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Two Discrete-time Age-based Replacement Problems with/without Discounting

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Abstract
This paper considers two classical age-based replacement models within a discrete-time framework: a standard age replacement model and an opportunistic age replacement model. Specifically, our analysis incorporates the concept of replacement priority in situations where failure replacement and preventive replacement occur at a given age or opportunity. We explore two priority cases within each replacement model. First, we formulate optimal preventive replacement policies aimed at minimizing the associated expected cost rate in the age replacement model and the opportunistic age replacement model by the familiar renewal reward argument. Next, we extend the findings presented earlier to scenarios involving discounting. We develop formulations for the expected total discounted costs over an infinite time horizon and obtain optimal preventive replacement policies minimizing these total expected costs. Additionally, we explore unified models incorporating probabilistic priority. To provide practical insights, we present numerical illustrations using real failure data from pole air switches, comparing the performance of these optimal preventive policies.

Keywords- Age replacement, Opportunistic replacement, Replacement option priority, Discrete-time models, Probabilistic priority.

1. Introduction
For stable operations of industrial systems, it is quite important to plan the maintenance schedules with economic justification. The preventive maintenance models play a significant role to give the solutions by keeping balancing between the corrective replacement and the pre-scheduled replacement cost. In more detail, the corrective (failure) replacement is that if the industrial system breaks down, it is promptly renewed with a new one. For another thing, the pre-scheduled replacement refers that the non-failed system is replaced preventively in advance. Typically, the cost of pre-scheduled replacement is lower than that of corrective replacement. Especially, the age replacement model shave received much attention to trigger the proactive failure management in industry. Nevertheless, opportunistic age replacement models have not been extensively explored, except in certain specific cases where opportunities for preventive replacement occur randomly. Given the advancements in supply chain network and the constraints on transportation time for spare parts, there is often the possibility to acquire spare parts at more cost-economic prices a tun predictable timings. In other words, opportunistic replacement is making

preventive replacements at a randomly occurring opportunity. In such a background, some researchers have developed several opportunity-based replacement models. The early models were supported by the relevant works (Radner and Jorgenson, 1963; Berg, 1978; Pullen and Thomas, 1986; Zheng, 1995; Zheng and Fard, 1991; Dekker and Smeithink, 1991, 1994; Dekker and Dijkstra, 1992; Iskandar and Sandoh, 1999; Jhang and Sheu, 1999). More concretely, Radner and Jorgenson (1963) first considered the opportunity arrival in age replacement model. Based on Radner and Jorgenson (1963), the preventive replacement models consisting of multiple units were studied by Berg (1978), Pullen and Thomas (1986) and Zheng (1995). Dekker and Dijkstra (1992), Dekker and Smeitink (1991) and Dekker and Smeitink (1994) took an opportunity consideration into the age and block replacement models and developed the classical control-limit policies. Iskandar and Sandoh (1999) also considered the opportunity-based age replacement model and analyzed impact of the warranty in replacement policy. Jhang and Sheu (1999) extended opportunistic age replacement policies with major and minimal repairs. Recently, different opportunistic age replacement models were developed by Cavalcante et al. (2018), Laggoune et al. (2010), Wang et al. (2021), Najafi et al. (2021). For instance, Cavalcante et al. (2018) examined the impact of opportunities in a maintenance model that integrates inspection and replacement. Their finding suggests that opportunities can enhance performance in maintenance planning. Similarly, Wang et al. (2021) investigated imperfect maintenance policies in a complex system, formulating condition-based and age-based maintenance policies, respectively.

It is worth mentioning that almost existing opportunity-based age replacement models just focused on continuous-time models. In practice, the discrete-time modeling is quite useful in scheduling the preventive replacement. For instance, some Japanese electrical power companies often reported their failure data of pole air switches by month or year. So, the continuous-time models may not work for such an annual maintenance planning. Nakagawa and Osaki (1977) first considered age replacement model in discrete-time setting as an analogy of the common continuous-time model. Since the seminal work in Nakagawa and Osaki (1977), more preventive replacement models in discrete time (see, e.g., Nakagawa 1984, 1985) were investigated from the viewpoints of their optimality structures. Castro and Alfa (2004) discussed the optimal discrete time preventive policy from two approaches; replacing the unit and repairing the unit. Eryilmaz (2021) revisited age replacement models when the lifetime of the system is modeled by a discrete phase-type distribution. Chien (2012) also formulated a discrete-time age replacement models and examined the impacts of a free-repair warranty in replacement policies.

It is noted that discrete-time replacement models face a technical challenge in formulation, as there is a probability that two or more replacement options may arrive simultaneously. For example, in the standard age replacement model in Nakagawa and Osaki (1977), two replacement options; pre-scheduled re placement and failure (corrective) replacement, may occur simultaneously at the same time. In this case, the authors only considered that the corrective replacement is prioritized to the pre-scheduled replacement, because the failure replacement is considered as are active action for the failure event. On the other hand, if the replacement options are selective, then the preventive replacement with cheaper replacement cost will be justified economically rather than the failure replacement with a more expensive replacement cost. Hence, in discrete-time model setting, the priority of replacement options has to be clarified in advance, and strongly decided on the calculation of the expected costs. For the challenge problem in discrete-time replacement modeling, Dohi et al. (2005) introduced the concept of replacement option priority and reformulated an opportunistic age replacement model. More concretely, replacement option priority is defined that if two or more replacement options occur simultaneously, our model will choose the option with higher priority. Recently, Wu et al. (2024) extended the above model by taking account of replacement first and last disciplines introduced by Zhao and Nakagawa (2012). However, we should note that the standard age-based replacement models have not been yet considered by taking into account of the replacement priority. This fact penalizes us to apply the existing discrete-time replacement models in practice. Our motivation in this paper is to reformulate two standard age-based replacement models in discrete-time, taking into account of the replacement priority.

Most optimal policies in the preventive maintenance models above are formulated by optimizing the expected costs rate. In other words, the time value of cost is not considered. Nowadays, since the global economic environment is strongly unstable, the net present value (NPV) method is accurate to estimate the maintenance cost. Fox (1966) was the first work on the age replacement model with NPV method. Nakagawa and Osaki (1977) explored the age replacement model with discounting in discrete time. Chen and Savits (1988) formulated age and block replacement policies with discounting and discussed their interrelationship. In comprehensive surveys, Nakagawa (2005, 2014) summarized age, periodic, block, and random replacement models with discounting using the NPV approach. van den Boomen et al. (2018) introduced a modified NPV method to calculate age and interval replacement models. Marais and Saleh (2009) developed an analytical framework to capture the NPV in a multi-states failure system. Zhang et al. (2023) analyzed the impact of mission and discounted rate in replacement first and last models. Unfortunately, almost all models mentioned above are formulated in continuous time, except Nakagawa and Osaki (1977).

On the one hand, Nakagawa and Osaki (1977) and Nakagawa (1984, 1985) implicitly assume that failure replacement is always chosen when both failure replacement and pre-scheduled replacement occur simultaneously. However, this assumption is valid in only rare cases, as the cost of pre-scheduled replacement is typically lower than that of failure replacement. In this paper, we introduce the concept of replacement priority to prioritize between these two replacement policies in discrete time. Additionally, we extend the application of the replacement priority concept to the discrete-time opportunistic age replacement model, building upon the work of Dekker and Dijkstra (1992). On the other hand, the contemporary economic environment is characterized by instability due to economic shocks. Existing opportunity-based models, including those by Dekker and Smeithink (1991, 1994), Dekker and Dijkstra (1992), Iskandar and Sandoh (1999), Jhang and Sheu (1999), and Laggoune et al. (2010), do not account for the time value of cost. These circumstances motivate us to analyze the models using the Net Present Value (NPV) method. In comparison with existing models, we briefly summarize the main contributions of this paper as follows:

(i) The concept of replacement option priority is introduced to the discrete-time replacement model.
(ii) The NPVs in the fundamental age replacement and opportunistic age replacement models are analyzed.
(iii) The optimal preventive replacement policies in each model are compared comprehensively.

This rest of paper is organized as follows. Section 2 describes two discrete-time age-based replacement models: a standard age replacement by Nakagawa and Osaki (1977) and an opportunistic age replacement model by Dekker and Dijkstra (1992). We introduce the priority of two replacement options in discrete time and derive the optimal replacement policies with/without discounting via the renewal reward and NPV approaches. In section 3, we introduce the probabilistic priority and unify two replacement options in each age-based replacement model, where each replacement option may occur with a probability. Section 4 compares the optimal preventive replacement times and their related expected cost rate with replacement priority through numerical examples. In these examples, we consider a real-world replacement problem involving pole air switches in a Japanese power company. Finally, section 5 ends the paper with some conclusions.

2. Model Formulation

2.1 Assumptions

In discrete-time setting, we revisit the classical age replacement model and the opportunistic age replacement model, separately. It is general that the lifetime of the system, *Y*, is integer-value and independent and identically distributed (i.i.d.) random variable shaving the common probability mass function (p.m.f.) $f_Y(n)$ $(n = 1, 2, \cdots)$ with the survivor function $\Pr\{Y \geq n\} = \overline{F}_Y(n-1)$. In the classical age replacement problem, if the failure occurs before time $n_o (= 1, 2 \cdots)$, the failed item is renewed at the failure point, otherwise, it is renewed by new one at time n_o .It is general that $\overline{G}(\cdot) = 1 - G(\cdot)$ and $\overline{F}(\cdot) = 1 - F(\cdot)$ .On the other hand, in the opportunity-based age replacement problem, we assume that the arrivals of replacement opportunity arrival, *X*, arise obeying the i.i.d. geometric distribution $\Pr\{X = x\} = g_X(x) = p(1-p)^{x-1}$ $(n = 1, 2 \cdots; 0 < p < 1)$. Even if the opportunities arrive before time n_o , the pre-scheduled replacement is not performed. However, if the failure does not occur before time n_o , the pre-scheduled replacement is made at the first arrival of the replacement opportunity after the time n_o . In this case, if the failure occurs before time n_o , the failure replacement is carried out. All the notations are shown in Table 10 in the Appendix.

The cost parameters in our modeling are given by:

c_1 : failure (corrective) replacement cost for each failed item,

c_2 : preventive replacement cost at a pre-scheduled time,

c_3 : preventive replacement cost at a random opportunity.

For the above notations, we suppose that:

***Assumption* 1:** $c_1 > c_2 \geq c_3$.

It is noted that one of two options; failure replacement F_a or pre-scheduled replacement S_c (or opportunistic replacement O_p), may occur as a simultaneous event in discrete-time setting. We introduce the definition of the replacement option priority.

***Definition* 1:** The option *P* has a priority to the option *Q*, if $P \succ Q$.

From this definition, we know that if two replacement options occur simultaneously, the option with higher priority should be chosen. In this study, the following four different models should be considered:

(i) Model 1: $S_c \succ F_a$,

(ii) Model 2: $F_a \succ S_c$,

(iii) Model 3: $O_p \succ F_a$,

(iv) Model 4: $F_a \succ O_p$.

Nakagawa and Osaki (1977) implicitly assumed Model 2 in their age replacement in discrete time. For another thing, if the replacement option is selective, it may be better to consider Model 1 because the pre-scheduled replacement with cheaper cost is prioritized to the failure replacement. In general, the simultaneous occurrence of F_a and S_c is a rare event and may not be selective in actual maintenance management. Hence, it would be valuable to consider two priority cases; Model 1 and Model 2 (Model 3 and Model 4) in the age replacement (opportunistic replacement) in discrete time.

2.2 Renewal Reward Approach

We define the discrete-time age replacement (AR) model. In this model, we consider two replacement options. If the item fails at time n, it is promptly replaced. Otherwise, a non-failed item is replaced at a pre-scheduled time $n = n_a$ with a new unit. The decision involves a tradeoff between the cost of replacing failed units and the cost of pre-scheduled replacements. The objective in this problem is to find the optimal pre-scheduled replacement time $n = n_a$ that minimizes the expected cost per unit time in a steady state (expected cost rate). The discrete-time AR model is illustrated in Figure 1.

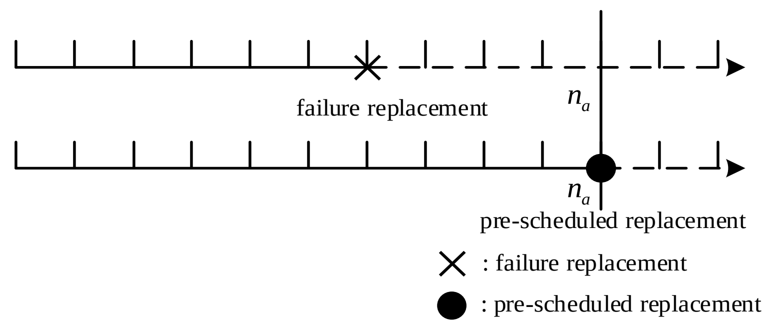

Figure 1. The discrete-time AR model.

For Model 1 and Model 2, we can calculate the probability that an item is renewed by new one at time $n(=1,2\cdots)$ as

$$h_{aj}(n) = \begin{cases} f_Y(n), & 1 \le n \le n_a - 1 \\ \overline{F}_Y(n-1), & n = n_a \\ 0, & n \ge n_a + 1 \end{cases} \tag{1}$$

where, $\sum_{n=1}^{\infty} h_{aj}(n) = 1$ $(j = 1, 2)$.

Let $A_{aj}(n_a)$ $(j = 1, 2)$ denote the meantime lengths during one cycle. Obviously, they are exactly the same in the two models, where,

$$\begin{aligned} A_{aj}(n_a) &= \sum_{n=1}^{n_a-1} n f_Y(n) + n_a \overline{F}_Y(n_a - 1) \\ &= \sum_{n=1}^{n_a} \overline{F}_Y(n-1) \end{aligned} \tag{2}$$

The total expected costs of one cycle, $B_{aj}(n_a)$, for Model j $(j = 1, 2)$, are given by,

$$\begin{aligned} B_{a1}(n_a) &= c_1 \sum_{n=1}^{n_a-1} f_Y(n) + c_2 \overline{F}_Y(n_a - 1) \\ &= c_2 + (c_1 - c_2) F_Y(n_a - 1) \end{aligned} \tag{3}$$

$$B_{a2}(n_a) = c_1 \sum_{n=1}^{n_a} f_Y(n) + c_2 \bar{F}_Y(n_a) \\ = c_2 + (c_1 - c_2) F_Y(n_a) \tag{4}$$

Based on the renewal reward theorem (Ross, 2013), the expected cost per unit time in the steady state (expected cost rates) $C_{aj}(n_a)$ for Model $j(=1,2)$ are formulated as,

$$C_{aj}(n_a) = \lim_{n\to\infty} \frac{\mathrm{E[total\ cost\ on}\ \ (0,n]]}{n} = \frac{B_{aj}(n_a)}{A_{aj}(n_a)} \tag{5}$$

and our interest is to find the optimal n_a^* minimizing $C_{aj}(n_a)$.

Define the non-linear functions:

$$q_{a1}(n_a) = R_Y(n_a) \sum_{n=1}^{n_a} \bar{F}_Y(n-1) - F_Y(n_a - 1) \tag{6}$$

$$q_{a2}(n_a) = r_Y(n_a + 1) \sum_{n=1}^{n_a} \bar{F}_Y(n-1) - F_Y(n_a) \tag{7}$$

where, $R_Y(n_a) = f_Y(n_a) / \bar{F}_Y(n_a)$.

For more detailed relationship between $R_Y(n_a)$ and $r_Y(n_a)$, see Lemma 1 in the Appendix.

***Theorem* 2.1** (i) *If the lifetime Y is strictly increasing failure rate (IFR), then $q_{aj}(n)$ (j= 1,2) is strictly increasing in n.*

- *In addition to (i), if $q_{aj}(\infty) > c_2/(c_1 - c_2)$, then it has at least one (at most two) optimal AR time n_a^* $(1 \le n_a^* < \infty)$ satisfying*

$$q_{aj}(n_a^* - 1) < c_2/(c_1 - c_2) \text{ and } q_{aj}(n_a^*) \ge c_2/(c_1 - c_2) \tag{8}$$

- *In addition to (i), if $q_{aj}(\infty) \le c_2/(c_1 - c_2)$, then the optimal AR time is $n_a^* \to \infty$, and the decision-maker only select the failure replacement.*

(ii) *If the lifetime Y is decreasing failure rate(DFR), then $q_{aj}(n)$ (j=1,2) is decreasing in n, and the optimal AR time is given by $n_a^* = 1$ or $n_a^* \to \infty$.*

To access the proof of Theorem 2.1, consult the Appendix. Theorem 2.1 highlights distinctions between our models and those proposed by Nakagawa and Osaki (1977). Notably, Model 1 demonstrates greater economic efficiency compared to Model 2. This is attributed to the lower cost associated with pre-scheduled replacement when contrasted with failure replacement expenses. Furthermore, our study delves into scenarios where the lifetime follows a decreasing failure rate (DFR). We find that it is different from the continuous age model, because $n_a^* = 1$ *or* $n_a^* \to \infty$ should be considered.

The following result is straight forward from Theorem 2.1.

***Theorem* 2.2** *For Model j(=1, 2), if lifetime* Y *is strictly IFR and* $q_{aj}(\infty) > c_2/(c_1 - c_2)$ *. Then the minimum expected cost rates* $C_{aj}(n_a^*)$ *can be obtained as*

$$U_{aj}(n_a^* - 1) < C_{aj}(n_a^*) \le U_{aj}(n_a^*) \tag{9}$$

where,

$$U_{a1}(n_a) = (c_1 - c_2) R_Y(n_a) \tag{10}$$

$$U_{a2}(n_a) = (c_1 - c_2) r_Y(n_a + 1) \tag{11}$$

Next, the opportunity-based age replacement (OR) model is considered. In the above AR model, the assumption was made that replacement is viable at any given moment. However, in practical scenarios, especially during busy periods, replacement may not always be readily available. In such instances, replacement commonly occurs at random intervals or trimmings. Dekker and Dijkstra (1992) explored an OR model where replacement opportunities follow a Poisson process. Their OR model demonstrates that a control-limit policy is optimal, ensuring replacement is triggered when the first opportunity arises just after a pre-scheduled replacement time limit $n = n_0$. Here, we reformulate the OR model in discrete-time setting. The discrete-time OR model is depicted in Figure 2.

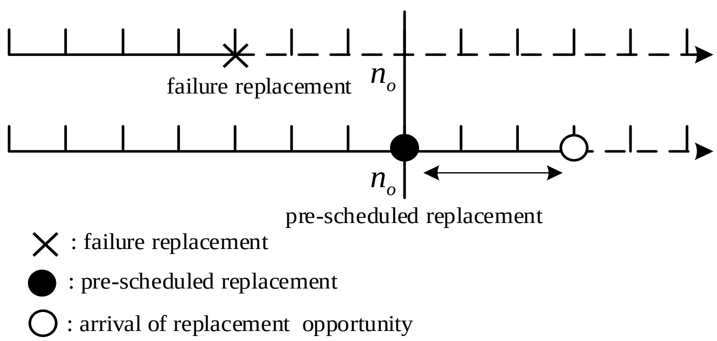

Figure 2. The discrete-time OR model.

For Model 3 and Model 4, we can give the probability that an item is renewed at time n $(= 0, 1, \cdots)$ as,

$$h_{oj}(n) = \begin{cases} f_Y(n), & 0 \le n \le n_o \\ f_Y(n)(1-p)^{n-n_o} + p(1-p)^{n-n_o-1}\bar{F}_Y(n-1), & n \ge n_o + 1 \end{cases} \tag{12}$$

where, $\sum_{n=0}^{\infty} h_{oj}(n) = 1$ $(j = 3, 4)$.

In this model, $A_{oj}(n_o)$ is the meantime lengths of one cycle, for Model j (= 3, 4), i.e.,

$$\begin{aligned} A_{oj}(n_o) &= \sum_{n=0}^{n_o} n f_Y(n) + \sum_{n=n_o+1}^{\infty} n\left[f_Y(n)(1-p)^{n-n_o} + p(1-p)^{n-n_o-1}\bar{F}_Y(n-1)\right] \\ &= \sum_{n=1}^{n_o} \bar{F}_Y(n-1) + \sum_{n=n_o+1}^{\infty} \bar{F}_Y(n-1)(1-p)^{n-n_o-1} \end{aligned} \tag{13}$$

$B_{oj}(n_o)$ are the expected total costs of one cycle in Model j (= 3, 4), where,

$$\begin{aligned} B_{o3}(n_o) &= c_1 \sum_{n=0}^{n_o} f_Y(n) + c_1 \sum_{n=n_o+1}^{\infty} f_Y(n)(1-p)^{n-n_o} + c_3 \sum_{n=n_o+1}^{\infty} p(1-p)^{n-n_o-1} \bar{F}_Y(n-1) \\ &= c_3 + (c_1 - c_3)\left[F_Y(n_o) + \sum_{n=n_o+1}^{\infty} f_Y(n)(1-p)^{n-n_o} \right] \end{aligned} \tag{14}$$

$$\begin{aligned} B_{o4}(n_o) &= c_1 \sum_{n=0}^{n_o} f_Y(n) + c_1 \sum_{n=n_o+1}^{\infty} f_Y(n)(1-p)^{n-n_o-1} + c_3 \sum_{n=n_o+1}^{\infty} p(1-p)^{n-n_o-1} \bar{F}_Y(n) \\ &= c_3 + (c_1 - c_3)\left[F_Y(n_o) + \sum_{n=n_o+1}^{\infty} f_Y(n)(1-p)^{n-n_o-1} \right] \end{aligned} \tag{15}$$

The expected costs rate, $C_{oj}(n_o)$, for Model j (=3,4) are

$$C_{oj}(n_o) = \lim_{n\to\infty} \frac{\mathrm{E}[\text{total cost on } (0,n]]}{n} = \frac{B_{oj}(n_o)}{A_{oj}(n_o)} \tag{16}$$

It is of interest to obtain the pre-scheduled replacement time limit n_o^* minimizing $C_{oj}(n_o)$. Define the non-linear functions:

$$q_{o3}(n_o) = H(n_o)A_{o3}(n_o) - \left[F_Y(n_o) + \sum_{n=n_o+1}^{\infty} f_Y(n)(1-p)^{n-n_o} \right] \tag{17}$$

$$q_{o4}(n_o) = h(n_o+1)A_{o3}(n_o) - \left[F_Y(n_o) + \sum_{n=n_o+1}^{\infty} f_Y(n)(1-p)^{n-n_o-1} \right] \tag{18}$$

where,

$$H(n_o) = \frac{\sum_{n=n_o+1}^{\infty} f_Y(n)(1-p)^{n-n_o}}{\sum_{n=n_o+1}^{\infty} \bar{F}_Y(n)(1-p)^{n-n_o}} \tag{19}$$

$$h(n_o+1) = \frac{\sum_{n=n_o+1}^{\infty} f_Y(n+1)(1-p)^{n-n_o}}{\sum_{n=n_o+1}^{\infty} \bar{F}_Y(n)(1-p)^{n-n_o}} \tag{20}$$

For additional information regarding the monotonic relationship between $R_Y(n)$ ($r_Y(n)$) and $H(n)$ ($h(n)$), refer to Lemma 2, Lemma 3, and Corollary 1 in the appendix. The following theorem describes the optimal pre-scheduled replacement time limit n_o^* minimizing the expected costs rate, $C_{oj}(n_o)$, for Model j(= 3,4), where the proof is given in the appendix.

***Theorem* 2.3** (i) *If the lifetime Y is strictly IFR, then $q_{oj}(n)$ (j=3,4) is strictly increasing in n.*

- *In addition to (i), if $q_{oj}(\infty) > c_3/(c_1 - c_3)$, then it has at least one (at most two) optimal OR time limit n_o^* $(0 \le n_o^* < \infty)$ satisfying*

$$q_{oj}(n_o^* - 1) < c_3/(c_1 - c_3) \text{ and } q_{oj}(n_o^*) \ge c_3/(c_1 - c_3) \tag{21}$$

- *In addition to (i), if $q_{oj}(\infty) \le c_3/(c_1 - c_3)$, then the optimal OR time limit is $n_o^* \to \infty$ and the decision-makers only select the failure replacement.*

(ii) If the lifetime Y is DFR, then $q_{oj}(n)$ (j=3,4) is decreasing in n, and the optimal OR time limit is $n_o^* = 0$ or $n_o^* = \infty$.

Compared with the classical model in Dekker and Dijkstra (1992), we further study the relationship between $R_Y(n)$ ($r_Y(n)$) and $H(n)$ ($h(n)$) and related corollaries. We also study the case that the lifetime is DFR.

***Theorem* 2.4** *For Model j(=3,4), if the lifetime Y is strictly IFR and $q_{oj}(\infty) > c_3/(c_1 - c_3)$. Then the minimal expected cost rates $C_{oj}(n_o^*)$ are given by*

$$U_{oj}(n_o^* - 1) < C_{oj}(n_o^*) \le U_{oj}(n_o^*) \tag{22}$$

where,

$$U_{o3}(n_o) = (c_1 - c_2)H(n_o) \tag{23}$$

$$U_{o4}(n_o) = (c_1 - c_2)h(n_o + 1) \tag{24}$$

2.3 NPV Approach

We denote the discount factor α $(0 < \alpha < 1)$ to represent the expected NPV of the unit cost. We first derive the preventive replacement policies in AR model. In the NPV formulation, the expected total discounted costs over an infinite time horizon, $C_{aj}(n_a, \alpha)$, for Model j(= 1, 2) are given by

$$C_{aj}(n_a, \alpha) = \left[c_1 + C_{aj}(n_a, \alpha)\right] \sum_{n=1}^{n_a - 1} \alpha^n f_Y(n) + \left[c_2 + C_{aj}(n_a, \alpha)\right] \alpha^{n_a} \bar{F}_Y(n_a - 1) \tag{25}$$

From a few algebraic manipulations, we can obtain,

$$C_{aj}(n_a, \alpha) = \frac{B_{aj}(n_a, \alpha)}{1 - A_{aj}(n_a, \alpha)} \tag{26}$$

where, the function $A_{aj}(n_a, \alpha)$ $(j = 1, 2)$ is the NPV of one unit cost during the renewal cycle. i.e.,

$$A_{aj}(n_a, \alpha) = \frac{1 - \alpha}{\alpha} \sum_{n=1}^{n_a} \alpha^n \bar{F}_Y(n - 1) \tag{27}$$

The functions $B_{aj}(n_a, \alpha)$ for Model j (=1,2) are the expected total discounted costs during the renewal cycle and they can be calculated by

$$B_{a1}(n_a,\alpha)=c_1\sum_{n=1}^{n_a-1}\alpha^n f_Y(n)+c_2\alpha^{n_a}\bar{F}_Y(n_a-1) \tag{28}$$

$$B_{a2}(n_a,\alpha)=c_1\sum_{n=1}^{n_a}\alpha^n f_Y(n)+c_2\alpha^{n_a}\bar{F}_Y(n_a) \tag{29}$$

It is evident from the well-known L'Hopital's theorem that

$$C_{aj}(n_a)=\lim_{\alpha\to 1}(1-\alpha)C_{aj}(n_a,\alpha) \tag{30}$$

Next, we define the non-linear functions for a fixed α:

$$q_{a1}(n_a|\alpha)=\left[\frac{(c_1-c_2)}{(1-\alpha)}R_Y(n_a)-c_2\right]\left[1-A_{a1}(n_a,\alpha)\right]-B_{a1}(n_a,\alpha) \tag{31}$$

$$q_{a2}(n_a|\alpha)=\left[\frac{\alpha(c_1-c_2)}{(1-\alpha)}r_Y(n_a+1)-c_2\right]\left[1-A_{a1}(n_a,\alpha)\right]-B_{a2}(n_a,\alpha) \tag{32}$$

In the NPV formulation, the optimal AR polices can be obtained (see for the proof in Appendix).

***Theorem* 2.5** (i) *If the lifetime* Y *is strictly IFR, then* $q_{aj}(n_a|\alpha)$ *(j=1,2) is strictly increasing in* n.

- *In addition to (i), if* $q_{aj}(\infty|\alpha)>0$, *then it has at least one (at most two) optimal AR time* n_a^* $(1\le n_a^*<\infty)$, *satisfying*

$$q_{aj}(n_a^*-1|\alpha)<0 \text{ and } q_{aj}(n_a^*|\alpha)\ge 0 \tag{33}$$

- *In addition to (i), if* $q_{aj}(\infty|\alpha)\le 0$, *then the optimal AR time is* $n_a^*\to\infty$ *and the decision-makers only perform the failure replacement.*

(iii) *If the lifetime* Y *is DFR, then* $q_{aj}(n_a|\alpha)$ *(j=1,2) is decreasing in* n, *and the optimal AR time becomes* $n_a^*=1$ *or* $n_a^*\to\infty$.

***Theorem* 2.6** *For Model j (=1,2), if the lifetime* Y *is strictly IFR and* $q_{aj}(\infty|\alpha)>0$. *Then the minimum* $C_{aj}(n_a^*|\alpha)$ *for a fixed* α *are given by*

$$U_{aj}(n_a^*-1|\alpha)<C_{aj}(n_a^*|\alpha)\le U_{aj}(n_a^*|\alpha) \tag{34}$$

where,

$$U_{a1}(n_a|\alpha)=\frac{(c_1-c_2)}{(1-\alpha)}R_Y(n_a)-c_2 \tag{35}$$

$$U_{a2}(n_a|\alpha)=\frac{\alpha(c_1-c_2)}{(1-\alpha)}r_Y(n_a+1)-c_2 \tag{36}$$

Next, we calculate the OR model with NPV approach. The expected NPV value of one unit cost during the renewal cycle, $A_{oj}(n_o,\alpha)$, for Model *j* (=3,4) are obtained as

$$A_{oj}(n_o,\alpha)=\sum_{n=0}^{n_o}\alpha^n f_Y(n)+\sum_{n=n_o+1}^{\infty}\alpha^n\left[f_Y(n)(1-p)^{n-n_o}+p(1-p)^{n-n_o-1}\bar{F}_Y(n-1)\right] \tag{37}$$

The expected total discounted costs during one cycle, $B_{oj}(n_o,\alpha)$, for Model *j* (=3,4) are given by,

$$B_{o3}(n_o,\alpha)=c_1\sum_{n=0}^{n_o}\alpha^n f_Y(n)+c_1\sum_{n=n_o+1}^{\infty}\alpha^n f_Y(n)(1-p)^{n-n_o}+c_3\sum_{n=n_o+1}^{\infty}\alpha^n p(1-p)^{n-n_o-1}\bar{F}_Y(n-1) \tag{38}$$

$$B_{o4}(n_o,\alpha)=c_1\sum_{n=0}^{n_o}\alpha^n f_Y(n)+c_1\sum_{n=n_o+1}^{\infty}\alpha^n f_Y(n)(1-p)^{n-n_o-1}+c_3\sum_{n=n_o+1}^{\infty}\alpha^n p(1-p)^{n-n_o-1}\bar{F}_Y(n) \tag{39}$$

Then, we obtain the NPV value of expected total costs $C_{oj}(n_o,\alpha)=B_{oj}(n_o,\alpha)/\left[1-A_{oj}(n_o,\alpha)\right]$ for Model *j* (=3,4).

It is evident to check that

$$C_{oj}(n_o)=\lim_{\alpha\to 1}(1-\alpha)C_{oj}(n_o,\alpha) \tag{40}$$

Define the non-linear functions for a fixed α:

$$q_{o3}(n_o|\alpha)=\left[\frac{(c_1-c_3)}{1-\alpha}H(n_o,\alpha)-c_3\right][1-A_{o3}(n_o,\alpha)]-B_{o3}(n_o,\alpha) \tag{41}$$

$$q_{o4}(n_o|\alpha)=\left[\frac{\alpha(c_1-c_3)}{1-\alpha}h(n_o+1,\alpha)-c_3\right][1-A_{o3}(n_o,\alpha)]-B_{o4}(n_o,\alpha) \tag{42}$$

where,

$$H(n_o,\alpha)=\frac{\sum_{n=n_o+1}^{\infty}f_Y(n)\alpha^n(1-p)^{n-n_o}}{\sum_{n=n_o+1}^{\infty}\bar{F}_Y(n)\alpha^n(1-p)^{n-n_o}} \tag{43}$$

$$h(n_o+1,\alpha)=\frac{\sum_{n=n_o+1}^{\infty}f_Y(n+1)\alpha^n(1-p)^{n-n_o}}{\sum_{n=n_o+1}^{\infty}\bar{F}_Y(n)\alpha^n(1-p)^{n-n_o}} \tag{44}$$

For additional information regarding the monotonic relationship $R_Y(n)$ ($r_Y(n)$) and $H(n,\alpha)$ ($h(n,\alpha)$), refer to Lemma 4 and Corollary 2 in Appendix. In the NPV formulation, the optimal OR policies can be described as follows (see the proof in Appendix).

Theorem 2.7 (i) *If the lifetime Y is strictly IFR, then $q_{oj}(n|\alpha)$ $(j=3,4)$ is strictly increasing in n.*

- *In addition to (i), if $q_{oj}(\infty|\alpha)>0$, then it has at least one (at most two) optimal OR time limit*

n_o^* $(0 \le n_o^* < \infty)$, *satisfying*

$$q_{oj}(n_o^* - 1|\alpha) < 0 \text{ and } q_{oj}(n_o^*|\alpha) \ge 0 \tag{45}$$

- *In addition to (i), if* $q_{oj}(\infty|\alpha) \le 0$, *then the optimal OR time limit is* $n_o^* \to \infty$ *and the decision-maker only choose failure replacement.*

(ii) *If the lifetime* Y *is DFR, then* $q_{oj}(n|\alpha)$ $(j = 3,4)$ *is decreasing in* n, *and the optimal OR time limit is* $n_o^* = 0$ *or* $n_o^* \to \infty$.

***Theorem* 2.8** *For Model j (=3,4),if the lifetime* Y *is strictly IFR and* $q_{oj}(n|\alpha) > 0$ *.Then the minimum* $C_{oj}(n_o^*|\alpha)$ *for a fixed* α *are given by*

$$U_{oj}(n_o^* - 1|\alpha) < C_{oj}(n_o^*|\alpha) \le U_{oj}(n_o^*|\alpha) \tag{46}$$

where,

$$U_{o3}(n_o|\alpha) = \frac{c_1 - c_3}{1-\alpha} H(n_o, \alpha) - c_3 \tag{47}$$

$$U_{o4}(n_o|\alpha) = \frac{\alpha(c_1 - c_3)}{1-\alpha} h(n_o + 1, \alpha) - c_3 \tag{48}$$

3. Unified Models with Probabilistic Priority

In the preceding section dealing with discrete-time age-based preventive replacement policies with renewal rewards and NPV approaches, we categorized two potential replacement priorities and determined the optimal pre-scheduled replacement times and pre-scheduled replacement times time limits for each case. Nevertheless, it is essential to acknowledge that the replacement option priority may not always be deterministic. In other words, the occurrence of priority may be probabilistic and subject to change at each decision point for replacement. For instance, if the decision-maker randomly selects one of the replacement options, this scenario holds true. We assume that each priority associated with Model *j* $(=1,2,3,4)$ happens with probability p_j $(0 \le p_j \le 1)$, where $p_1 + p_2 = 1$ and $p_3 + p_4 = 1$.

3.1 Renewal Reward Approach

First, we calculate discrete-time AR model with probabilistic priority. In unified model, the expected time length with probability p_j can be calculated by $A_{a3}(n_a) = A_{aj}(n_a)$ in Equation (2). In addition, the expected total cost of one cycle, $B_{a3}(n_a)$,with the probabilistic priority is given by $B_{a3}(n_a) = \sum_{j=1}^{2} p_j B_{aj}(n_a)$ with Equations (3) and (4). The underlying problem is simply formulated as $\min_{n_a} C_{a3}(n_a) = B_{a3}(n_a) / A_{aj}(n_a)$.

Define $q_{a3}(n_a) = \sum_{j=1}^{2} p_j U_{aj}(n_a) A_{aj}(n_a) - B_{aj}(n_a)$ with Equations (10) and (11). Then it can be seen that $q_{a3}(n_a + 1) - q_{a3}(n_a) = \sum_{j=1}^{2} p_j \{q_{aj}(n_a + 1) - q_{aj}(n_a)\} A_{aj}(n_a + 1)$. Hence, for $p_j \ne 0$, necessary conditions of strictly increasing $q_{aj}(n_a)$ are to hold all conditions in Theorem 2.1.

***Theorem* 3.1** *If the lifetime Y is strictly IFR and $q_{a3}(\infty) > c_2/(c_1 - c_2)$. Then the optimal $C_{a3}(n_a)$* in *unified model with probabilistic priority is obtained as*

$$U_{a3}(n_a^* - 1) < C_{a3}(n_a^*) \le U_{a3}(n_a^*) \tag{49}$$

where,

$$U_{a3}(n_a) = \sum_{j=1}^{2} p_j U_{aj}(n_a) \tag{50}$$

Next, the discrete-time OR model with probabilistic priority is consider. We have the expected time length $A_{o5}(n_o) = A_{oj}(n_o)$ in Equation (13) and the expected total cost during one cycle $B_{o5}(n_o) = \sum_{j=3}^{4} p_j B_{oj}(n_o)$ with Equation (14) and (15). Define $q_{o5}(n_o) = \sum_{j=3}^{4} p_j U_{oj}(n_o) A_{oj}(n_o) - B_{o5}(n_o)$ with Equation (23) and (24). Then, one has $q_{o5}(n_o+1) - q_{o5}(n_o) = \sum_{j=3}^{4} p_j \{q_{oj}(n_o+1) - q_{oj}(n_o)\} A_{oj}(n_o+1)$ .We calculate the underlying model $\min_{n_o} C_{o5}(n_o) = B_{o5}(n_o)/A_{o5}(n_o)$ and derive those necessary conditions of strictly increasing $q_{o5}(n_o)$ are to hold all conditions in Theorem 2.3.

***Theorem* 3.2** *If the lifetime Y is strictly IFR and $q_{o5}(\infty) > c_3/(c_1 - c_3)$. Then the optimal $C_{o5}(n_o^*)$ in unified model with probabilistic priority is given by*

$$U_{o5}(n_o^* - 1) < C_{o5}(n_o^*) \le U_{o5}(n_o^*) \tag{51}$$

where,

$$U_{o5}(n_a) = \sum_{j=3}^{4} p_j U_{oj}(n_o) \tag{52}$$

3.2 NPV Approach

Since the expected NPV of one unit cost during the renewal cycle and the expected total cost during one cycle are given by $A_{a3}(n_a, \alpha) = A_{aj}(n_a, \alpha)$ in Equation (27) and $B_{a3}(n_a, \alpha) = \sum_{j=1}^{2} p_j B_{aj}(n_a, \alpha)$ with Equation (28) and (29).We define $q_{a3}(n_a, \alpha) = \sum_{j=1}^{2} p_j U_{aj}(n_a, \alpha) A_{aj}(n_a, \alpha) - B_{aj}(n_a, \alpha)$ with Equation (35) and (36). Then it can be seen that $q_{a3}(n_a+1, \alpha) - q_{a3}(n_a, \alpha) = \sum_{j=1}^{2} p_j \{q_{aj}(n_a+1, \alpha) - q_{aj}(n_a, \alpha)\} A_{aj}(n_a+1, \alpha)$. Hence, for $p_j \ne 0$, necessary conditions of strictly increasing $q_{aj}(n_a, \alpha)$ are to hold all conditions in Theorem 2.5.

***Theorem* 3.3** *If the lifetime Y is strictly IFR and $q_{a3}(\infty|\alpha) > 0$. Then the minimum $C_{a3}(n_a^*|\alpha)$ for a fixed α is obtained as:*

$$U_{a3}(n_a^* - 1|\alpha) < C_{a3}(n_a^*|\alpha) \le U_{a3}(n_a^*|\alpha) \tag{53}$$

where,

$$U_{a3}(n_a|\alpha)=\sum_{j=1}^{2}p_jU_{aj}(n_a|\alpha) \tag{54}$$

Finally, we formulate the OR model with discounting. The expected NPV value of one unit cost and the expected total cost are given by $A_{o5}(n_o,\alpha)=A_{oj}(n_o,\alpha)$ in Equation (37) and $B_{o5}(n_o,\alpha)=\sum_{j=3}^{4}p_jB_{oj}(n_o,\alpha)$ with Equation (38) and (39). Define $q_{o5}(n_o,\alpha)=\sum_{j=3}^{4}p_jU_{oj}(n_o,\alpha)A_{oj}(n_o,\alpha)-B_{o5}(n_o,\alpha)$ with Equation (47) and (48). Then, one has $q_{o5}(n_o+1,\alpha)-q_{o5}(n_o,\alpha)=\sum_{j=3}^{4}p_j\{q_{oj}(n_o+1,\alpha)-q_{oj}(n_o,\alpha)\}A_{oj}(n_o+1,\alpha)$, and derives those necessary conditions of strictly increasing $q_{o5}(n_o|\alpha)$ are to hold all conditions in Theorem 2.7.

***Theorem* 3.4** *If the lifetime Y is strictly IFR and $q_{o5}(\infty|\alpha)>0$. Then the optimal $C_{o5}(n_o^*|\alpha)$ for a fixed α is given by:*

$$U_{o5}(n_o^*-1|\alpha)<C_{o5}(n_o^*|\alpha)\le U_{o5}(n_o^*|\alpha) \tag{55}$$

where,

$$U_{o5}(n_a|\alpha)=\sum_{j=3}^{4}p_jU_{oj}(n_o|\alpha) \tag{56}$$

4. Numerical Illustrations and Discussion

Holland and McLean (1975) conducted an empirical study on the preventive replacement of electrical devices within a continuous-time framework. Dohi et al. (2005) addressed a discrete-time optimal preventive replacement problem concerning section switches equipped with telegraph poles over a predetermined time period. More recently, Wu et al. (2024) expanded upon the above model, incorporating the replacement last discipline introduced by Zhao and Nakagawa (2012). Here, we computed the optimal AR time n_a^* satisfying the inequalities (9) and (35), the optimal OR time limit n_o^* satisfying the in Equations (23), (47) and their related $C(n_a^*)$ and $C(n_o^*)$ for section switches equipped with telegraph poles. Following we suppose that $f_Y(n)$ is a discrete Weibull p.m.f. with $r=0.9995$, $\beta=2.8547$ and *MTTE=E[Y]*=13.4:

$$f_Y(n)=r^{(n-1)^\beta}-r^{n^\beta} \tag{57}$$

where, $n=1,2,\cdots$.Based on the above parameters, the corresponding survival function and failure rate function are shown as

$$\bar{F}_Y(n-1)=r^{(n-1)^\beta} \tag{58}$$

and

$$r_Y(n)=1-r^{n^\beta-(n-1)^\beta} \tag{59}$$

Furthermore, suppose that $g_X(n)$ is a geometric p.m.f. $g_X(n)=0.05(1-0.05)^{n-1}$. Other model parameters are set as $c_2=1.0$, $c_3\in\{0.4,1.0\}$ and $c_1\in\{1.5,2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0,10.0\}$. In our unified models with probabilistic priorities, we set $p_1 = 0.4, p_2 = 0.6, p_3 = p_4 = 0.5$. The discount factor is set as $\alpha = 0.6$, 0.9.

Tables 1-3 present the optimal AR time n_a^* and the optimal OR time limit n_o^*, and their associated $C(n_a^*)$ and $C(n_o^*)$ for Model 1 to Model 4. We also give the results for the unified models. From these results, we obtain the following lessons learned from the numerical illustrations.

Lesson (1): When c_1 increases, both the optimal AR time n_a^* and the optimal OR time limit n_o^* become small. This is because the preventive replacement tends to be set earlier if the corrective replacement cost is higher.

Lesson (2): When $c_2 = c_3$, AR policy is better than OR policy. In addition, AR time is larger than OR time limit, i.e., $n_a^* > n_o^*$.

Lesson (3): When $c_2 > c_3$, OR policy is better than AR policy in some cases where c_1 is relatively smaller. For example, when $c_1 = 1.5$, it is easy to confirm that OR policy is better than AR policy. In our actual application, under the assumption of $c_2 = 2c_3$, if $c_2 < c_1 < 1.5c_2$, the decision-maker should consider opportunity in the preventive replacement, otherwise, i.e., $c_2 > 1.5c_2$, the decision-maker should consider only AR policy instead of OR policy.

Lesson (4): In most cases, the optimal preventive replacement times for each priority model tend to converge to the same values. This phenomenon arises from the discretization of replacement times into integer values and the relatively subtle differences in replacement priorities.

Lesson (5): Comparing Tables 1 and 2 with Table 3, notable discrepancies in the preventive replacement times are not evident. Moreover, the associated expected costs tend to converge towards similar values.

Table 1. Optimal n_a^*, n_o^* and their related $C(n_a^*)$ and $C(n_o^*)$ between Model 1 and Model 3, when $r = 0.9995$, $\beta = 2.8547$ and $p = 0.05$.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a1}(n_a^*)$	n_o^*	$C_{o3}(n_o^*)$	n_a^*	$C_{a1}(n_a^*)$	n_o^*	$C_{o3}(n_o^*)$
1.5	15	0.1083	4	0.1016	15	0.1083	11	0.1132
2.0	12	0.1296	3	0.1313	12	0.1296	8	0.1465
3.0	10	0.1575	2	0.1900	10	0.1575	5	0.2088
4.0	8	0.1769	2	0.2482	8	0.1769	4	0.2690
5.0	8	0.1926	2	0.3064	8	0.1926	3	0.3283
6.0	7	0.2049	1	0.3644	7	0.2049	3	0.3871
7.0	7	0.2166	1	0.4223	7	0.2166	2	0.4458
8.0	6	0.2264	1	0.4802	6	0.2264	2	0.5041
9.0	6	0.2345	1	0.5381	6	0.2345	2	0.5623
10.0	6	0.2427	1	0.5960	6	0.2427	2	0.6205

Table 2. Optimal n_a^*, n_o^* and their related $C(n_a^*)$ and $C(n_o^*)$ between Model 2 and Model 4, when r = 0.9995, β = 2.8547 and p = 0.05.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a2}(n_a^*)$	n_o^*	$C_{o4}(n_o^*)$	n_a^*	$C_{a2}(n_a^*)$	n_o^*	$C_{o4}(n_o^*)$
1.5	16	0.1111	4	0.1045	16	0.1111	13	0.1139
2.0	12	0.1367	3	0.1356	12	0.1367	9	0.1488
3.0	9	0.1716	2	0.1969	9	0.1716	5	0.2141
4.0	8	0.1968	2	0.2578	8	0.1968	4	0.2769
5.0	7	0.2175	2	0.3186	7	0.2175	3	0.3389
6.0	7	0.2352	2	0.3795	7	0.2352	3	0.4005
7.0	6	0.2503	1	0.4397	6	0.2503	2	0.4618
8.0	6	0.2638	1	0.5003	6	0.2638	2	0.5527
9.0	6	0.2773	1	0.5608	6	0.2773	2	0.5835
10.0	5	0.2893	1	0.6213	5	0.2893	2	0.6444

Table 3. Optimal n_a^*, n_o^* and their related $C(n_a^*)$ and $C(n_o^*)$ with unified models, when r = 0.9995, β = 2.8547 and p = 0.05.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a3}(n_a^*)$	n_o^*	$C_{o5}(n_o^*)$	n_a^*	$C_{a3}(n_a^*)$	n_o^*	$C_{o5}(n_o^*)$
1.5	16	0.1095	4	0.1131	16	0.1095	13	0.1135
2.0	12	0.1323	3	0.1335	12	0.1323	9	0.1476
3.0	9	0.1623	2	0.1934	9	0.1623	5	0.2114
4.0	8	0.1849	2	0.2530	8	0.1849	4	0.2730
5.0	7	0.2029	2	0.3125	7	0.2029	3	0.3366
6.0	7	0.2170	2	0.3720	7	0.2170	3	0.3938
7.0	6	0.2310	1	0.4310	6	0.2310	2	0.4538
8.0	6	0.2413	1	0.4918	6	0.2413	2	0.5284
9.0	6	0.2517	1	0.5495	6	0.2517	2	0.5729
10.0	6	0.2620	1	0.6087	6	0.2620	2	0.6325

Tables 4-6 present the optimal AR time n_a^* and OR time limit n_o^*, and their expected total discounted costs $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ for Model 1 to Model 4, when the discounted factor is given by α = 0.90. By observing the results carefully, we could obtain the following findings:

Lesson (6): The lessons (1)~(5) always hold in models with discounting.

Lesson (7): In terms of the optimal time in AR and OR policies, the optimal replacement time with discounting is longer than that without discounting. That is, when the economic environment is unstable, the decision-makers will shorten the replacement times for their equipment.

Lesson (8): When the discount factor is relatively small, such as $\alpha = 0.90$, the optimal preventive replacement times for each priority model often converge to similar values in most cases. In this scenario, the discount factor has a minimal impact on the optimal replacement times.

Table 4. Optimal n_a^*, n_o^* and their related $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ between Model 1 and Model 3, when *r* = 0.9995, *β* = 2.8547, *p* = 0.05 and *α* = 0.90.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a1}(n_a^*\|0.9)$	n_o^*	$C_{o3}(n_o^*\|0.9)$	n_a^*	$C_{a1}(n_a^*\|0.9)$	n_o^*	$C_{o3}(n_o^*\|0.9)$
1.5	18	0.5800	5	0.5411	18	0.5800	15	0.5828
2.0	14	0.7410	4	0.6965	14	0.7410	10	0.7679
3.0	11	0.9802	3	0.9971	11	0.9802	7	1.1100
4.0	9	1.1548	2	1.2910	9	1.1548	5	1.4312
5.0	8	1.2968	2	1.5825	8	1.2968	4	1.7413
6.0	8	1.4195	1	1.8715	8	1.4195	3	2.0464
7.0	7	1.5190	1	2.1585	7	1.5190	3	2.3341
8.0	7	1.6131	1	2.4454	7	1.6131	3	2.6418
9.0	6	1.7028	1	2.7323	6	1.7028	2	2.9361
10.0	6	1.7706	1	3.0192	6	1.7706	2	3.2276

Table 5. Optimal n_a^*, n_o^* and their related $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ between Model 2 and Model 4, when *r* = 0.9995, *β* = 2.8547, *p* = 0.05 and *α* = 0.90.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a2}(n_a^*\|0.9)$	n_o^*	$C_{o4}(n_o^*\|0.9)$	n_a^*	$C_{a2}(n_a^*\|0.9)$	n_o^*	$C_{o4}(n_o^*\|0.9)$
1.5	22	0.5834	7	0.5672	22	0.5834	20	0.5835
2.0	15	0.7560	5	0.7366	15	0.7560	12	0.7748
3.0	11	1.0523	4	1.0597	11	1.0523	7	1.1346
4.0	9	1.2736	3	1.3732	9	1.2736	5	1.4732
5.0	8	1.4559	2	1.6817	8	1.4559	4	1.7998
6.0	7	1.6182	2	1.9882	7	1.6182	4	2.1196
7.0	7	1.7511	1	2.2939	7	1.7511	3	2.4339
8.0	7	1.8839	1	2.5958	7	1.8839	3	2.7465
9.0	6	1.9933	1	2.8978	6	1.9933	2	3.0568
10.0	6	2.0973	1	3.1998	6	2.0973	2	3.3634

Table 6. Optimal n_a^*, n_o^* and their related $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ in unified models, *r* = 0.9995, *β* = 2.8547, *p* = 0.05 and *α* = 0.90.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a3}(n_a^*\|0.9)$	n_o^*	$C_{o5}(n_o^*\|0.9)$	n_a^*	$C_{a3}(n_a^*\|0.9)$	n_o^*	$C_{o5}(n_o^*\|0.9)$
1.5	19	0.5818	6	0.5500	19	0.5818	15	0.5835
2.0	14	0.7512	5	0.7172	14	0.7512	11	0.7716
3.0	11	1.0090	3	1.0289	11	1.0090	7	1.1223
4.0	9	1.2023	2	1.3330	9	1.2023	5	1.4522
5.0	8	1.3604	2	1.6321	8	1.3604	4	1.7705
6.0	8	1.4991	2	1.9311	8	1.4991	4	2.0830
7.0	7	1.6118	1	2.2261	7	1.6118	3	2.3890
8.0	7	1.7214	1	2.5206	7	1.7214	3	2.6942
9.0	6	1.8190	1	2.8150	6	1.8190	2	2.9964
10.0	6	1.9013	1	3.1095	6	1.9013	2	3.2955

Tables 7-9 present the optimal AR time n_a^* and OR time n_o^*, and their expected total discounted costs $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ for Model 1 to Model 4, when the discounted α = 0.60.

Lesson (9) The discount factor is not sensitive in our example with $\alpha = 0.90$ which is almost unity. However, if the discount factor is much smaller (i.e., α = 0.60), two expected costs with/without discounting give the remarkable differences. More specially, when the discount factor is relatively smaller, such as α = 0.60, the optimal times in Model 1 (or Model 3) is much bigger than ones in Model 2 (or Model 4).

The Figure 3 graphically presents the difference between the AR models without discounting and with discounting.

The Figure 4 graphically presents the difference between the OR models without discounting and with discounting.

Table 7. Optimal n_a^*, n_o^* and their related $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ between Model 1 and Model 3, when r = 0.9995, β = 2.8547, p = 0.05 and α = 0.90.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a1}(n_a^*\|0.6)$	n_o^*	$C_{o3}(n_o^*\|0.6)$	n_a^*	$C_{a1}(n_a^*\|0.6)$	n_o^*	$C_{o3}(n_o^*\|0.6)$
1.5	25	0.0181	11	0.0181	25	0.0181	23	0.0181
2.0	20	0.0241	9	0.0241	20	0.0241	19	0.0241
3.0	15	0.0361	7	0.0359	15	0.0361	13	0.0362
4.0	12	0.0480	5	0.0474	12	0.0480	10	0.0482
5.0	11	0.0595	5	0.0588	11	0.0595	9	0.0601
6.0	10	0.0707	4	0.0699	10	0.0707	8	0.0720
7.0	9	0.0814	4	0.0810	9	0.0814	7	0.0838
8.0	9	0.0918	3	0.0919	9	0.0918	6	0.0954
9.0	8	0.1014	3	0.1027	8	0.1014	6	0.1070
10.0	8	0.1110	3	0.1134	8	0.1110	5	0.1186

Table 8. Optimal n_a^*, n_o^* and their related $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ between Model 2 and Model 4, when r = 0.9995, β = 2.8547, p = 0.05 and α = 0.60.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a2}(n_a^*\|0.6)$	n_o^*	$C_{o4}(n_o^*\|0.6)$	n_a^*	$C_{a2}(n_a^*\|0.6)$	n_o^*	$C_{o4}(n_o^*\|0.6)$
1.5	31	0.0181	16	0.0181	31	0.0181	27	0.0181
2.0	27	0.0241	12	0.0241	27	0.0241	23	0.0241
3.0	22	0.0362	9	0.0361	22	0.0362	20	0.0362
4.0	17	0.0482	7	0.0480	17	0.0482	15	0.0482
5.0	14	0.0602	6	0.0598	14	0.0602	12	0.0603
6.0	12	0.0722	5	0.0715	12	0.0722	11	0.0723
7.0	11	0.0839	5	0.0831	11	0.0839	9	0.0843
8.0	10	0.0956	4	0.0945	10	0.0956	8	0.0962
9.0	10	0.1070	4	0.1059	10	0.1070	8	0.1082
10.0	9	0.1182	3	0.1172	9	0.1182	7	0.1201

Table 9. Optimal n_a^*, n_o^* and their related $C(n_a^*|\alpha)$ and $C(n_o^*|\alpha)$ in unified models, when r = 0.9995, β = 2.8547, p = 0.05 and α = 0.60.

C_1	C_3= 0.4				C_3= 1			
	AR		OR		AR		OR	
	n_a^*	$C_{a3}(n_a^*\|0.6)$	n_o^*	$C_{o5}(n_o^*\|0.6)$	n_a^*	$C_{a3}(n_a^*\|0.6)$	n_o^*	$C_{o5}(n_o^*\|0.6)$
1.5	28	0.0181	13	0.0180	28	0.0181	24	0.0181
2.0	23	0.0241	10	0.0240	23	0.0241	21	0.0242
3.0	16	0.0361	7	0.0360	16	0.0361	15	0.0362
4.0	14	0.0481	6	0.0478	14	0.0481	12	0.0482
5.0	12	0.0599	5	0.0593	12	0.0599	12	0.0602
6.0	11	0.0715	4	0.0708	11	0.0715	9	0.0722
7.0	10	0.0827	4	0.0821	10	0.0827	8	0.0841
8.0	9	0.0935	4	0.0933	9	0.0935	7	0.0959
9.0	9	0.1042	3	0.1044	9	0.1042	7	0.1077
10.0	8	0.1142	3	0.1153	8	0.1142	6	0.1194

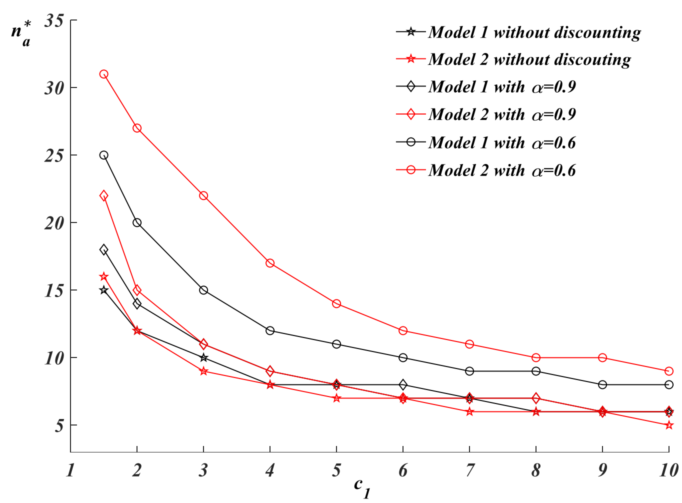

Figure 3. The optimal pre-scheduled replacement time in AR model.

In this paper, through the experimental results, we can observe the performance differences on various replacement strategies under different contexts. We emphasize on lessons (1), (3) and (9).

Firstly, we talk the lesson (1) in more detail. In the engineering perspective, if the cost of replacing an industrial asset after it fails is higher with corrective replacement, then there's a tendency to replace the asset proactively before it fails with preventive replacement. This is motivated by avoiding the higher costs associated with unexpected failures, which could include downtime, repairs, or even safety risks.

Secondly, we pay more attention on lesson (3). Lesson (3) tells us that the age-based replacement strategy generally exhibits better performance, in situations with relatively high failure replacement costs. This is because, in such cases, extending the equipment's operational lifespan can effectively reduce maintenance costs, while opportunistic strategy might lead to premature equipment replacement, thereby increase of overall maintenance expenses. In contrast, when failure replacement costs are relatively low, the advantage of opportunistic strategy becomes more pronounced, so the age-based replacement strategy might lead to premature equipment replacement, resulting in unnecessary maintenance expenses.

Thirdly, we further analyze the lesson (9). i.e., we further investigated economic discounting in preventive replacement policies. When the discount factor is relatively large such as α = 0.90, the preventive replacement times for each priority model often tend to align in most cases. This tendency is similar to the models without discounting. However, when the discount factor is relatively small such as α = 0.60, the optimal replacement time with discounting is longer than that without discounting. This implies that decision-makers, in uncertain economic conditions, may decide to postpone equipment replacement due to the higher value placed on the present costs and benefits compared to future ones. Meanwhile, the optimal pre-scheduled replacement times in Model 2 and Model 4 are much longer than the ones in Model 1 and Model 3. So, when the economic environment is not stable, the replacement cost should be calculated by the NPV method.

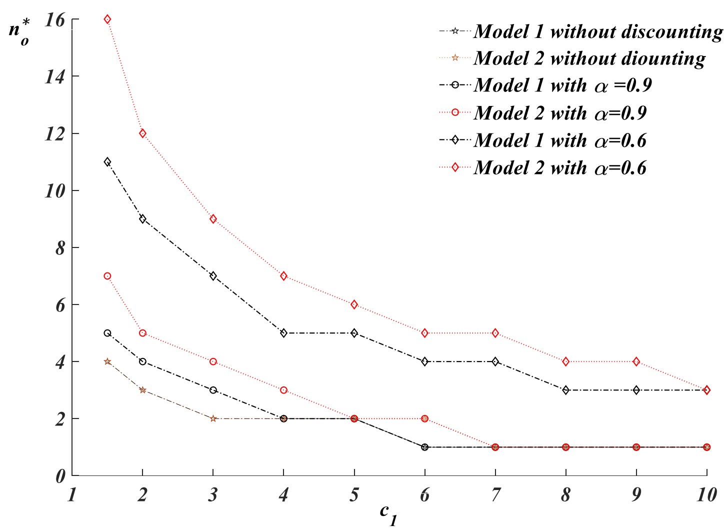

Figure 4. The optimal pre-scheduled replacement time limit in OR model.

5. Conclusion

We have focused on discrete-time age replacement models and opportunity-based age replacement models, considering the prioritization of two replacement options. The optimal pre-scheduled preventive replacement times have been determined using the renewal reward and net present value methods. Additionally, we have formulated unified age-based replacement models with probabilistic priority by introducing probabilistic replacement options. Through numerical illustrations, we compared optimal age replacement policies with optimal opportunistic replacement policies in various scenarios. Our observations indicate that opportunistic replacement policies outperformed age replacement policies in cases where the failure replacement cost is relatively smaller.

It is worth mentioning that there are some limitations in this paper from the following reasons:

(i) The lesson (3) is valid in the engineering perspective when $c_1 / c_2 = 1.5$. In future work, we will further examine the other cases on the cost balance more comprehensively.

(ii) In our experiments, we just used the model parameters in the existing empirical work by Dohi et al. (2005). In other words, we did not analyze the other discrete lifetime data in actual applications. More detailed statistical properties of the discrete lifetime data should be investigated by checking the goodness-of-fit to the discrete Weibull distribution.

(iii) It is important to note that while the several trends were observed in the experiments, real-world decision-making often requires to consider other factors, including equipment importance, availability requirements, and maintenance cycles. Such complex modeling should be considered in the future research. Jhang and Sheu (1999) formulated different opportunistic age replacement policies with major and minimal repairs. In our future work, we will reformulate their models in discrete-time setting and investigate effects of replacement priorities as well.

Appendix

Table 10. Notations in this paper.

Y	Lifetime of the unit
$\Pr\{X = n\} = f_Y(n)$	Probability mass function
$\Pr\{X \geq n\} = \overline{F}_Y(n-1)$	Reliability function
$\Pr\{X < n\} = F_Y(n-1)$	Cumulative probability function
$r_Y(n) = f_Y(n)/\overline{F}_Y(n-1)$	Failure rate function
$R_Y(n) = f_Y(n)/\overline{F}_Y(n)$	Shifted failure rate function
$f_Y(0) = 0$	The probability of lifetime at $n = 0$
X	Interarrival of two successive opportunities, which obeys geometric distribution
$\Pr\{X = n\} = g_Y(n) = p(1-p)^{n-1}$	Probability mass function
$\Pr\{X \geq n\} = \overline{G}_X(n-1) = (1-p)^{n-1}$	Survival function
$\Pr\{X < n\} = G_X(n-1) = 1-(1-p)^{n-1}$	Cumulative probability function
$g_X(0) = 0$	The probability of opportunity arrival at $n = 0$
$H(n_0) = \sum_{n=n_0+1}^{\infty} f_Y(n)(1-p)^{n-n_0} / \sum_{n=n_0+1}^{\infty} \overline{F}_Y(n)(1-p)^{n-n_0}$	Failure rate function in discrete-time opportunity-based model
$h(n_0) = \sum_{n=n_0+1}^{\infty} f_Y(n+1)(1-p)^{n-n_0} / \sum_{n=n_0+1}^{\infty} \overline{F}_Y(n)(1-p)^{n-n_0}$	Shifted failure rate function in discrete-time opportunity-based model

***Lemma* 1.** *The $R_Y(n)$ is a strictly increasing (decreasing) function of n, then $r_Y(n)$ is a strictly increasing (decreasing) function of n.*

Proof.

$$R_Y(n) = \frac{f_Y(n)}{\overline{F}_Y(n)} = \frac{\overline{F}_Y(n-1)}{\overline{F}_Y(n)} \cdot \frac{f_Y(n)}{\overline{F}_Y(n-1)}$$
$$= \left\{\frac{\overline{F}_Y(n-1) - f_Y(n)}{\overline{F}_Y(n-1)}\right\}^{-1} r_Y(n) = \{1 - r_Y(n)\}^{-1} r_Y(n) \tag{60}$$

Further difference yields,

$$R_Y(n+1)-R_Y(n)=\frac{r_Y(n+1)-r_Y(n)}{\{1-r_Y(n+1)\}\{1-r_Y(n)\}} \tag{61}$$

Since $0<r_Y(n)<1$ for $n=1,2,\cdots$, the proof is completed.

***Lemma* 2.** *If $R_Y(n)$ $(r_Y(n))$ is a strictly increasing (decreasing) function of n, then H(n)(h(n)) is a strictly increasing (decreasing) function of n.*

Proof. If $R_Y(n)$ is a *strictly* increasing function of *n*, then we have,

$$\frac{f_Y(n)}{\bar{F}_Y(n)}<\frac{f_Y(n+k)}{\bar{F}_Y(n+k)} \tag{62}$$

where, $k\geq 1$ is an arbitrary integer. Since

$$f_Y(n)\bar{F}_Y(n+k)<f_Y(n+k)\bar{F}_Y(n) \tag{63}$$

we can obtain the following inequality:

$$f_Y(n)\sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k<\bar{F}_Y(n)\sum_{k=1}^{\infty}f_Y(n+k)(1-p)^k \tag{64}$$

Further, we have

$$\begin{aligned}&f_Y(n)\sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k+\sum_{k=1}^{\infty}f_Y(n+k)(1-p)^k\sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k\\&<\bar{F}_Y(n)\sum_{k=1}^{\infty}f_Y(n+k)(1-p)^k+\sum_{k=1}^{\infty}f_Y(n+k)(1-p)^k\sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k\end{aligned} \tag{65}$$

From Equation (66), it holds that

$$\frac{\sum_{k=0}^{\infty}f_Y(n+K)(1-p)^k}{\sum_{k=0}^{\infty}\bar{F}_Y(n+k)(1-p)^k}<\frac{\sum_{k=1}^{\infty}f_Y(n+k)(1-p)^k}{\sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k} \tag{66}$$

Hence, we have

$$H(n-1)<H(n) \tag{67}$$

The proof for $h(n)$ is similar to $H(n)$.

From Lemma 1 and Lemma 2, we can obtain the following lemmas directly.

***Lemma* 3.** *If the function H(n) is a strictly increasing (decreasing) function of n, then h(n) is a strictly increasing (decreasing) function of n.*

***Lemma* 4.** *If the function $R_Y(n)(r_Y(n))$ is a strictly increasing (decreasing) function of n, then $H(n,\alpha)(h(n,\alpha))$ is a strictly increasing (decreasing) function of n.*

Proof. Similar to the proof of Lemma 2.

From Lemma 1 and Lemma 4, we can obtain the following lemma without the proofs.

***Lemma* 5.** *If the function $H(n,\alpha)$ is a strictly increasing (decreasing) function of n, then r(n,α) is a strictly increasing (decreasing) function of n.*

***Corollary* 1.** If the function $g_X(n)$ is an arbitrary discrete p.m.f., then Lemma 2 holds.

***Corollary* 2.** If the function $g_X(n)$ is an arbitrary discrete p.m.f., then Lemma 4 holds.

***Proof of Theorem* 2.1**

Here, we give the proof for Model. From the inequality $C_{a1}(n_a+1)-C_{a1}(n_a)\geq 0$, we get

$$R_Y(n_a)\sum_{n=1}^{n_a}\overline{F}_Y(n-1)-F_Y(n_a-1)\geq\frac{c_2}{c_1-c_2} \tag{68}$$

Let $q_{a1}(n_a)$ denoting the left-hand side of Equation (68) and further taking the difference yield,

$$q_{a1}(n_a+1)-q_{a1}(n_a)=\left[R_Y(n_a+1)-R(n_a)\right]\sum_{n=1}^{n_a+1}\overline{F}_Y(n-1) \tag{69}$$

If the lifetime Y is strictly IFR, then $q_{a1}(n_a)$ is a strictly increasing function in n_a, where,

$$\lim_{n_a\to\infty}q_{a1}(n_a)=q_{a1}(\infty)=R_Y(\infty)\sum_{n=1}^{\infty}\overline{F}_Y(n-1)-1 \tag{70}$$

If $q_{a1}(\infty)>c_2/(c_1-c_2)$, then it has at least one (at most two) n^*_a satisfying Equation (8). If $q_{a1}(\infty)\leq c_2/(c_1-c_2)$,the function $C_{a1}(n_a)$ is monotonically decreasing. Then the optimal Artime becomes $n^*_a\to\infty$.

For an, if $R_Y(n_a)$ is decreasing, then the function $C_{a1}(n_a)$ is concave in n_a . Thus, if $C_{a1}(1)=c_2<C_{a1}(\infty)=c_1/\sum_{n=1}^{\infty}\overline{F}_Y(n-1)$, then $n^*_a=1$, otherwise, $n^*_a\to\infty$ .The proof for Model 2 is similar to that for Model 1.

***Proof of Theorem* 2.3**

Here, we give the proof for Model 3. From the inequality $C_{o3}(n_o+1)-C_{o3}(n_o)\geq 0$, we get

$$H(n_o)A_{o3}(n_o)-\left[F_Y(n_o)+\sum_{n=n_o+1}^{\infty}f_Y(n)(1-p)^{n-n_o}\right]\geq\frac{c_3}{c_1-c_3} \tag{71}$$

Denoting the left-hand side of Equation (71) by $q_{o3}(n_o)$ and further taking the difference yield,

$$q_{o3}(n_o+1)-q_{o3}(n_o)=\left[H(n_o+1)-H(n_o)\right]A_{o3}(n_o+1) \tag{72}$$

If the lifetime Y is strictly IFR, then the function $q_{o3}(n_o)$ is strictly increasing in n_o, where,

$$\lim_{n_o \to \infty} q_{o3}(n_o) = q_{o3}(\infty) = R_Y(\infty)\sum_{n=1}^{\infty} \bar{F}_Y(n-1) - 1 \tag{73}$$

If $q_{o3}(\infty) > c_3/(c_1 - c_3)$, then it has at least one (at most two) n^*_o satisfying Equation (21). If $q_{o3}(\infty) \le c_3/(c_1 - c_3)$, the function $C_{o3}(n_o)$ is monotonically decreasing. Then the optimal OR time limit becomes $n^*_o \to \infty$.

For another thing, if $H(n_0)$ is decreasing, then the function $C_{o3}(n_o)$ is concave in n_o. Thus, if $C_{o3}(0) < C_{o3}(\infty)$, then $n^*_o = 0$, otherwise, $n^*_o \to \infty$.The proof for Model 4 is similar to that for Model 3.

***Proof of Theorem* 2.5**

Consider Model 1. From the inequality $C_{a1}(n_a + 1, \alpha) - C_{a1}(n_a, \alpha) \ge 0$, we have

$$\left[\frac{c_1 - c_2}{(1-\alpha)} R_Y(n_a) - c_2\right]\left[1 - A_{a1}(n_a, \alpha)\right] - B_{a1}(n_a, \alpha) \ge 0 \tag{74}$$

Letting $q_{a1}(n_a|\alpha)$ denote the left-hand side of Equation (74) and taking the difference yield

$$q_{a1}(n_a + 1|\alpha) - q_{a1}(n_a|\alpha) = \frac{c_1 - c_2}{1-\alpha}\left[R_Y(n_a + 1) - R_Y(n_a)\right]\left[1 - A_{a1}(n_a + 1, \alpha)\right]. \tag{75}$$

If $R_Y(n_a)$ is a strictly increasing in n_a, the function $C_{a1}(n_a, \alpha)$ is strictly convex in n_a for a fixed α. Further, if $q_{a1}(\infty|\alpha) > 0$, then it has at least one (at most two) n_a^* which satisfies Equation (33). Conversely, if $q_{a1}(\infty|\alpha) \le 0$, then $C_{a1}(n_a, \alpha)$ is a monotonically decreasing function. Then the optimal pre-scheduled time n_a^* is $n_a^* \to \infty$.

For another thing, if $R_Y(n_a)$ is decreasing, then the function $C_{a1}(n_a, \alpha)$ is concave in n_a for a fixed α. Thus, if $C_{a1}(1|\alpha) < C_{a1}(\infty|\alpha)$, then $n_a^* = 1$, otherwise, $n_a^* \to \infty$.The proof for Model 2 is similar to that for Model 1.

Conflict of Interest

The authors confirm that this article's contents have no conflict of interest.

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