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Quartic B-Spline Method for Non-Linear Second Order Singularly Perturbed Delay Differential Equations

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Abstract
This paper introduces a novel computational approach utilizing the quartic B-spline method on a uniform mesh for the numerical solution of non-linear singularly perturbed delay differential equations (NSP-DDE) of second-order with a small negative shift. These types of equations are encountered in various scientific and engineering disciplines, including biology, physics, and control theory. We are using quartic B-spline methods to solve NSP-DDE without linearizing the equation. Thus, the set of equations generated by the quartic B-spline technique is non-linear and the obtained equations are solved by Newton-Raphson method. The success of the approach is assessed by applying it to a numerical example for different values of perturbation and delay parameter parameters, the maximum absolute error (MAE) is obtained via the double mesh principle. The convergence rate of the proposed method is four. Obtained numerical results are compared with existing numerical techniques in literature and observe that the proposed method is superior with other numerical techniques. The quartic B-spline method provides the numerical solution at any point of the given interval. It is easy to implement on a computer and more efficient for handling second-order NSP-DDE.

Keywords- Singularly perturbed delay differential equations, Non-linear, Quartic B-spline method, Newton-Raphson method.

1. Introduction

Time delay is often observed in a variety of models, generally in the systems involving feedback analysis. Such systems give rise to differential-difference equations (DDEs). A negative shift in these equations is also called as delay. The time delay can have different significances, such as delay in transport, incubation time, response time, etc. The DDEs involving perturbation parameter ε with the highest order derivative term are called "singularly perturbed differential-difference equations" (SPDDEs). The word singular indicates the abrupt change in behavior of the solution as $\varepsilon \to 0$. The occurrence of DDEs and SPDDEs can be observed in the mathematical modeling of different scientific and engineering phenomena (Derstine et al., 1982; Glizer, 2000; Liao, 2005; Mackey and Glass, 1977; Stein, 1965). Kyrychko and Hogan (2010) have conducted a survey of applications of time delay in the area of engineering. SPDDEs also arise in optics and psychology (Longtin and Milton, 1988; Mallet-Paret and Nussbaum, 1989).

Lange and Miura (1982, 1994a, 1994b) were the first to introduce the analytical approach for SPDDEs. They studied a class of second-order equations and observed that even a tiny change in the delay parameter increases the width of the oscillation. The popular approach to deal with singularly perturbed differential equations (SPDEs) and SPDDEs is to obtain approximate solutions using different numerical methods. The most famous methods are the finite difference method (FDM), finite element method (FEM), spline techniques, etc. Farrel et al. (2000) have discussed different difference methods, such as central difference scheme, upwind difference scheme, and fitted operator method for various convection-diffusion problems. For an approximation of second-order SPDEs, Kadalbajoo et al. (2008) carried out a comparative analysis using fitted mesh FDM, B-spline technique, and FEM. Kadalbajoo and Gupta (2010) surveyed numerical methods used in the literature for SPDEs from 2000 to 2009. Malge and Lodhi (2022) have presented a review of the analytical and numerical techniques applied for SPDDEs based on the recent literature. Roos et al. (2008) also have discussed FDM and FEM for second-order singular boundary value problem (BVP), elliptical and parabolic initial-boundary value problems, and the incompressible Navier-Stokes equation.

An added advantage of the spline techniques is that they give the solution at any interval point, and the higher-order splines provide a better order of convergence. As a result, we found many spline methods applied for SPDEs and SPDDEs. Kadalbajoo and Arora (2009) have employed the cubic B-spline method for second-order SPDE. An SPDDE with a turning point is solved using the cubic B-spline collocation method (Kumar, 2018). Kanth and Kumar (2020) have applied a method that included spline in tension for the boundary region and midpoint approximation in the inner region for a second-order SPDDE. Vaid and Arora (2019b) have used the trigonometric B-spline technique. Flaherty and Mathon (1980) introduced polynomial and tension spline for SPDE. A fourth-order SPDE was approximated by Lodhi and Mishra (2018). An FDM with cubic spline in tension was developed by Chakravarthy et al. (2017) for SPDDE with a large delay. Vaid and Arora (2019a) applied the quintic-trigonometric spline method for third-order SPDDE. Goh at el. (2012) approximated SPDE using the quartic B-spline method. Prasad et al. (2022) applied the exponential spline method for NSP-DDE of order two with large delay using a special mesh. Roul and Kumari (2022) have solved a nonlinear singular BVP. After modifying it at singular points, the authors have applied the quartic trigonometric B-spline method. A variety of other spline techniques are applied by researchers for different types of BVPs (Ersoy Hepson, 2021; Mittal et al., 2020; Rani et al., 2022; Tamsir et al., 2022). By using a de Boor-like algorithm, one frequency trigonometric splines are constructed and analyzed by Albrecht et al. (2023).

Non-linear SPDEs and SPDDEs are mathematical models that describe many critical real-world phenomena, from chemical reactions to population dynamics (Chang and Howes, 2012; Kruthika et al., 2017). There has been growing interest in studying these equations in recent years due to their ability to capture the intricate dynamics of systems with delays and non-linearities. Quasilinearization is a common method to convert non-linear BVPs into a series of linear BVPs (Bellman et al., 1965a; Bellman, 1965b) and solve it by applying a suitable numerical method. To manage the delay term, researchers often use Taylor's series expansion. This process converts a SPDDEs into a SPDEs. Kadalbajoo and Kumar (2010) treated NSP-DDE using the cubic B-spline method, and Kanth and Murali (2018) used an exponentially fitted spline method for the same BVP. Lodhi and Mishra (2017) found the approximate solution of non-linear SPDE by the quintic B-spline method. The exponential polynomial single step method was applied by Fadugba et al. (2022) for an NSP-DDE of first order with fixed delay.

The study of NSP-DDE is still under exploration and very little literature is available particularly for the second order NSP-DDE. In this study, we have used the quartic B-spline method (QBSM) on a uniform mesh to treat NSP-DDEs of the retarded kind. The main objective behind applying the QBSM is to obtain a more accurate solution with a higher convergence rate. This method generates a set of equations that are solved by Newton-Raphson method. Our approach is similar to the one applied by Roul and Thula (2019). Here, the authors have found approximate solutions to Bratu-type and Lane-Emden problems. Kadalbajoo and Kumar (2010) have discussed the same NSP-DDE. Authors have first converted non-linear equation into a series of linear equations by means of quasi linearization. Further it is solved using cubic B-spline collocation method which is proved to have almost second order convergence.

This article is organized as follows. Problem statement with all the constraints is given in section 2. The basics of B-spline function, QBSM and the derivation of the technique for NSP-DDE are explained in section 3. Error estimates are discussed in section 4. Section 5 determines the method's efficiency by applying it to a numerical problem. The findings of this research are presented in the final section.

2. Problem Statement

Consider the following second order NSP-DDE:

$$\varepsilon x'' = g\big(\wp, x, x'(\wp-\delta)\big),\ 0<\wp<1 \tag{1}$$

$$x(\wp)=\varphi(\wp), \text{ on } -\delta \le t \le 0,\ x(b)=\gamma \tag{2}$$

where, $0<\varepsilon<<1$, $\delta$is a delay parameter with $\delta = o(\varepsilon)$. It is considered that the solution of Equations (1)-(2) is continuous on [0,1] and differentiable on (0,1). If $R=\{(w_1,w_2,w_3): 0\le x\le 1, -\infty < w_2, w_3 < \infty\}$ then the function $g(w_1,w_2,w_3)$ is smooth on region R. It also satisfies the following conditions.

(i) $g_{w_2}(w_1,w_2,w_3)\ge 0$and $g_{w_3}(w_1,w_2,w_3)\le 0$.

(ii) $g_{w_2}(w_1,w_2,w_3)-g_{w_3}(w_1,w_2,w_3)\ge \lambda>0$ where, λ is constant.

(iii) $g_{w_2}(w_1,w_2,w_3)=O(|w_3|^2)$ as $w_3\to\infty$ for all $w_1\in[0,1]$ and all real w_2 and w_3.

where, $g_{w_2}(w_1,w_2,w_3)$ and $g_{w_3}(w_1,w_2,w_3)$ are partial derivatives of the function $g(w_1,w_2,w_3)$ with respective to w_2 and w_3. If $\delta=0$ and the above conditions are satisfied then BVPs (1)-(2) possess a unique solution (Lange and Miura, 1991).

We approximate the delay term using Taylor's expansion as:

$$x'(\wp-\delta)=x'(\wp)-\delta x''(\wp) \tag{3}$$

Equations (1)-(2) becomes:

$$(\varepsilon+\delta p(\wp))x''=q\big(\wp,x(\wp),x'(\wp)\big),\ 0<\wp<1 \tag{4}$$

$$x(0)=\varphi(0) \text{ and } x(1)=\gamma \tag{5}$$

3. Method Description

In this section, we have presented the basis of B-spline function, quartic B-spline method and the QBSM to solve Equations (4) and (5).

3.1 Basics of B-spline Method

B-splines are polynomial interpolation functions commonly used to find approximate solutions of various BVPs. A B-spline of degree k is a linear combination of $k+1$ basis functions of degree $k-1$. Define a partition of [0,1] as $\Pi_N: \wp_0=0<\wp_1<\wp_2<\cdots,\wp_m,\cdots<\wp_{N-1}<\wp_N=1$, where $\wp_m=\wp_0+mh$, $h=1/N$ and define a set $P_{k,\Pi_N}$of all polynomials of degree $\le k$in the interval$[\wp_m,\wp_{m+1}]$for $m=0:1:N$of Π_N. Then P_{k,Π_N} forms a linear space, and the set S_k of all functions $r(\wp)\in P_{k,\Pi_N}\cap C^{k-1}[0,1]$ is a subspace of P_{k,Π_N}.

B-Spline of order 0 is defined as

$$ß_{m,0}(\wp) = \begin{cases} 1, \wp \in \wp_m, \wp_{m+1}) \\ 0, \text{otherwise} \end{cases}.$$

The B-splines of order $k \geq 1$ are derived from the recurrence relation:

$$ß_{m,k}(\wp) = \frac{\wp - \wp_m}{\wp_{m+k} - \wp_m} ß_{m,k-1}(\wp) + \frac{\wp_{m+k+1} - \wp}{\wp_{m+k+1} - \wp_{m+1}} ß_{m+1,k-1}(\wp), \quad \wp_m \leq \wp \leq \wp_{m+k+1}.$$

For $N+1$ B-splines, there are $N+k+1$ knots. Thus, depending on value of k, few knots are added on both sides of partition P_{k,Π_N}. B-splines have the following properties (Schumaker, 2007):

- The basis functions $ß_{m,k}(\wp) > 0$ on $\wp_m \leq \wp \leq \wp_{m+k+1}, k \geq 1$ and 0 otherwise.
- $\sum_{i=0}^{N} ß_{m,k}(\wp) = 1, \quad \wp \in [0,1]$.
- $ß_{m,k}(\wp)$ are piecewise polynomials functions having knots at $\wp_m{}'s$.
- $ß_{m,k}(\wp)$ is bell shaped curve symmetric about $\wp = \wp_m$.
- $ß_{m,k}(\wp)$ are $k-1$ times continuously differentiable functions.

3.2 Description of Quartic B-spline Method

Here, we discuss the QBSM to solve Equations (4) and (5) on a uniform mesh. For this, consider the partition defined by Π_N. Let S_4 denotes the space of all quartic B-splines having knots at $\wp_m{}'s. S_4 = \{p(\wp)|p(\wp) \in C^3[0,1]$is a polynomial of degree 4 on $\Pi_N\}$.

Quartic B-spline are defined as:

$$Q_{4,m}(\wp) = \frac{1}{24h^4}\begin{cases} (\wp - \wp_{m-2})^4, \ \wp \in [\wp_{m-2}, \wp_{m-1}] \\ h^4 + 4h^3(\wp - \wp_{m-1}) + 6h^2(\wp - \wp_{m-1})^2 + 4h(\wp - \wp_{m-1})^3 - 4(\wp - \wp_{m-1})^4, \ \wp \in [\wp_{m-1}, \wp_m] \\ 11h^4 + 12h^3(\wp - \wp_m) - 6h^2(\wp - \wp_m)^2 - 12h(\wp - \wp_m)^3 + 6(\wp - \wp_m)^4, \ \wp \in [\wp_m, \wp_{m+1}] \\ h^4 + 4h^3(\wp_{m+2} - \wp) + 6h^2(\wp_{m+2} - \wp)^2 + 4h(\wp_{m+2} - \wp)^3 - 4(\wp_{m+2} - \wp)^4, \ \wp \in [\wp_{m+1}, \wp_{m+2}] \\ (\wp_{m+3} - \wp)^4, \ \wp \in [\wp_{m+2}, \wp_{m+3}] \\ 0, \text{otherwise} \end{cases} \quad (6)$$

One can easily observe that each $Q_{4,m}(\wp)$is a fourth-degree piecewise polynomial having knots at Π_N. Add four knots on each side of the partition P_N as $\wp_{-4} < \wp_{-3} < \wp_{-2} < \wp_{-1}$and $\wp_{N+1} < \wp_{N+2} < \wp_{N+3} < \wp_{N+4}$. Then $\overline{Q} = \{Q_{4,-2}, Q_{4,-1}, Q_{4,0}, Q_{4,1}, \dots, Q_{4,N+1}\}$ is linearly independent on [0,1]. The vector space $Q*(P_N)$ of all linear combinations of quartic B-splines in $\overline{Q}$ is of dimension $N+4$and $Q*(P_N) = S_4$ (Prenter, 1975). Table 1 gives the values of quartic B-splines and its derivative at the mesh points.

Table 1. Values of quartic B-splines at nodal points.

	$\wp_{m-3}$	$\wp_{m-2}$	$\wp_{m-1}$	$\wp_m$	$\wp_{m+1}$	$\wp_{m+2}$
$Q_{4,m}(\wp)$	0	$1/24$	$11/24$	$11/24$	$1/24$	0
$Q_{4,m}{}'(\wp)$	0	$1/6h$	$3/2h$	$-3/2h$	$-1/6h$	0
$Q_{4,m}{}''(\wp)$	0	$1/2h^2$	$-1/2h^2$	$-1/2h^2$	$1/2h^2$	0
$Q_{4,m}{}'''(\wp)$	0	$1/h^3$	$-3/h^3$	$3/h^3$	$-1/h^3$	0

By using Table 1 values, we get,

$$\mathbb{S}(\wp_m) = \frac{1}{24}\alpha_{m-2} + \frac{11}{24}\alpha_{m-1} + \frac{11}{24}\alpha_m + \frac{1}{24}\alpha_{m+1} \tag{7}$$

$$\mathbb{S}'(\wp_m) = -\frac{1}{6h}\alpha_{m-2} - \frac{1}{2h}\alpha_{m-1} + \frac{1}{2h}\alpha_m + \frac{1}{6h}\alpha_{m+1} \tag{8}$$

$$\mathbb{S}''(\wp_m) = \frac{1}{2h^2}\alpha_{m-2} - \frac{1}{2h^2}\alpha_{m-1} - \frac{1}{2h^2}\alpha_m + \frac{1}{2h^2}\alpha_{m+1} \tag{9}$$

$$\mathbb{S}'''(\wp_m) = -\frac{1}{h^3}\alpha_{m-2} + \frac{3}{h^3}\alpha_{m-1} - \frac{3}{h^3}\alpha_m + \frac{1}{h^3}\alpha_{m+1} \tag{10}$$

***Lemma* 3.1:** The quartic B-splines $\{Q_{4,m}\}_{m=-2}^{N+1}$ satisfy the following inequality:
$\sum_{m=-2}^{N+1}|Q_{4,m}(\wp)| \le \frac{35}{24}$.

***Proof*:** Using the values of quartic B-splines form Table 1, at a mesh point $\wp = \wp_m$, we have
$|Q_{4,m-2}(\wp_m)| = \frac{1}{24}, |Q_{4,m-1}(\wp_m)| = \frac{11}{24}, |Q_{4,m}(\wp_m)| = \frac{11}{24}, |Q_{4,m+1}(\wp_m)| = \frac{1}{24}$,

and $Q_{4,j}(\wp_m) = 0$ for other values of m. Thus, at a mesh point $\wp = \wp_m$.
$\sum_{m=-2}^{N+1}|Q_{4,m}(\wp)| = |Q_{4,m-2}(\wp_m)| + |Q_{4,m-1}(\wp_m)| + |Q_{4,m}(\wp_m)| + |Q_{4,m+1}(\wp_m)| = 1.$

Also, if $\wp \in [\wp_{m-1}, \wp_m]$, then
$|Q_{4,m-3}(\wp)| \le \frac{1}{24}, |Q_{4,m-2}(\wp)| \le \frac{11}{24}, |Q_{4,m-1}(\wp)| \le \frac{11}{24}, |Q_{4,m}(\wp)| \le \frac{11}{24}, |Q_{4,m+1}(\wp)| \le \frac{1}{24}$.

Thus,
$\sum_{m=-2}^{N+1}|Q_{4,m}(\wp)| = |Q_{4,m-3}(\wp)| + |Q_{4,m-2}(\wp)| + |Q_{4,m-1}(\wp)| + |Q_{4,m}(\wp)| + |Q_{4,m+1}(\wp)| \le \frac{35}{24}$.

Hence proved.

3.3 Formulation of QBSM for NSP-DDE

In this part of the paper, we apply QBSM to obtain numerical solution of non-linear SPDDE defined in section 2. Define the quartic B-spline function $\mathbb{S}(\wp)$, which approximates $x(\wp)$as follow at the mesh points $\wp_m$ as:

$$\mathbb{S}(\wp) = \sum_{m=-2}^{N+1}\alpha_m Q_{4,m}(\wp) \tag{11}$$

The spline function also satisfies the interpolation constraints

$$\mathbb{S}^{(k)}(\wp_m) = x^{(k)}(\wp_m), \quad m = 0:1:N \text{ and } k = 0,1,2,3 \tag{12}$$

Substitute (7) in BVP (4)-(5), we get,

$$(\varepsilon + \delta p(\wp))\mathbb{S}''(\wp) = q(\wp, \mathbb{S}(\wp), \mathbb{S}'(\wp)), \; 0 < \wp < 1 \tag{13}$$

$$\mathbb{S}(0) = \varphi(0) \text{ and } \mathbb{S}(1) = \gamma \tag{14}$$

Discretizing Equation (12) at grid points, we have

$$(\varepsilon + \delta p(\wp_m))\mathbb{S}''(\wp_m) = q(\wp_m, \mathbb{S}(\wp_m), \mathbb{S}'(\wp_m)), \; m = 0:1:N \tag{15}$$

and the boundary conditions (9) yields

$$\mathbb{S}(\wp_0) = \varphi(\wp_0) \text{ and } \mathbb{S}(\wp_N) = \gamma \tag{16}$$

The Equations (15)-(16) are a set of $N+3$ equations in $N+4$ unknowns; namely, $\wp_{-2}, \wp_{-1}, \wp_0, \dots, \wp_{N+1}$. To obtain one more equation, we differentiate Equation (8) and consider its value at $\wp = \wp_N$. This results in a set of $N+4$ equations with an equal number of unknowns that are stated in matrix form as:

$$CX = D \tag{17}$$

where, the matrix C is given by,

$$C = \begin{bmatrix} \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{P_0}{2h^2} & \frac{-P_0}{2h^2} & \frac{-P_0}{2h^2} & \frac{P_0}{2h^2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{P_1}{2h^2} & \frac{-P_1}{2h^2} & \frac{-P_1}{2h^2} & \frac{P_1}{2h^2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{P_2}{2h^2} & \frac{-P_2}{2h^2} & \frac{-P_2}{2h^2} & \frac{P_2}{2h^2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & & & \cdots & \frac{P_{N-1}}{2h^2} & \frac{-P_{N-1}}{2h^2} & \frac{-P_{N-1}}{2h^2} & \frac{P_{N-1}}{2h^2} & 0 \\ 0 & 0 & 0 & \cdots & & & & \cdots & 0 & \frac{P_N}{2h^2} & \frac{-P_N}{2h^2} & \frac{-P_N}{2h^2} & \frac{P_N}{2h^2} \\ 0 & 0 & 0 & \cdots & & & & & \cdots & \frac{-P_N}{h^3} & \frac{3P_N}{h^3} & \frac{-3P_N}{h^3} & \frac{P_N}{h^3} \\ 0 & 0 & 0 & \cdots & & & & & \cdots & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} \end{bmatrix}.$$

and D is the right-hand side matrix,

$$D^T = [\varphi(0) \quad K_0 \quad K_1 \quad \cdots \quad \cdots \quad \cdots \quad K_{N-1} \quad K_N \quad K'_N \quad \gamma].$$

where, $P_m = \varepsilon + \delta a(\wp_m), K_m = q(\wp_m, \mathbb{S}_m, \mathbb{S}(\wp_m)), m = 0{:}1{:}N$ and $K'_N = q'(\wp_N, \mathbb{S}_N, \mathbb{S}(\wp_N))$.

As the considered BVP is non-linear, we have applied Newton-Raphson method to get the solution of (17).

4. Error Analysis

Various error bounds on the solution $x(\wp)$ and its derivatives at the nodal points $\wp_0, \wp_1, \dots, \wp_N$ are obtained in this part of the manuscript. We prove the following result using a similar approach as in Kadalbajoo and Kumar (2007).

***Lemma* 4.1:** Let $x(\wp) \in C^8[0,1]$ and $\mathbb{S}(\wp)$be approximate of $x(\wp)$ given by the quartic B-spline which satisfies the conditions (12), then

$$\mathbb{S}'(\wp_m) = x'(\wp_m) + \frac{1}{720}h^4 x^{(5)}(\wp_m) - \frac{1}{2016}h^6 x^{(7)}(\wp_m) + O(h^8) \tag{18}$$

$$\mathbb{S}''(\wp_m) = x''(\wp_m) - \frac{1}{240}h^4 x^{(6)}(\wp_m) + \frac{1}{6048}h^6 x^{(8)}(\wp_m) + O(h^8) \tag{19}$$

$$\mathbb{S}'''(\wp_m) = x'''(\wp_m) - \frac{1}{12}h^2 x^{(5)}(\wp_m) + \frac{1}{240}h^4 x^{(7)}(\wp_m) + O(h^6) \tag{20}$$

$$\mathbb{S}^{(4)}(\wp_{m+}) = 24\left[\mathbb{S}(\wp_{m-1}) + \mathbb{S}(\wp_m) - 2\mathbb{S}^{(4)}(\wp_{m+1})\right] + 6h[\mathbb{S}'(\wp_{m-1}) + 8\mathbb{S}'(\wp_m) + 3\mathbb{S}'(\wp_{m+1})] \tag{21}$$

$$\mathbb{S}^{(4)}(\wp_{m-}) = 24\left[\mathbb{S}(\wp_{m+1}) + \mathbb{S}(\wp_m) - 2\mathbb{S}^{(4)}(\wp_{m-1})\right] - 6h[\mathbb{S}'(\wp_{m+1}) + 8\mathbb{S}'(\wp_m) + 3\mathbb{S}'(\wp_{m-1})] \tag{22}$$

where, $\mathbb{S}^{(4)}(\wp_{m+})$ presents the value of $\mathbb{S}^{(4)}(\wp_m)$ in $[\wp_m, \wp_{m+1}]$, and $\mathbb{S}^{(4)}(\wp_{m-})$ is the value of $\mathbb{S}^{(4)}(\wp_m)$ in $[\wp_{m-1}, \wp_m]$.

***Proof*:** Using Equations (7)-(10), we obtain

$$h[\mathbb{S}'(\wp_{m-2}) + 11\mathbb{S}'(\wp_{m-1}) + 11\mathbb{S}'(\wp_m) + \mathbb{S}'(\wp_{m+1})] = 4[\mathbb{S}(\wp_{m+1}) + 3\mathbb{S}(\wp_m) - 3\mathbb{S}(\wp_{m-1}) - \mathbb{S}(\wp_{m-2})] \quad (23)$$

$$h^2\mathbb{S}''(\wp_m) = 2[\mathbb{S}(\wp_{m+1}) - 2\mathbb{S}(\wp_m) + \mathbb{S}(\wp_{m-1})] - \frac{h}{2}[\mathbb{S}'(\wp_{m+1}) - \mathbb{S}'(\wp_{m-1})] \quad (24)$$

$$h^3\mathbb{S}'''(\wp_m) = 12[\mathbb{S}(\wp_{m+1}) - \mathbb{S}(\wp_{m-1})] - 3h[\mathbb{S}'(\wp_{m+1}) + 6\mathbb{S}'(\wp_m) + \mathbb{S}'(\wp_{m-1})] \quad (25)$$

Using operator $E(\mathbb{S}(\wp_m)) = \mathbb{S}(\wp_{m+1})$ in Equation (23), we get

$$h[E^{-2} + 11E^{-1} + 11 + E]\mathbb{S}'(\wp_m) = 4[E + 3 - 3E^{-1} - E^{-2}]\mathbb{S}(\wp_m) \quad (26)$$

since $E = e^{hD}$ where $D = \frac{d}{d\wp}$, we get

$$h[e^{-2hD} + 11e^{-hD} + 11 + e^{hD}]\mathbb{S}'(\wp_m) = 4[e^{hD} + 3 - 3e^{-hD} - e^{-2hD}]\mathbb{S}(\wp_m) \quad (27)$$

Expressing $E = e^{hD}$ in Taylor's series expansion in powers of hD, we obtain

$$24h\left(1 - \frac{1}{2}hD + \frac{1}{3}h^2D^2 - \frac{1}{8}h^3D^3 + \frac{7}{144}h^4D^4 + \cdots\cdots\right)\mathbb{S}'(\wp_m) = 24h\left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 - \frac{1}{8}h^3D^4 + \cdots\cdots\right)x(\wp_m) \quad (28)$$

or

$$S'(\wp_i) = \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 - \frac{1}{8}h^3D^4 + \cdots\right)\left(1 + \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 - \frac{1}{8}h^3D^3 + \cdots\right)\right)^{-1} x(\wp_m)$$

$$= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 - \cdots\right)\left(1 - \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 - \cdots\right) + \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 - \cdots\right)^2 - \cdots\right)x(\wp_m)$$

$$= \left(D + \frac{1}{720}h^4D^5 - \frac{1}{2016}h^6D^7 + \frac{1}{17280}h^8D^9 + \cdots\right)x(\wp_m).$$

Simplifying this equation, Equation (18) gets proved.

To prove Equation (19), substitute Equation (18) in Equation (24) and using the interpolation constraints (8), we obtain,

$$h^2\mathbb{S}''(\wp_m) = 2[x(\wp_{m+1}) - 2x(\wp_m) + x(\wp_{m-1})] - \frac{h}{2}\Big\{x'(\wp_{m+1}) + \frac{1}{720}h^4x^{(5)}(\wp_{m+1}) - \frac{1}{2016}h^6x^{(7)}(\wp_{m+1}) + O(h^8) - \Big(x'(\wp_{m-1}) + \frac{1}{720}h^4x^{(5)}(\wp_{m-1}) - \frac{1}{2016}h^6x^{(7)}(\wp_{m-1}) + O(h^8)\Big)\Big\}, m = 1:1:N.$$

Applying Taylor's series to $x^{(k)}(\wp_{m\pm1})$ for $k = 0,1,5,7$, we get (19). Using the same approach in Equations (20)-(22). This completes the proof of lemma.

To find the error, define $e(\wp) = x(\wp) - \mathbb{S}(\wp)$, substitute Equations (18)-(22) in the Taylor's approximation of $e(\wp_m + \theta h)$; we obtain

$$e(\wp_m + \theta h) = -\frac{(10\theta^2-1)\theta}{720}h^5x^{(5)}(\wp_m) + \frac{(5\theta^2-3)\theta^2}{1440}h^6x^{(6)}(\wp_m) + \frac{(7\theta^2-5)\theta}{10080}h^7x^{(7)}(\wp_m) + O(h^9) \quad (29)$$

Hence proved.

Following theorem gives the global error bounds.

***Theorem*:** If the conditions in Lemma 4.1 are satisfied, then
$\|x(\wp) - \mathbb{S}(\wp)\|_\infty = O\left(h^{5-k}\right)$ for $k = 0, 1, 2, 3, 4$.

***Proof*:** For proof, see (Roul and Thula, 2019).

5. Numerical Illustration and Discussion

We use the technique on a numerical example to prove the accuracy and to get the numerical value of rate of convergence. The numerical simulations in this study were conducted using MATLAB (version R2023b, MathWorks Inc.). MATLAB was chosen for its robust suite of numerical computation libraries and its suitability for solving differential equations.

$$\varepsilon x''(\wp) + 2x'(\wp - \delta) - e^{x(\wp)} = 0,$$
$$x(\wp) = 0,\ -\delta < x < 0 \text{ and } x(1) = 0.$$

Due to non-linearity of the BVP, the exact solution cannot be determined. Hence, the MAE is found by the double mesh principle, which is given by,

$$e^N = \max_{0 \le m \le N} |\mathbb{S}(\wp_m) - \mathbb{S}(\wp_{2m})|.$$

We have also determined the rate of convergence (RCGT) using the formula.

$$c = \frac{ln(e^N/e^{2N})}{ln\,2}.$$

Table 2 presents calculated MAE, RCGT, and run time for various values of N, and ε with $\delta = 0.4\varepsilon$. The tabulated values show that the MAE decreases as Nincreases. From Table 3, it is seen that the results obtained by QBSM are better as compared with Kadalbajoo and Kumar (2010). Figure 1 is the numerical solution for many values of δ and Figure 2 for different values of ε. In both the figures, a left boundary layer is observed. From Figure 1, we can see the decrease in the width of the boundary region as δ increases. Also, Figure 2 shows that the width of the boundary layer and the oscillations decrease as ε tends to zero.

Table 2. MAE, rate of convergence and run time for $\delta = 0.4\varepsilon$.

ε	N=50	RCGT	Time in Sec.	N=100	Time in Sec.	RCGT	N=200	Time in Sec.
2^{-1}	2.0316E-05	3.9536E+00	2.961	1.3114E-06	10.904	3.9941E+00	8.2298E-08	43.166
2^{-2}	2.9896E-04	3.9238E+00	2.980	1.9698E-05	11.831	3.9497E+00	1.2748E-06	42.437
2^{-3}	3.6852E-03	3.6456E+00	2.959	2.9447E-04	11.271	3.9223E+00	1.9423E-05	42.556
2^{-4}	2.1380E-02	2.5426E+00	3.180	3.6694E-03	11.107	3.6505E+00	2.9220E-04	42.957

Table 3. Comparison of MAE with Kadalbajoo and Kumar (2010) for $\varepsilon = 10^{-1}$

N	MAE by Kadalbajoo and Kumar (2010)	MAE
64	5.688 E – 02	3.413 E – 03
128	1.754 E – 02	2.665 E – 04
256	5.580 E – 02	1.772 E – 05
1024	1.276 E – 02	6.210 E – 06

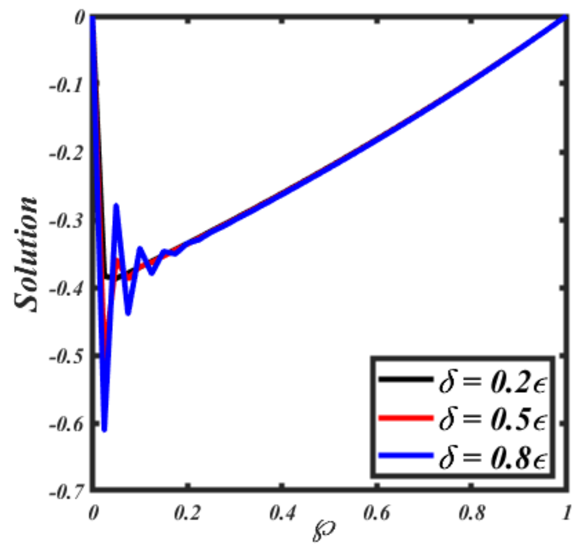

Figure 1. Solution of example 1 for various δvalues.

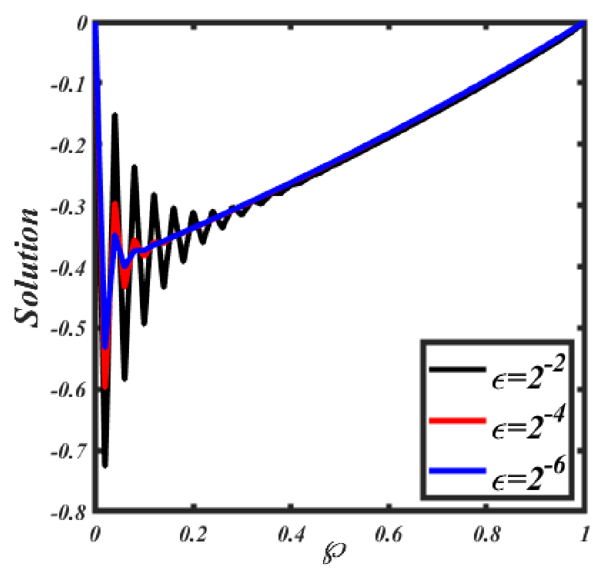

Figure 2. Solution of example 1 for various ε values.

6. Conclusions and Future Scope

We conclude that the QBSM is efficient for the NSP-DDE. The accuracy of the approach is demonstrated through numerical experiments with a minimal number of grid points, the method can precisely capture the solution. Additionally, it is noted that the MAE's value decreases as N increases. The calculated order of convergence is almost four. We have also explained the error analysis for the discussed method. Overall, the quartic B-spline method is computationally easy to implement and can be extended to solve a variety of SPDEs as well as SPDDEs with different delay structures and boundary conditions. It has been noticed that very few researchers have addressed NSP-DDE, so there is a lot of scope to solve these equations using variety of numerical methods tailored specifically for solving complex NSP-DDE. This includes exploring adaptive mesh refinement strategies, higher-order accurate methods, and hybrid approaches combining various numerical schemes to enhance accuracy and efficiency.

Conflict of Interest

The authors have confirmed that there are no conflicts of interest to report with this research.

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