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Nonpolynomial Spline for Numerical Solution of Singularly Perturbed Convection-Diffusion Equations with Discontinuous Source Term

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Abstract
This research addresses the numerical solution of singularly perturbed convection-diffusion kind boundary value problem of second-order with a discontinuity term. Due to the perturbation parameter and discontinuity term, the problem solution has a boundary layer and an interior layer. A nonpolynomial cubic spline method is utilized to solve the boundary value problem. A specific set of parameters associated with nonpolynomial spline is used to tailor the method. A comprehensive analysis of the stability and convergence of the recommended method is presented which gives second-order convergence results. The suggested method is implemented on two examples, and the obtained results are contrasted with an existing method, highlighting the precision and efficacy of the proposed method, which would enhance the method's novelty.

Keywords- Boundary value problem, Nonpolynomial cubic spline, Discontinuity term, Stability, Convergence analysis.

1. Introduction

A singularly perturbed differential equation (SPDE) with a discontinuous source term (DST) is a differential equation that typically involves a perturbation parameter multiplying the highest differential term and includes a discontinuity in its source term. The numerical result of SPDE with DST is a challenging problem owing to the occurrence of boundary and interior layers in the solution. In overcoming these challenges, Nonpolynomial spline (NPS) methods offer a promising way for such differential equations. Nonpolynomial splines employ nonpolynomial basis functions to represent the solution. NPS method stands out for its precision, effectiveness, adaptability, stability and convergence. However, it is important to consider its limitations in terms of computational complexity and parameter selection. Singularly perturbed boundary value problems (SPBVP) mainly arise in fluid dynamics, chemical reaction engineering, electronics and circuit design, control systems, heat transfer, etc. (Doolan et al., 1980; Roos et al., 2008). Over the past two decades, to solve the SPBVP with DST, researchers have presented a large number of numerical techniques, such as Finite Difference Method (FDM) (Cen et al., 2017; Chandru and Shanti, 2015; Chandru et al., 2017; Clavero et al., 2017; Farrell et al., 2004a; Farrell et al., 2004b; Munyakazi, 2015; Shanthi et al., 2006), Finite Element Method (Chin and Krasny, 1983; Roos and Zarin, 2010), Boundary Value Technique (Chandru and Shanti, 2014; Shanti and Ramanujam, 2004) and spline approach (Kadalbajoo and Jha, 2012; Mane and Lodhi, 2023; Pathan and Vembu, 2017). A fourth-order convergent approach on a uniform mesh has been developed by Rashidinia et al. (2010) using a NPS of degree five for SPBVP. Aziz and Khan (2002) applied a cubic spline approach for SPBVP of second order and increased the convergence of the process. Rashidinia et al. (2008) presented the Nonpolynomial cubic spline method (NPCSM) to solve the second-order SPBVP, demonstrating that the approach is second-order convergent for different parameter values. Debela and Duressa (2021) recently applied NPS to second-order SPBVP with DST. Zhang et al. (2022) developed a cubic B-spline method to solve SPBVP of ninth-order for linear and non-linear types with outstanding estimates of the exact solution. Khalid et al. (2021) proposed NPCSM and cubic spline methods to solve non-linear sixth and eighth-order SPBVP. Basha and Shanthi (2020) obtained quadratic uniform convergence of a second-order weakly coupled system on a Shishkin mesh, using central FDM, by applying the condition of the mean of the source term at the discontinuity point. Alam et al. (2020) obtained fourth-order convergence for the numerical treatment of second-order SPBVP with a turning point. Alinia and Zarebnia (2018) solved third-order SPDE by applying the tension B-spline method and improved the accuracy. Gupta and Kumar (2012) presented a quintic B-spline technique for the numerical treatment of SPBVP of fourth order. Thula (2022) recently obtained sixth-order convergence using the optimal quintic B-spline method. For a bottomless study of the theory of perturbation and spline methods, one can refer to (Prenter, 1989; Nayfeh, 1981).

In the context of SPBVP with a discontinuous source term, a prominent gap exists in the literature regarding the application of spline methods. Spline methods have proved accomplishment in the treatment of linear SVBVP with smooth data types (Kadalbajoo and Gupta, 2010; Khan and Khandelwal, 2014; Khan and Khandelwal, 2019; Kumar et al., 2007; Lodhi and Mishra, 2017; Lodhi and Mishra, 2018), but requires further study with discontinuous data type. This gap provides an opportunity to develop an accurate numerical technique to address the challenges posed by singular perturbations and discontinuous source terms in the context of spline-based methods. This paper suggests a new approach based on NPCSM to solve the second-order SPBVP convection-diffusion with DST. This technique depends on an NPS function containing a trigonometric and a polynomial part. The specifically chosen parameter is the primary key for increasing the accuracy. NPCSM has advantages over existing methods in terms of accuracy and efficiency, which motivated to solve the SPBVP. Compared to other existing methods in the literature, the spline method provides the solution at any point in the domain and is simple to apply. This article mainly intends to analyze both layers' behavior and provide a layer-resolving approach with sufficient accuracy.

We denote, $\Upsilon = (0\ ,1)$, $\Upsilon^- = (0\ ,k)$ and $\Upsilon^+ = (k,1)$ where k is the discontinuity point and $k \in \Upsilon$.

The second-order SPBVP is given as:

$$L\,r(t) \equiv \varepsilon r''(t) + c(t)r'(t) = g(t), \forall t \in (0,k) \cup (k,1) \quad (1)$$

$$r(0) = r_0\,, r(1) = r_1 \quad (2)$$

where, $c, g \in C^2(\Upsilon^- \cup \Upsilon^+), r \in C^1(\Upsilon) \cap C^2(\Upsilon^- \cup \Upsilon^+)$ and a parameter $0 < \varepsilon \ll 1$. The functions $c(t)$ and $g(t)$ has a discontinuity $k \in \Upsilon$ at one point. Due to the discontinuity point k, the answer $r \notin C^2(\Upsilon)$.

Following is a description of the manuscript's structure: Section 2 elucidates the scheme's development and derivation, while convergence and stability of the proposed way are addressed in Section 3. The technique's efficacy is addressed in Section 4 by presenting two numerical examples, and a conclusion and future scope are summarized in Section 5.

2. Development of Scheme

This section describes the Nonpolynomial cubic spline method to solve SPBVP given by Equations (1)-(2). We divide the main interval $\Upsilon = (0\ ,1)$ into two subintervals Υ^- and Υ^+. Further, each subinterval Υ^- and Υ^+ is divided into $\frac{N}{2}$ equal points with step-size $\theta_1 = \frac{2k}{N}$ and $\theta_2 = \frac{2(1-k)}{N}$ respectively. Let $\pi_1 = \left\{0 = t_0, t_1, \dots, t_{\frac{N}{2}}\right\},\ \pi_2 = \left\{t_{\frac{N}{2}+1}, t_{\frac{N}{2}+2}, \dots, t_N = 1\right\}$ be the partition of Υ^- and Υ^+ respectively with $\pi = \pi_1 \cup \pi_2$. Define step size θ as:

$$\theta = \begin{cases} \theta_1, 1 \le i \le \frac{N}{2} \\ \theta_2, \left(\frac{N}{2}\right) + 1 \le i \le N \end{cases}.$$

A nonpolynomial spline function $S_\pi(t)$, is a twice continuously differentiable function in $[a, b]$ that depends on the term ω and interpolates $r(t)$ in each nodal point t_i, $i = 0(1)N$. Also $S_\pi(t)$ tends to spline of degree three in the interval $[a, b]$ as $\omega \to 0$. The proposed spline function has the following form:

$N_3 = \text{span}\{1, t, \cos\omega\, t, \sin\omega\, t\}$ where ω is the trigonometric component frequency of the spline function.

We define nonpolynomial spline $S_\pi(t)$ in the subdomain $[t_i, t_{i+1}]$, $i = 0(1)N-1$, as:

$$S_\pi(t) = u_i + v_i(t - t_i) + w_i \sin\omega(t - t_i) + x_i \cos\omega(t - t_i),\ \ i = 0(1)N \tag{3}$$

where, ω is the free parameter and u_i, v_i, w_i and x_i are unknown constants.

Let r_i be an approximation to $r(t_i)$. To determine the unknown coefficients in Equation (3), we require interpolating conditions at nodal points $t_{i\ ,}t_{i+1}$ and continuity of the first derivative condition at the standard points (t_i, r_i).

We denote the following notations:

$$S_\pi(t_i) = r_i, S_\pi(t_{i+1}) = r_{i+1}, S_\pi{}''(t_i) = Z_i, S_\pi{}''(t_{i+1}) = Z_{i+1} \tag{4}$$

In Equation (3), the unknown coefficients are obtained through algebraic manipulations. Hence, the value u_i, v_i, w_i and x_i is given as follows:

$$u_i = r_i + \frac{Z_i}{\omega^2}, v_i = \frac{r_{i+1} - r_i}{\theta} + \frac{Z_{i+1} - Z_i}{\omega\xi}, w_i = \frac{Z_i \cos\xi - Z_{i+1}}{\omega^2 \sin\xi}, x_i = \frac{-Z_i}{\omega^2} \tag{5}$$

where, $\xi = \omega\theta$.

Applying the criteria of continuity of the first differential at the nodal points (t_i, r_i), i.e.,

$$S'_{\pi_{i-1}}(t_i) = S'_{\pi_i}(t_i).$$

We obtain the following relation:

$$\alpha Z_{i+1} + 2\beta Z_i + \alpha Z_{i-1} = \frac{1}{\theta^2}(r_{i+1} - 2r_i + r_{i-1}), i = 0, 1, \dots, N-1 \tag{6}$$

where, $\alpha = \frac{1}{\theta^2}(\xi \operatorname{cosec}\xi - 1)$ and $\beta = \frac{1}{\theta^2}(1 - \xi\cot\xi)$.

When $\omega \to 0$, then $\alpha \to \frac{1}{6}$ and $\beta \to \frac{1}{3}$, therefore Equation (6) gives the cubic spline relation given below:

$$Z_{i+1} + 4Z_i + Z_{i-1} = \frac{6}{\theta^2}(r_{i+1} - 2r_i + r_{i-1}) \tag{7}$$

We discretize the differential Equations (1)-(2) as follows:

$$\varepsilon r_i'' + c_i r_i' = g_i \tag{8}$$

where, $c_i = c(t_i)$, $g_i = g(t_i)$.

Using the moment of spline in Equation (8), we get,

$$\varepsilon Z_i + c_i r_i' = g_i \tag{9}$$

Therefore,

$$Z_i = \frac{1}{\varepsilon}(g_i - c_i r_i') \tag{10}$$

Using an approximation for the first derivative of r:

$$r_i' = \frac{r_{i+1} - r_{i-1}}{2\theta} \tag{11}$$

$$r_{i+1}' = \frac{3r_{i+1} - 4r_i + r_{i-1}}{2\theta} \tag{12}$$

$$r_{i-1}' = \frac{-r_{i+1} + 4r_i - 3r_{i-1}}{2\theta} \tag{13}$$

Substituting the value of Equations (10)-(13) in Equation (6) gives the following linear system of equations:

$$L_i r_{i-1} + M_i r_i + N_i r_{i+1} = P_i \tag{14}$$

with boundary conditions, $r(0) = r_0$, $r(1) = r_1$.

where L_i, M_i, N_i and P_i is given as:

$$\begin{cases} L_i = \left(-\frac{3\alpha\theta c_{i-1}}{2} - \beta\theta c_i + \frac{\alpha\theta c_{i+1}}{2} + \varepsilon\right) \\ M_i = (2\theta\alpha c_{i-1} - 2\theta\alpha c_{i+1} - 2\varepsilon), \\ N_i = \left(-\frac{\alpha\theta c_{i-1}}{2} + \beta\theta c_i + \frac{3\alpha\theta c_{i+1}}{2} + \varepsilon\right) \\ P_i = \theta^2(\alpha g_{i-1} + 2\beta g_i + \alpha g_{i+1}). \end{cases} \tag{15}$$

which gives the approximations $r_1, r_2, \dots, r_{N-1}$ of solution $r(t)$ at the nodal points $t_1, t_2, \dots, t_{N-1}$.

To tackle the discontinuity point $t_{\frac{N}{2}} = k$, we used a second-order hybrid difference operator, i.e.,

$$L_t^N r_{\frac{N}{2}} = \frac{-r_{\frac{N}{2}+2} + 4r_{\frac{N}{2}+1} - 3r_{\frac{N}{2}}}{2\theta} - \frac{r_{\frac{N}{2}-2} - 4r_{\frac{N}{2}-1} + 3r_{\frac{N}{2}}}{2\theta} = 0 \tag{16}$$

3. Stability and Convergence Analysis

The matrix given by Equations (14)-(16) is not an M-matrix. The equation is converted into a new one to obtain the matrix monotonicity property. The value of $r_{\frac{N}{2}-2}$ and $r_{\frac{N}{2}+2}$ can be easily obtained from Equations (14)-(15), given below.

$$r_{\frac{N}{2}-2} = \frac{1}{L_{\frac{N}{2}-1}}\left(P_{\frac{N}{2}-1} - M_{\frac{N}{2}-1} r_{\frac{N}{2}-1} - N_{\frac{N}{2}-1} r_{\frac{N}{2}}\right),$$

$$r_{\frac{N}{2}+2} = \frac{1}{N_{\frac{N}{2}-1}}\left(P_{\frac{N}{2}+1} - L_{\frac{N}{2}+1} r_{\frac{N}{2}} - M_{\frac{N}{2}+1} r_{\frac{N}{2}+1}\right).$$

Inserting the value of $r_{\frac{N}{2}-2}$ and $r_{\frac{N}{2}+2}$ in L_t^N which gives:

$$L_T^N r_{\frac{N}{2}} \equiv \left(\frac{M_{\frac{N}{2}-1}}{L_{\frac{N}{2}-1}} + 4\right) r_{\frac{N}{2}-1} + \left(\frac{N_{\frac{N}{2}-1}}{L_{\frac{N}{2}-1}} + \frac{L_{\frac{N}{2}+1}}{N_{\frac{N}{2}+1}}\right) r_{\frac{N}{2}} + \left(\frac{M_{\frac{N}{2}+1}}{N_{\frac{N}{2}+1}} + 4\right) r_{\frac{N}{2}+1} = \frac{P_{\frac{N}{2}+1}}{N_{\frac{N}{2}+1}} + \frac{P_{\frac{N}{2}-1}}{L_{\frac{N}{2}-1}}.$$

Hence, we acquire a new system of linear equations given as:

$$\begin{cases} L_i r_{i-1} + M_i r_i + N_i r_{i+1} = P_i, \text{ for } \left\{0 \le i \le \frac{N}{2} - 1\right\} \cup \left\{\frac{N}{2} + 1 \le i \le N\right\}. \\ L_T^N r_{\frac{N}{2}} \equiv \left(\frac{M_{\frac{N}{2}-1}}{L_{\frac{N}{2}-1}} + 4\right) r_{\frac{N}{2}-1} + \left(\frac{N_{\frac{N}{2}-1}}{L_{\frac{N}{2}-1}} + \frac{L_{\frac{N}{2}+1}}{N_{\frac{N}{2}+1}}\right) r_{\frac{N}{2}} + \left(\frac{M_{\frac{N}{2}+1}}{N_{\frac{N}{2}+1}} + 4\right) r_{\frac{N}{2}+1} = \frac{P_{\frac{N}{2}+1}}{N_{\frac{N}{2}+1}} + \frac{P_{\frac{N}{2}-1}}{L_{\frac{N}{2}-1}}, \text{ for } i = \frac{N}{2}. \end{cases} \tag{17}$$

The matrix associated with the new linear system given by Equation (17) is tridiagonal, an invertible and diagonally dominant matrix. Further, for small values of, $\theta(i.e.\ \theta \to 0)$, we have $L_i \neq 0$, $N_i \neq 0$, $\forall i = 1(1)N - 1$.

$$\text{i.e.,} \left(\frac{M_{\frac{N}{2}-1}}{L_{\frac{N}{2}-1}} + 4\right) \neq 0\, , \left(\frac{M_{\frac{N}{2}+1}}{N_{\frac{N}{2}+1}} + 4\right) \neq 0.$$

Hence, the matrix is irreducible, as given in Varga (2000). Therefore, NPCSM is a stable method based on these two conditions, as given in Kadalbajoo and Reddy (1989).

We describe the convergence using NPCSM for second-order SPVBPs with DST. The system of linear equations provided by Equations (14)-(15) can be expressed in matrix form, given as:

$$QR + \theta^2 EJ = K \tag{18}$$

Equation (18) contains a tridiagonal matrix Q of order $N - 1$ satisfying diagonally dominance property given as follows:

$$Q = P + \theta FG \tag{19}$$

Here $P = (p_{ij})$ is tridiagonal matrix written as,

$$p_{ij} = \begin{cases} -2\varepsilon\, , i = j = \{1, \dots, N-1\} - \{N/2\}, \\ \ \varepsilon\, , |i - j| = 1, i \neq N/2, \\ -6\, , i = j = N/2, \\ \ 4\, , i = N/2\, , j = (N/2) - 1, (N/2) + 1, \\ -1\, , i = N/2\, , j = (N/2) - 2, (N/2) + 2, \\ \ 0, \text{ otherwise} \end{cases} \tag{20}$$

and $FG = (q_{ij})$ is a tridiagonal matrix specified by

$$q_{ij} = \begin{cases} 2\alpha(c_0 - c_1)\, , i = j = 1, \\ -\frac{\alpha}{2} c_{i-1} + \beta c_i + \frac{3}{2}\alpha c_{i+1}\, , \mathrm{i} > \mathrm{j}\, , \mathrm{i} \neq N/2, \\ 2\alpha(c_{i-1} - c_i)\, , i = j, \mathrm{i} \neq N/2, \\ -\frac{3}{2}\alpha c_{i-1} - \beta c_i + \frac{\alpha}{2} c_{i+1}\, , \mathrm{i} < \mathrm{j}\, , \mathrm{i} \neq N/2, \\ 2\alpha(c_{N-2} - c_{N-1})\, , i = j = N - 1 \end{cases} \tag{21}$$

and

$$J = (g_1, g_2, \dots, g_{N-1})^t,$$
$$R = (r_1, r_2, \dots, r_{N-1})^t.$$

The matrix E, which is tridiagonal, is given as:

$$E = \begin{bmatrix} 2\beta & \alpha & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha & 2\beta & \alpha & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha & 2\beta & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha & 2\beta & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & 0 & \cdots & 0 & 0 & \alpha & 2\beta & \alpha \\ 0 & 0 & \cdots & 0 & 0 & 0 & \alpha & 2\beta \end{bmatrix} \tag{22}$$

and $K = (k_1, 0, \dots 0, k_{N-1})^t$,
$k_1 = \theta^2 \alpha g_0 + \left(\frac{-\alpha}{2}\theta c_2 + \beta\theta c_1 + \frac{3\alpha}{2}\theta c_0 - \varepsilon\right)$,
$k_i = 0, i = 2(1)N-2$,
$k_{N-1} = \theta^2 \alpha g_N + \left(\frac{-3\alpha}{2}\theta c_N - \beta\theta c_{N-1} + \frac{\alpha}{2}\theta c_{N-2} - \varepsilon\right)$.

Let $\overline{R}$ be the exact solution of problem (1)-(2) at the mesh point t_i, $0 \le i \le N-1$, , except $t_i = k$, we have,
$$Q\overline{R} + \theta^2 EJ = T(\theta) + K \tag{23}$$

where, $T = \left(T(t_1), T(t_2), \dots, T(t_{N-1})\right)^t$ is the local truncation error (TE) associated with the Equations (14)-(15). Using Taylor's series expansion at the point t_i in Equation (14) and using Equation (8), the local TE is given as follows:
$$T_i(\theta) = [1 - 2(\alpha + \beta)]\varepsilon\theta^2 r''(\eta_i) + [-\alpha c_{i-1} + \beta c_i - \alpha c_{i+1}]\frac{\theta^4}{3} r'''(\eta_i) + \frac{1}{12}[(1 - 12\alpha) + \alpha\theta(c_{i-1} - c_{i+1})]\theta^4 r^{iv}(\eta_i) + O(\theta^5) \tag{24}$$

where, $t_{i-1} < \eta_i < t_{i+1}$.

using Equations (18) and (23), we get,
$$Q\left(\overline{R} - R\right) = Q\bar{E} = T(\theta) \tag{25}$$

where,
$$\bar{E} = \overline{R} - R = (e_1, e_2, \dots, e_{N-1})^t \tag{26}$$

The following lemma is required to calculate the bound for $\|\bar{E}\|$.

***Lemma* 3.1** If H is a square matrix of order $N$and $\|H\| < 1$, then $(I + H)^{-1}$ exists and $\|(I + H)^{-1}\| < \frac{1}{1-\|H\|}$.

From Equation (25), we have,
$$\bar{E} = Q^{-1} T,$$
$$\bar{E} = (P + \theta FG)^{-1} T,$$

$$\bar{E} = [I + \theta P^{-1}FG]^{-1}P^{-1}T,$$
$$\text{and } \|\bar{E}\| \leq \|[I + \theta P^{-1}FG]^{-1}\|\|P^{-1}\|\|T\|.$$

Then,
$$\|\bar{E}\| \leq \frac{\|P^{-1}\|\|T\|}{1-\theta\|P^{-1}\|\|FG\|} \tag{27}$$

Provided, $\theta\|P^{-1}\|\|FG\| \leq 1$.

Following the method in Henrici (1962), we have,
$$\|P^{-1}\| \leq \frac{(b-a)^2}{8\theta^2} \tag{28}$$

So, we have,
$$\|FG\| \leq q(8\alpha + 2\beta) \tag{29}$$

where, $q = \max|q(t_i)|$, $a < t_i < b$.

For $\alpha + \beta = \frac{1}{2}$ and $\alpha \neq \frac{1}{12}$, the truncation error is given as:

$$\|T\| \leq \zeta_1\theta^4 M_4, \text{ where } M_4 = \max|r^4(\eta)| \tag{30}$$

Hence, we obtain the truncation error given as:
$$\|\bar{E}\| = O(\theta^2) \tag{31}$$

However, for the choice of parameter $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$, the truncation error is given by,
$$T_i(\theta) = \frac{1}{36}(-c_{i+1} + 5c_i + c_{i-1})\theta^4 r'''(\eta_i) + \frac{1}{144}(c_{i+1} - c_{i-1})\theta^4 r^{iv}(\eta_i) + O(\theta^5),\ t_{i-1} < \eta_i < t_{i+1} \tag{32}$$

Hence,
we obtain the optimal second-order convergent method for the parameter choice $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

At the discontinuity node $t_{\frac{N}{2}} = k$, using the condition given by Equation (16), the truncation error $\left\|\bar{E}_{\frac{N}{2}}\right\|$ is given by,
$$\left\|\bar{E}_{\frac{N}{2}}\right\| = \left\|-r\left(t_{\frac{N}{2}+2}\right) + 4r\left(t_{\frac{N}{2}+1}\right) - 6r\left(t_{\frac{N}{2}}\right) + 4r\left(t_{\frac{N}{2}-1}\right) - r\left(t_{\frac{N}{2}-2}\right)\right\| \tag{33}$$

Expanding the term $r\left(t_{\frac{N}{2}+2}\right)$,$r\left(t_{\frac{N}{2}+1}\right)$,$r\left(t_{\frac{N}{2}-1}\right)$,$r\left(t_{\frac{N}{2}-2}\right)$ using Taylor's theorem around the point $t_{\frac{N}{2}}$ and simplifying the expression, we obtain the truncation error given as:
$$\left\|\bar{E}_{\frac{N}{2}}\right\| = O(\theta^2).$$

Hence, the current approach using NPCSM gives second-order convergence.

We summarise the above findings as follows:

Remark: The NPCSM given by Equations (14)-(15) for addressing the problem (1)-(2) proves the convergence order two for any value of parameter α, β with their sum equal to 0.5 and achieves optimal convergence order two for the particular choice of parameter $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

The above findings suggest that the method's performance is not contingent on a specific parameter choice and remains accurate across various parameter values α, β with $\alpha + \beta = \frac{1}{2}$.

4. Numerical Illustration and Discussion

This section demonstrates the relevance of the proposed numerical method. The suggested approach calculates the numerical rate of convergence (ROC) and maximum absolute error (MAE) of two numerical problems for various values of perturbation parameter, mesh point and parameters α and β.

***Example* 1.** The linear SPBVP with discontinuous source term:
$\varepsilon r''(t) + r'(t) = g(t),\ t \in \Upsilon^- \cup \Upsilon^+,$
$r(0) = -1, r(1) = 0.$

where,

$$g(t) = \begin{cases} -9,\ t \le \frac{1}{3} \\ 9(t-1)^2,\ t > \frac{1}{3} \end{cases}.$$

***Example* 2.** The linear SPBVP with discontinuous source term:
$\varepsilon r''(t) + r'(t) = g(t),\ t \in \Upsilon^- \cup \Upsilon^+,$
$r(0) = 1, r(1) = 1.$
where,

$$g(t) = \begin{cases} -1,\ t \le \frac{1}{2} \\ \ 1\ ,\ t > \frac{1}{2} \end{cases}.$$

We define the maximum absolute error (MAE) given in Mane and Lodhi (2023), as:

$$\bar{E}_\varepsilon^N = \max_{t_i \in I^- \cup I^+} \left| r_\varepsilon^N - \bar{r}_\varepsilon^{4096} \right| \text{ and } E^N = \max_{\varepsilon} E_\varepsilon^N \tag{34}$$

The Rate of Convergence (ROC) provided by Das and Natesan (2013) is given as:
$p_N = log_2 \frac{E^N}{E^{2N}}.$

Tables 1, 2, and 4 show the MAE calculated for examples 1 and 2, respectively, for the parameter values $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$ and $\alpha = \frac{1}{24}$ and $\beta = \frac{11}{24}$ with $\alpha + \beta = \frac{1}{2}$, contrasted to the current technique Farrell et al. (2004a) in Table 3 for Example 1. In comparison to Farrell et al. (2004a), it was found that the MAE decreases as nodal point *N* increases. The convergence of the method is analysed. The findings presented in Tables 1, 2, and 4 indicate that the current technique has second-order convergence based on computational ROC and theoretical results. Figures 1 and 3 depict the error graph of examples 1 and 2, and Figures 2, and 4 portray the numerical solution plot of examples 1, and 2 for various perturbation parameters and mesh points. All the mathematical computations are accomplished in a MATLAB environment.

Table 1. MAE and ROC of example 1 for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	2.9132E-03	1.7865E-03	9.5404E-04	4.6663E-04	2.0463E-04
2^{-2}	3.3363E-03	2.4075E-03	1.3627E-03	6.8362E-04	3.0337E-04
2^{-3}	1.4271E-03	2.1653E-03	1.4447E-03	7.7261E-04	3.5287E-04
2^{-4}	4.6822E-03	8.8606E-04	1.1143E-03	7.0074E-04	3.4160E-04
2^{-5}	1.6619E-02	3.5262E-03	4.6878E-04	5.1975E-04	2.9591E-04
2^{-6}	5.3119E-02	1.3365E-02	3.0513E-03	5.4395E-04	2.0978E-04
2^{-7}	2.1324E-01	4.7537E-02	1.1993E-02	2.8340E-03	5.9082E-04
2^{-8}	6.8411E-01	2.0528E-01	4.4758E-02	1.1222E-02	2.6156E-03
E^N	6.8411E-01	2.0528E-01	4.4758E-02	1.1222E-02	2.6156E-03
p	1.7366E+00	2.1974E+00	1.9959E+00	2.1011E+00	

Table 2. MAE and ROC of example 1 for $\alpha = \frac{1}{24}$ and $\beta = \frac{11}{24}$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	6.2455E-04	6.7900E-04	4.2279E-04	2.1981E-04	9.9086E-05
2^{-2}	4.1326E-04	7.2189E-04	5.5879E-04	3.1122E-04	1.4437E-04
2^{-3}	3.4529E-03	1.5524E-04	4.6606E-04	3.2183E-04	1.6094E-04
2^{-4}	8.6975E-03	1.6845E-03	1.2742E-04	2.2691E-04	1.4085E-04
2^{-5}	2.0777E-02	5.2866E-03	1.1099E-03	1.4860E-04	9.2423E-05
2^{-6}	5.8178E-02	1.5271E-02	3.8479E-03	8.6730E-04	1.5763E-04
2^{-7}	2.1999E-01	5.0003E-02	1.2830E-02	3.1918E-03	7.3069E-04
2^{-8}	6.9606E-01	2.0859E-01	4.5955E-02	1.1616E-02	2.7649E-03
E^N	6.9606E-01	2.0859E-01	4.5955E-02	1.1616E-02	2.7649E-03
p	1.7356E+00	2.1838E+00	1.9875E+00	2.0714E+00	

Table 3. MAE and ROC of example 1 in Farrell et al. (2004a).

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.01309	0.00663	0.00325	0.00152	0.00065
2^{-2}	0.02770	0.01406	0.00691	0.00325	0.00139
2^{-3}	0.04356	0.02196	0.01075	0.00504	0.00217
2^{-4}	0.03212	0.02210	0.01265	0.00592	0.00254
2^{-5}	0.04478	0.02044	0.02044	0.00398	0.00398
2^{-6}	0.06132	0.02907	0.01354	0.00609	0.00252
2^{-7}	0.07358	0.03464	0.01630	0.00747	0.00314
2^{-8}	0.08445	0.04068	0.01927	0.00885	0.00375
E^N	0.08445	0.04068	0.01927	0.00885	0.00375
p	1.0537	1.0775	1.1222	1.2383	

Table 4. MAE and ROC of example 2 for $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	8.9781E-04	4.5247E-04	2.2154E-04	1.0399E-04	4.4699E-05
2^{-1}	1.6339E-03	8.3841E-04	4.1413E-04	1.9525E-04	8.4105E-05
2^{-2}	2.5293E-03	1.3391E-03	6.7161E-04	3.1903E-04	1.3794E-04
2^{-3}	2.9027E-03	1.6107E-03	8.2771E-04	3.9810E-04	1.7321E-04
2^{-4}	2.4654E-03	1.5012E-03	8.1103E-04	4.0038E-04	1.7654E-04
2^{-5}	1.6325E-03	1.2057E-03	7.2172E-04	3.7603E-04	1.7048E-04
2^{-6}	1.0353E-03	7.8704E-04	5.7364E-04	5.7364E-04	1.5878E-04
2^{-7}	2.6606E-03	5.4469E-04	3.6551E-04	2.5882E-04	1.3781E-04
2^{-8}	4.0482E-03	1.3541E-03	2.9620E-04	1.5706E-04	1.0371E-04
2^{-9}	5.2096E-03	2.0235E-03	6.9512E-04	1.6615E-04	5.6926E-05
2^{-10}	5.6329E-03	2.6048E-03	1.0110E-03	3.5604E-04	9.1553E-05
E^N	5.6329E-03	2.6048E-03	1.0110E-03	3.5604E-04	9.1553E-05
p	1.1127E+00	1.3653E+00	1.5057E+00	1.9594E+00	

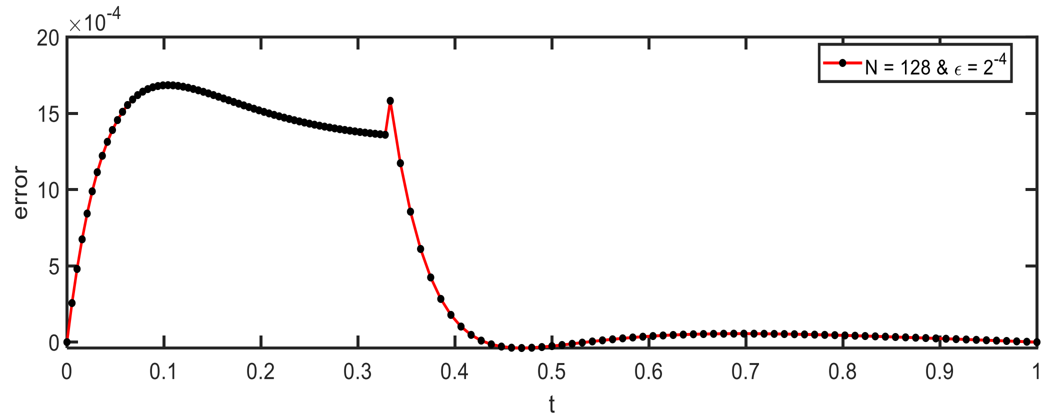

Figure 1. Example 1 error plot for N = 128 and $\varepsilon = 2^{-4}$.

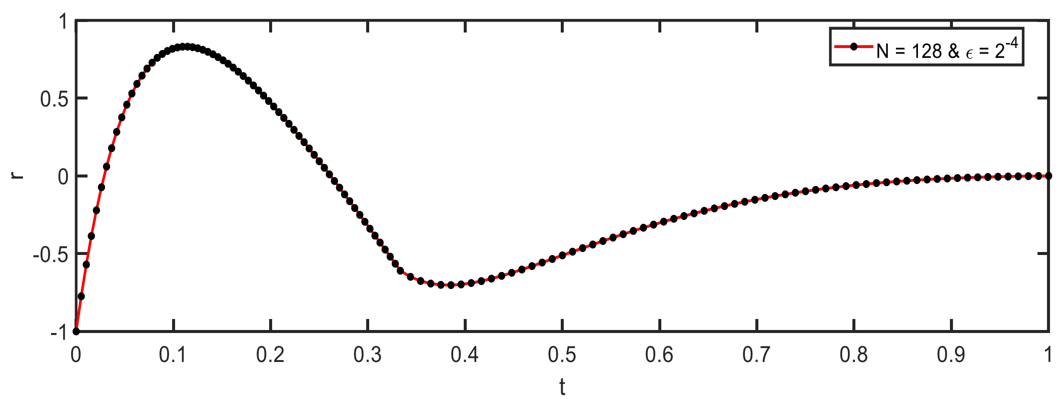

Figure 2. The nature of example 1 numerical solution for N = 128 and $\varepsilon = 2^{-4}$.

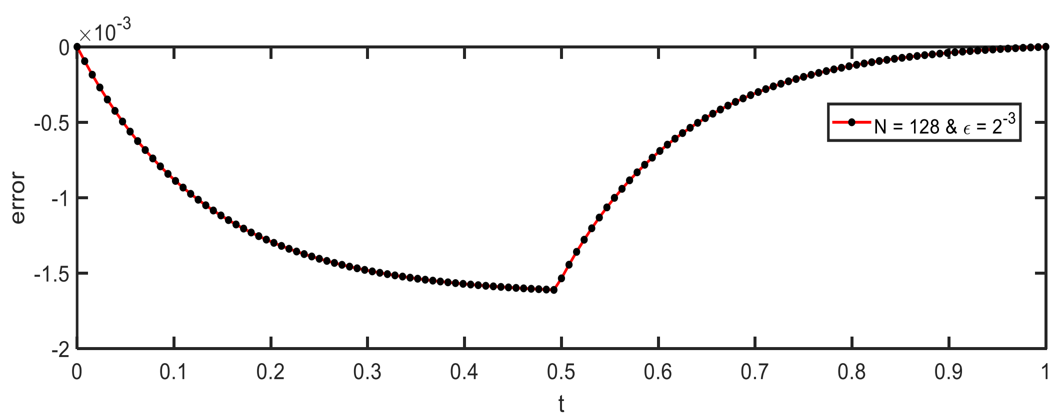

Figure 3. Example 2 error plot for N = 128 and $\varepsilon = 2^{-3}$.

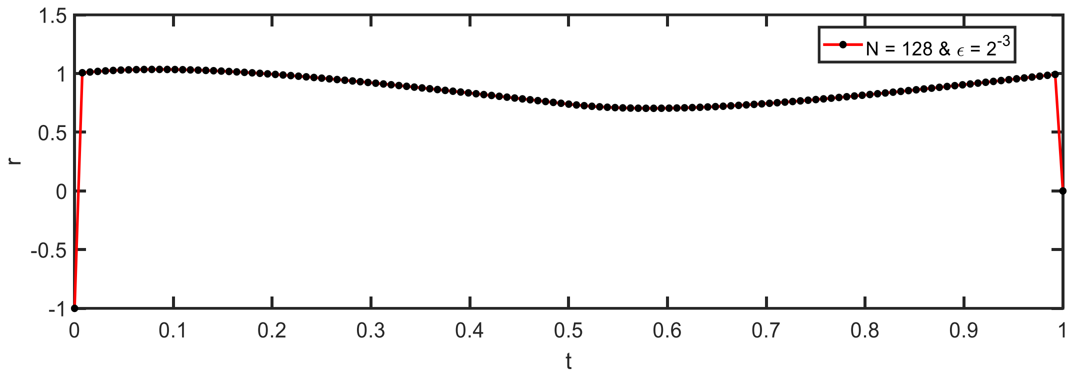

Figure 4. The nature of example 2 numerical solution for N $= 128$ and $\varepsilon = 2^{-3}$.

5. Conclusions and Future Scope

The research article addresses NPCSM for solving second-order SPBVP with a DST. We proved that the suggested method attained quadratic convergence under various parameter values of α and β having conditions $\alpha + \beta = 0.5$. At the point of discontinuity, a second-order hybrid difference operator is used. Based on the obtained values of errors and ROC, the comparative studies reveal the improved results than the existing method. The numerical illustration divulges the accuracy and efficiency of the proposed method. Also, nonpolynomial spline techniques show potential in addressing the challenges that arise from singular perturbations and discontinuous source terms.

The present finding in this article provides the platform for future research using nonpolynomial spline methods to solve higher-order SPBVP with DST.

Conflict of Interest

For this publication, the authors do not have any conflicts of interest.

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