$. Hence the percentiles of $\lambda$ are essential for this purpose. However we adopted a different simulation study to get the proportion of correct decision vs. a wrong decision as given in Section 3.

3. Simulation Study

Samples of size 5, 10, 15, 20 and 25 are generated from Gamma 2 and at those samples the $\sigma$ of Gamma 2 as well as Rayleigh distribution is estimated by ML method. The respective likelihood and the LR criterion $\lambda$ are evaluated. The proportion of negative and positive $\lambda$’s are computed out of the used simulation runs. The details are given in the following Table 1.

<table>
<thead>
<tr>
<th>Sample Size ($n$)</th>
<th>Number of times $\lambda$ is negative</th>
<th>Number of times $\lambda$ is positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.38</td>
<td>0.62</td>
</tr>
<tr>
<td>10</td>
<td>0.57</td>
<td>0.43</td>
</tr>
<tr>
<td>15</td>
<td>0.63</td>
<td>0.37</td>
</tr>
<tr>
<td>20</td>
<td>0.69</td>
<td>0.31</td>
</tr>
<tr>
<td>25</td>
<td>0.75</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 1. Proportion of negative and positive $\lambda$’s

These results reveal that when sample size is more than 15 Rayleigh cannot be an alternative to Gamma 2. In small samples that are of size 5, 10 Rayleigh can be used as a reasonable alternative to Gamma 2. Specifically over increased simulation runs for sample sizes 5, 10 we observe the following Table 2.

<table>
<thead>
<tr>
<th>Sample Size ($n$)</th>
<th>Count of Negative $\lambda$</th>
<th>Count of Positive $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.59489</td>
<td>0.40511</td>
</tr>
<tr>
<td>10</td>
<td>0.43973</td>
<td>0.56027</td>
</tr>
</tbody>
</table>

Table 2. Increased simulation runs for sample sizes 5 and 10

This further shows a fifty-fifty choice of Gamma 2 vs. Rayleigh up to a sample of size 10.

4. Conclusion

We therefore conclude that in samples of size as small as up to 10, Gamma 2 and Rayleigh are equally likely models for a sample generated from either of the two. For other large samples they make a distinction.

References


