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Oscillation for Super Linear/Linear Second Order Neutral Difference Equations with Variable Several Delays

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Abstract
This article, is concerned with finding sufficient conditions for the oscillation and non oscillation of the solutions of a second order neutral difference equation with multiple delays under the forward difference operator, which generalize and extend some existing results.This could be possible by extending an important lemma from the literature.

Keywords- Oscillation, Non oscillation, Neutral difference equation, Asymptotic behavior.

1. Introduction

This article is concerned with finding sufficient conditions so that a solution of the neutral delay difference equation(NDDE in short)

$$\Delta^2(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}) + v_n G(y_{\sigma(n)}) = f_n, \tag{1}$$

which does not oscillate, tends to zero as $n \to \infty$. Here Δ is the forward difference operator, given by $\Delta x_n = x_{n+1} - x_n$, $p_n^{\{j\}}, v_n$ and f_n are members of infinite sequences of real numbers with $v_n > 0, G \in C(R, R)$. Further, we assume $\{\sigma(n)\}$ is an unbounded sequence such that $\sigma(n) \le n$ for every n. Different ranges of the $\{p_n^{\{j\}}\}$ for $j = 1,2,\dots,k$ are considered. The m_j for $j = 1,2,\dots,k$ are positive integers.

The following hypothesis are needed in the sequel,
(E1) $xG(x) > 0$ for $x \ne 0$.

(E2) $v_n > 0, \sum_{n=n_0}^{\infty} v_n = \infty$.

(E3) There exists $\{F_n\}$, a bounded sequence such that $\Delta^2 F_n = f_n$.

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(E4) The sequence F_n in (E3) satisfies $\lim_{n\to\infty} F_n = 0$.

(E5) $\sum_{n_0}^{\infty} v_n^* = \infty$, where $v_n^* = \min\{v_n, v_{n-m_1}, v_{n-m_2}\}$.

(E6) For $v > 0, w > 0, u > 0$, there exists a scalar $\beta > 0$, such that $G(v)G(w) \geq G(vw)$ and $G(v) + G(w) + G(u) \geq \beta G(v + w + u)$.

(E7) $\sum_{n=n_0}^{\infty} nv_n = \infty$.

(E8) $\sum_{j=1}^{\infty} (n_j)v_{n_j} = \infty$ where v_{nj} is any subsequence of v_n.

(E9) For $u > 0$ there exists $\delta > 0$ such that $G(u) \geq \delta u$. For $u < 0$ there exists $\delta > 0$ such that $G(u) \leq \delta u$.

(E10) $G(-u) = -G(u)$.

(E11) $\liminf_{n\to\infty} \sigma(n)/n > 0$.

Remark 1.1 By (E9), *if* $\liminf_{n\to\infty} u_n > 0$ *then* $\exists\ \delta > 0$ such that $\liminf_{n\to\infty} G(u_n)/u_n > \delta$.
We assume that $p_n^{\{j\}}, j = 1,2,\ldots,k$ are bounded and satisfies one of the following conditions. There exists positive constants b_j, $j = 1,2,\ldots,k$ and b such that

(R1) $b_j \geq p_n^{\{j\}} \geq 0, \forall\ \ j = 1,2,\ldots k$ and $\sum_{j=1}^{k} \liminf_{n\to\infty}\ p_n^{\{j\}} < \sum_{j=1}^{k} b_j = b < 1$.

(R2) $-b_j \leq p_n^{\{j\}} \leq 0, \forall\ \ j = 1,2,\ldots k$ and $\sum_{j=1}^{k} \liminf_{n\to\infty}\ p_n^{\{j\}} \geq \sum_{j=1}^{k} -b_j = -b > -1$.

$p_n^{\{j\}} \leq 0, \forall\ j = 1,2,\ldots k$ and $\exists\ i \in \{1,2,3,\ldots,k\}$ such that

(R3) $\limsup\ p_n^{\{i\}} - \sum_{j\neq i} \liminf\ p_n^{\{j\}} < -1$.

$p_n^{\{j\}} \geq 0\ \ \forall\ \ j = 1,2,\ldots k$ and $\exists\ i \in \{1,2,3,\ldots,k\}$ such that

(R4) $\liminf\ p_n^{\{i\}} - \sum_{j\neq i} \limsup\ p_n^{\{j\}} > 1$.

For easy understanding and convenience of writing the proofs of the results, the higher order NDDE

$$\Delta^2(y_n - p_n^{\{1\}} y_{n-m_1} - p_n^{\{2\}} y_{n-m_2}) + v_n G(y_{\sigma(n)}) = f_n \tag{2}$$

with two delay terms under the Δ^2 sign, is considered, instead of (1) and this is with out any loss of generality.

One of the following conditions are to be assumed on the bounded sequences $\{p_n^{\{j\}}\}$ for $j = 1,2$ while considering the neutal equation (2).

There exists positive constants $b, b_1, and\ \ b_2$ such that

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(R5) $b \geq p_n^{\{1\}} > 0, b \geq p_n^{\{2\}} \geq 0.$

(R6) $-b \leq p_n^{\{1\}} < 0, -b \leq p_n^{\{2\}} \leq 0.$

(R7) $1 > b_1 \geq p_n^{\{1\}} \geq 0, 1 > b_2 \geq p_n^{\{2\}} \geq 0,$ and $b_1 + b_2 = b < 1.$

(R8) $-1 < -b_1 \leq p_n^{\{1\}} \leq 0, -1 < -b_2 \leq p_n^{\{2\}} \leq 0$ and $b_1 + b_2 = b < 1.$

(R9) $b \geq p_n^{\{1\}} > 1, b \geq p_n^{\{2\}} \geq 0.$

(R10) $-b \leq p_n^{\{1\}} < -1, -b \geq p_n^{\{2\}} \leq 0.$

Note that (R1) and (R2) are equivalent to (R7) and (R8) respectively. Further note that (R6) is less restrictive than (R3) and (R5) is less restrictive than (R4). If $p_n = p_n^{\{1\}}$, then the ranges of p_n, which are obtained, by the substitution k=1, in (R1)−(R4), or $p_n^{\{2\}} = 0$, in (R7) – (R10), are considered (Parhi and Tripathy, 2003; Rath et al., 2010).

Let N_1 be a fixed non negative integer and $r = \max\{m_j : j = 1,2,\ldots,k\}$. Let $N_0 = \min\{N_1 - r, \sigma(N_1)\}$. A solution of (1), is defined as "a real sequence $\{y_n\}$, which is defined ∀ +ve integer $n \geq N_0$, and satisfies (1) for $n \geq N_1$. Further, if the initial values

$$y_n = a_n \quad \text{for} \quad N_0 \leq n \leq N_1 + 1, \tag{3}$$

are provided then the equation (1) has a unique solution satisfying the initial values (3). A non trivial solution $\{y_n\}$ of (1) is called oscillatory, if for any positive integer $n_0 \geq N_1$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory."

As is well known that, it is not always easy to solve a functional difference equation and find it's solution in closed form, therefore, qualitative theory of difference equations is developed rapidly, since here we assume that the solutions of the difference equation exist and concentrate to investigate its oscillatory behaviour . Recently, numerous articles on oscillation of solutions of neutral difference equations are published for example (Agarwal et al., 1996; Agarwal and Grace, 1999; Parhi and Tripathy, 2003; Zhou and Huang, 2003; Yildz and Ocalan, 2007; Karpuz et al., 2009a; Karpuz et al., 2009b; Yildiz et al., 2009; Yildiz, 2015) and the references cited therein. Thandapani et al. (1999) found non-oscillation and oscillation criteria for the equation

$$\Delta^m (y_n - p_n y_{n-l}) + v_n G(y_{n-r}) = f_n. \tag{4}$$

Here, we study the oscillatory behaviour of (1) and (2), which seems, not being considered by any author till date. This paper generalizes the study of the equation (4) for $m = 2$. We observe that while studying the NDDEs , the authors (Parhi and Tripathy, 2003; Rath and Padhy, 2005; Rath et al., 2010) have significantly used the Lemma 2.1 of Parhi and Tripathy (2003), which is the discrete analogue to the Lemma 1.5.2 of Gyori and Ladas (1991), for their results. It is further observed that, even, many results for the study of neutral differential equations (i.e; the continuous case) are dependent on a similar result, i.e; Lemma 1.5.2 of Gyori and Ladas (1991). However, the Lemma

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2.1 of Parhi and Tripathy (2003) cannot be applied to the study of (1) or that of (2). In this context, one may go through the "**open problem** 1.8, at page 31 of Gyori and Ladas (1991) which suggests to extend the lemma suitably, for its own sake and its application to the study of the neutral equations with several delays." In this article we extend the lemma for the said purpose in order to study the oscillatory behavior of (1) or (2), there by, improving, extending and generalizing some results of Parhi and Tripathy (2003); Rath and Behera (2018).

2. Some Lemmas

In this section first, we quote some results from different research articles , that would be helpful in the sequel.

Lemma 2.1 (Parhi and Tripathy, 2003) [Lemma 2.1] *" Let $\{f_n\},\{q_n\}$ and $\{p_n\}$ be real sequences defined for $n \geq N_0 > 0$ such that*

$$f_n = q_n - p_n q_{n-m}, n \geq N_0 + m$$

where $m \geq 0$ is an integer. Let b, b_1 and b_2 be reals such that p_n satisfies one of the three conditions below

(i) $-1 < -b \leq p_n \leq 0$, (ii) $-b_2 \leq p_n \leq -b_1 < -1$, (iii) $0 \leq p_n \leq b_2$.

If $q_n > 0$ for $n > N_0$, $\liminf_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} f_n = \delta$ exists then $\delta = 0$."

Lemma 2.2 (Royden, 1988) *" Let $\{u_n\}$ and $\{v_n\}$ be two real sequences defined for $n \geq n_0 > 0$.* Then

$$\liminf_{n\to\infty} u_n + \liminf_{n\to\infty} v_n \leq \liminf_{n\to\infty} (u_n + v_n) \leq \limsup_{n\to\infty} u_n + \liminf_{n\to\infty} v_n$$

$$(\text{or} \qquad \liminf_{n\to\infty} u_n + \limsup_{n\to\infty} v_n) \leq \limsup_{n\to\infty} (u_n + v_n) \leq \limsup_{n\to\infty} u_n + \limsup_{n\to\infty} v_n$$

provided that no sum is of the form $\infty - \infty$."

Lemma 2.3 (Royden, 1988) *" Let $\{u_n\}$ and $\{v_n\}$ be two non negative real sequences defined for $n \geq n_0 > 0$. Then*

$$\liminf_{n\to\infty} u_n \times \liminf_{n\to\infty} v_n \leq \liminf_{n\to\infty} (u_n \times v_n) \leq \limsup_{n\to\infty} u_n \times \liminf_{n\to\infty} v_n$$

$$(\text{or} \qquad \liminf_{n\to\infty} u_n \times \limsup_{n\to\infty} v_n) \leq \limsup_{n\to\infty} (u_n \times v_n) \leq \limsup_{n\to\infty} u_n \times \limsup_{n\to\infty} v_n \qquad (5)$$

provided that no product is of the form $0 \times \infty$."

Lemma 2.4 (Agarwal, 2000; Parhi and Tripathy, 2003) *"Let z_n be a real valued function defined for $n \in N(n_0) = \{n_0, n_0 + 1, \ldots\}, n_0 > 0$ and $z_n > 0$ with $\Delta^m z_n$ either +ve or –ve on $N(n_0)$ and not equal to zero.Then $\exists$ an integer p, $0 \leq p \leq m-1$, with $(m+p)$ even for $\Delta^m z_n \geq 0$, and $m + p$ odd for $\Delta^m z_n \leq 0$ such that*

$\Delta^i z_n > 0$ for $n \geq n_0, 0 \leq i \leq p,$

$(-1)^{p+i}\Delta^i z_n > 0,$ for $n \geq n_0, p+1 \leq i \leq m-1.$"

Remark 2.5 *From the above lemma, for* $m = 2$, *if* $\Delta^2 z_n \leq 0$ *and* $z_n \geq 0$ *then* $p = 1$*and* $\Delta z_n > 0$.

Definition 2.6 "Define the factorial function (Kelley and Peterson, 1991) by $n^{(k)} := n(n-1)\ldots(n-k+1)$,

where $k \leq n$ and $n \in Z$ and $k \in N$. Note that $n^{(k)} = 0$, if $k > n$."

Lemma 2.7 (Rath, et al., 2010) " *Let* $p \in N$ *and* $x(n)$ *be a +ve real sequence in* $[n_1, \infty)$ *for some large* n_1 . *If* $\exists$ *an integer* $p_0 \in \{0,1,\ldots,p-1\}$, *such that* $\Delta^i w(\infty) = 0$ *and* $\Delta^{p_0} w(\infty)$ *exits (finite) for all* $i \in \{p_0+1,\ldots,p-1\}$. *Then*

$$\Delta^p w(n) = -x(n), \tag{6}$$

implies

$$\Delta^{p_0} w(n) = \Delta^{p_0} w(\infty) + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!}\sum_{i=n}^{\infty} (i+p-p_0-1-n)^{(p-p_0-1)} x(i), \tag{7}$$

for all sufficiently large n."

Remark 2.8 Consider $\{w_n\}$ as a real sequence and L as a +ve scalar such that $w_n > L$ for $n \geq n_1$. If $z_n \geq w_n - \epsilon$ for $n \geq n_2 \geq n_1$, where ϵ is is any arbitrary pre assigned positive number, then $\exists$ a +ve scalar $C < L$ and a +ve integer $n_3 \geq n_2$ such that $n \geq n_3$ implies $z_n \geq C$.

Lemma 2.9 (Malik and Arora, 2008) "If $\sum u_n$ and $\sum v_n$ are two positive term series such that $\lim_{n\to\infty}(\frac{u_n}{v_n}) = l$, where l is a finite number and not equal to zero, then the two series diverge or converge together. If $l = \infty$ then divergence of $\sum v_n$ $\Rightarrow$ divergence of $\sum u_n$. If $l = 0$ then, convergence of $\sum v_n$ $\Rightarrow$ convergence of $\sum u_n$."

Remark 2.10 By Lemma 2.9, it follows that (E7) holds if and only if $\sum_{n=n_0}^{\infty}(n-n_0+1)v_n = \infty$. It is because $(n-r+1)^r < n^{(r)} < n^r$ for $r \leq n$.

Remark 2.11 The condition $|\sum_{n=n_0}^{\infty} nf_n| < \infty$ implies that (E3) and (E4) holds.In fact, if we define $F_n = \sum_{j=n}^{\infty}(j-n+1)f_j$ by Lemma 2.9 then, $\Delta^2 F_n = f_n$ and $\lim_{n\to\infty}F_n = 0$.

The following result extends and generalizes the Lemma 2.1.

Lemma 2.12 *Assume* $y_n > 0$ *for* $n \geq n_0$ *with* $\liminf_{n\to\infty} y_n = 0$. *Suppose that*

$$z_n = y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}. \tag{8}$$

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Further, assume that $lim_{n\to\infty} z_n = \delta$ *exists finitely. Then*

(a) If $p_n^{\{j\}} \geq 0$ *then* $\delta \leq 0$ *and if* $p_n^{\{j\}} \leq 0$ *then* $\delta \geq 0$.

(b) Further, suppose that y_n *is bounded and* $p_n^{\{j\}}, j = 1,2,\ldots,k$, *satisfy one of the four conditions* (R1), (R2), (R3) or (R4). *Then* $\delta = 0$ *and* $lim_{n\to\infty} y_n = 0$.

Proof. (a) Since $\lim_{n\to\infty} z_n = \delta$ exists finitely then $\liminf_{n\to\infty} z_n = \limsup_{n\to\infty} z_n = \delta$. If $p_n^{\{j\}} \geq 0$ then $z_n \leq y_n$ and $\liminf_{n\to\infty} z_n \leq \liminf_{n\to\infty} y_n$. This implies $\delta \leq 0$. Again if $p_n^{\{j\}} \leq 0$ then $z_n \geq y_n$ and this implies $\delta \geq 0$. Hence the result follows.

(b) Since y_n is bounded then $\liminf_{n\to\infty} y_n$ and $\limsup_{n\to\infty} y_n$ exists finitely.

Let us consider <u>case (i)</u> i.e; $p_n^{\{j\}}$ satisfy(R1). This implies $p_n^{\{j\}} \geq 0$. Hence we obtain $\delta \leq 0$. Then using Lemma 2.2 and 2.3 we have

$$
\begin{aligned}
0 \geq \delta = \limsup_{n\to\infty} z_n &= \limsup_{n\to\infty} \left(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j} \right) \\
&\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} \left(-\sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j} \right) \\
&\geq \limsup_{n\to\infty} y_n - \limsup_{n\to\infty} \left(\sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j} \right) \\
&\geq \limsup_{n\to\infty} y_n - \sum_{j=1}^{k} \limsup_{n\to\infty} \left(p_n^{\{j\}} y_{n-m_j} \right) \\
&\geq \limsup_{n\to\infty} y_n - \sum_{j=1}^{k} \limsup_{n\to\infty} p_n^{\{j\}} \limsup_{n\to\infty} y_{n-m_j} \\
&\geq \limsup_{n\to\infty} y_n \left(1 - \sum_{j=1}^{k} \limsup_{n\to\infty} p_n^{\{j\}} \right) \\
&\geq \limsup_{n\to\infty} y_n (1-b) \geq 0.
\end{aligned}
$$

Hence $\delta = 0$ and $\limsup_{n\to\infty} y_n \leq 0$, by (R1), which implies $\limsup_{n\to\infty} y_n = 0$. Hence $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} y_n = 0$.

Next consider <u>case (ii)</u> i.e; $p_n^{\{j\}}$ satisfy(R2). Clearly, $z_n \geq y_n$ due to (R2) and this implies $\delta \geq 0$. Further, using Lemma 2.2 and 2.3 we have

$$
\begin{aligned}
\delta = \liminf_{n\to\infty} z_n &= \liminf_{n\to\infty} \left(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j} \right) \\
&\leq \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} \left(\sum_{j=1}^{k} -p_n^{\{j\}} y_{n-m_j} \right) \\
&\leq \sum_{j=1}^{k} \limsup_{n\to\infty} \left(-p_n^{\{j\}} \right) \left(y_{n-m_j} \right)
\end{aligned}
$$

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$$\le \sum_{j=1}^{k} \limsup_{n\to\infty}\left(-p_n^{\{j\}}\right) \limsup_{n\to\infty}\left(y_{n-m_j}\right)$$
$$\le \sum_{j=1}^{k} -\liminf_{n\to\infty}\left(p_n^{\{j\}}\right) \limsup_{n\to\infty}\left(y_{n-m_j}\right)$$
$$\le b \limsup_{n\to\infty} y_n \le b\alpha.$$

Hence we get

$$\alpha \ge \frac{\delta}{b} > \delta. \tag{9}$$

Again

$$\delta = \limsup_{n\to\infty} z_n = \limsup_{n\to\infty}\left(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)$$
$$\ge \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}\left(-\sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)$$
$$\ge \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}\left(\sum_{j=1}^{k}\left(-p_n^{\{j\}}\right) y_{n-m_j}\right)$$
$$\ge \limsup_{n\to\infty} y_n + \sum_{j=1}^{k} \liminf_{n\to\infty}\left(\left(-p_n^{\{j\}}\right) y_{n-m_j}\right)$$
$$\ge \limsup_{n\to\infty} y_n + \sum_{j=1}^{k} \liminf_{n\to\infty}\left(-p_n^{\{j\}}\right) \liminf_{n\to\infty} y_{n-m_j}$$
$$\ge \limsup_{n\to\infty} y_n = \alpha.$$

Combining the above inequation with (9), it follows that $\alpha > \delta \ge \alpha$, a contradiction which implies $\delta = 0 = \alpha$.

Let us consider <u>case iii</u>: i.e; $p_n^{\{j\}}$ satisfy(R3). Clearly, $z_n \ge y_n$ due to (R3) and this implies $\delta \ge 0$. Further, using Lemma 2.2 and 2.3 we have

$$\delta = \liminf_{n\to\infty} z_n = \liminf_{n\to\infty}\left(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)$$
$$\le \limsup_{n\to\infty}\left(y_n + \sum_{j\ne i} -p_n^{\{j\}} y_{n-m_j}\right) + \liminf_{n\to\infty}\left(-p_n^{\{i\}} y_{n-m_i}\right)$$
$$\le \limsup_{n\to\infty} y_n + \limsup_{n\to\infty} \sum_{j\ne i} -p_n^{\{j\}} y_{n-m_j} + \limsup_{n\to\infty}\left(-p_n^{\{i\}}\right) \liminf_{n\to\infty}\left(y_{n-m_i}\right)$$
$$\le \limsup_{n\to\infty} y_n + \sum_{j\ne i} \limsup_{n\to\infty}\left(-p_n^{\{j\}} y_{n-m_j}\right)$$
$$\le \limsup_{n\to\infty} y_n + \sum_{j\ne i} \limsup_{n\to\infty}\left(-p_n^{\{j\}}\right) \limsup_{n\to\infty}\left(y_{n-m_j}\right)$$
$$\le \limsup_{n\to\infty}(y_n)\left[1 - \sum_{j\ne i} \liminf_{n\to\infty} p_n^{\{j\}}\right]. \tag{10}$$

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Again, we have

$$\begin{aligned}
\delta &= \limsup_{n\to\infty} z_n = \limsup_{n\to\infty}\left(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)\\
&\geq \liminf_{n\to\infty} y_n + \limsup_{n\to\infty}\left(-\sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)\\
&\geq 0 + \limsup_{n\to\infty}\left(-\sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)\\
&\geq \limsup_{n\to\infty}\left(-p_n^{\{i\}} y_{n-mi}\right) + \liminf_{n\to\infty}\sum_{j\neq i}\left(-p_n^{\{j\}} y_{n-m_j}\right)\\
&\geq \limsup_{n\to\infty} y_{n-m_i}\liminf_{n\to\infty}\left(-p_n^{\{i\}}\right) + \sum_{j\neq i}\liminf_{n\to\infty}\left(\left(-p_n^{\{j\}}\right) y_{n-m_j}\right)\\
&\geq \limsup_{n\to\infty} y_{n-m_i}\liminf_{n\to\infty}\left(-p_n^{\{i\}}\right) + \sum_{j\neq i}\left(\liminf_{n\to\infty}\left(-p_n^{\{j\}}\right)\liminf_{n\to\infty} y_{n-m_j}\right)\\
&\geq \limsup_{n\to\infty} y_n\left(-\limsup_{n\to\infty} p_n^{\{i\}}\right). \qquad (11)
\end{aligned}$$

From (10) and (11), it follows that

$$\limsup_{n\to\infty} y_n\left(\left(\sum_{j\neq i}\liminf_{n\to\infty} p_n^{\{j\}}\right) - 1 - \limsup_{n\to\infty} p_n^{\{i\}}\right) \leq 0.$$

Using (R3), we obtain $\limsup_{n\to\infty} y_n = \alpha = 0$. Then from (10) and (11) we have $\delta \leq 0$ and $\delta \geq 0$ respectively. This implies $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} y_n = 0$.

Let us consider case (iv) i.e; $p_n^{\{j\}}$ satisfy (R4). Then $p_n^{\{j\}} \geq 0$ and $\delta \leq 0$. By Lemma 2.2 and 2.3, we have

$$\begin{aligned}
\delta &= \liminf_{n\to\infty} z_n = \liminf_{n\to\infty}\left(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)\\
&\leq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}\sum_{j=1}^{k} - p_n^{\{j\}} y_{n-m_j}\\
&\leq \alpha + \liminf_{n\to\infty}\left(-p_n^{\{i\}} y_{n-m_i}\right) + \limsup_{n\to\infty}\sum_{j\neq i} - p_n^{\{j\}} y_{n-m_j}\\
&\leq \alpha - \limsup_{n\to\infty}\left(p_n^{\{i\}} y_{n-m_i}\right) + \sum_{j\neq i}\limsup_{n\to\infty}\left(-p_n^{\{j\}} y_{n-m_j}\right)\\
&\leq \alpha - \liminf_{n\to\infty} p_n^{\{i\}}\limsup_{n\to\infty} y_{n-m_i} - \left(\sum_{j\neq i}\liminf_{n\to\infty} p_n^{\{j\}} y_{n-m_j}\right)\\
&\leq \alpha - \alpha\liminf_{n\to\infty} p_n^{\{i\}} - \left(\sum_{j\neq i}\liminf_{n\to\infty} p_n^{\{j\}}\liminf_{n\to\infty} y_{n-m_j}\right)\\
&\leq \alpha\left(1 - \liminf_{n\to\infty} p_n^{\{i\}}\right) \qquad (12)
\end{aligned}$$

Again we have

$$\delta = \limsup_{n\to\infty} z_n = \limsup_{n\to\infty}\left(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)$$

$$\geq \liminf_{n\to\infty} \quad y_n + \limsup_{n\to\infty}\left(-\sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}\right)$$

$$\geq \limsup_{n\to\infty}\left(-p_n^{\{i\}} y_{n-m_i}\right) + \liminf_{n\to\infty} \sum_{j\neq i}\left(-p_n^{\{j\}} y_{n-m_j}\right)$$

$$\geq -\liminf_{n\to\infty}\left(p_n^{\{i\}} y_{n-m_i}\right) - \sum_{j\neq i} \limsup_{n\to\infty}\left(p_n^{\{j\}} y_{n-m_j}\right)$$

$$\geq -\liminf_{n\to\infty} \; y_{n-m_i}\left(\limsup_{n\to\infty} \; p_n^{\{i\}}\right) - \sum_{j\neq i}\left(\limsup_{n\to\infty} \; p_n^{\{j\}} \limsup_{n\to\infty} \; y_{n-m_j}\right)$$

$$\geq -\limsup_{n\to\infty} \; y_n\left(\sum_{j\neq i} \limsup_{n\to\infty} \; p_n^{\{j\}}\right)$$

$$\geq -\alpha\left(\sum_{j\neq i} \limsup_{n\to\infty} \; p_n^{\{j\}}\right). \tag{13}$$

From (12) and (13), it follows that

$$-\alpha\left(\sum_{j\neq i} \limsup_{n\to\infty} p_n^{\{j\}}\right) \leq \delta \leq \alpha\left(1 - \liminf_{n\to\infty} p_n^{\{i\}}\right).$$

By (R4), we obtain $\limsup_{n\to\infty} y_n \leq 0$. This implies $\limsup_{n\to\infty} y_n = 0$. By (13), it follows that $\delta \geq 0$. Using part (a) of this lemma, we obtain $\delta = 0$.Thus,the lemma is proved.

Lemma 2.13 *Assume* $y_n < 0$ *for* $n \geq n_0$ *with* $limsup_{n\to\infty} y_n = 0$. *Suppose that* z_n *is defined as in (8).*

Further, assume that $lim_{n\to\infty} z_n = \delta$ *exists finitely. Then*

(a) If $p_n^{\{j\}} \geq 0$ *then* $\delta \geq 0$ *and* $p_n^{\{j\}} \leq 0$ *then* $\delta \leq 0$.
(b) Further, suppose that y_n *is bounded and* $p_n^{\{j\}}, j = 1,2,\ldots,k$, *satisfy one of the four conditions* (R1), (R2), (R3) or (R4). *Then* $\delta = 0$ *and* $lim_{n\to\infty} y_n = 0$.

Proof:. The proof is similar to that of Lemma 2.12 and is therefore omitted.

Remark 2.14 The above Lemma 2.12 is an extension of Lemma 2.1. One may observe that u_n and v_n are not assumed to be bounded in Lemmas 2.1 or 2.2 or 2.3 . However, it is assumed in Lemma 2.12 that y_n and y_{n-m_j} are bounded. This is only to avoid the conditions that " provided that no sum is of the form $\infty - \infty$" in Lemma 2.2 and that "provided that no product is of the form $0 \times \infty$" in Lemma 2.3. However, if *each* $p_n^{\{j\}}$, satisfies (R2) or (R3) then the terms in z_n are positive when $y_n > 0$. Hence in the limiting case the sum cannot be of the form $\infty - \infty$. Further, if $\liminf_{n\to\infty} |p_n^j| > 0$, for each j, in the case when (R2) holds, then the product term in Lemma 2.3 cannot be of the form $0 \times \infty$. Therefore, we can relax the condition of boundedness on y_n. In Lemma 2.12 and state it as another lemma.

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Lemma 2.15 *Assume* $y_n > 0$ *for* $n \geq n_0$ *with* $\liminf_{n\to\infty} y_n = 0$. *Suppose that* z_n *is defined as in (8) and that* $lim_{n\to\infty} z_n = \delta$ *exists finitely. Let* $p_n^{\{j\}}$ *satisfy any one of the two conditions* (R2) *or* (R3). *Further, suppose that each* $p_n^{\{j\}}$ *satisfy* $\liminf_{n\to\infty} |p_n^j| > 0$, *if* (R2) *holds. Then* $\delta = 0$ *and* $lim_{n\to\infty} y_n = 0$.

Remark 2.16 Suppose z_n is as defined in (8) with $k = 2$. Then Lemmas 2.12, 2.13, and 2.15 hold if each $p_n^{\{j\}}$ satisfy one of the four conditions (R1), (R2), (R3), or (R4) with $k = 2$.

Before the last lemma in this section is stated, it is assumed that $y = y_n$ is non-oscillatory solution of (2) for $n \geq N_1$. Define for $n \geq n_0$,

$$z_n = y_n - p_n^{\{1\}} y_{n-m_1} - p_n^{\{2\}} y_{n-m_2} \tag{14}$$

and

$$w_n = z_n - F_n. \tag{15}$$

Lemma 2.17 *Suppose that each* $p_n^{\{j\}}$ *satisfies the condition,* (R4) *with* $k = 2$, *or* (R9). *Let* (E1), (E3), (E4),(E7), (E9) *and* (E11) *hold. Suppose that* y_n *is a solution of (2) in some interval* $[n_1 \infty)$. *Further assume that* z_n *and* w_n *as defined in (14) and (15) respectively. If* $y_n > 0$ *then either* $\lim_{n\to\infty} w_n = -\infty$ *or* $\lim_{n\to\infty} w_n = \lambda$ *(finite) and* $\lim_{n\to\infty} \Delta w_n = 0$ *with* $\Delta w_n > 0$. *If* $y_n < 0$ *then either* $lim_{n\to\infty} w_n = \infty$ *or* $lim_{n\to\infty} w_n = \lambda$ *(finite) and* $lim_{n\to\infty} \Delta w_n = 0$ *with* $\Delta w_n < 0$.

Proof. Let y_n be an eventually positive solution of (2) for $n \geq n_0 \geq N_1$. Then for $n \geq n_0$, using (14) and (15) in (2), we obtain

$$\Delta^2 w_n = -v_n G(y_{\sigma(n)}) \leq 0. \tag{16}$$

Hence w_n, Δw_n are monotonic for $n \geq n_1$ and of one sign. By (E3) and (E4) we have

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n = \lambda, \text{ where } \lambda \in [-\infty, \infty]. \tag{17}$$

If possible, let λ be equal to ∞. Then $w_n > 0$ and $\Delta w_n > 0$ for $n \geq n_1$. Hence $\lim_{n\to\infty} \Delta w_n = l$, exists. Application of Lemma 2.7 to (16), for $n \geq n_2$ yields

$$\Delta w_n = l + \sum_{i=n}^{\infty} v_i G(y_{\sigma(i)}). \tag{18}$$

This implies

$$\sum_{i=n}^{\infty} v_i G(y_{\sigma(i)}) < \infty, \qquad \text{for } n \geq n_2. \tag{19}$$

From this, it follows, due to (E7), that $\liminf_{n\to\infty} (G(y_{\sigma(n)})/n) = 0$. Hence $\liminf_{n\to\infty} (y_{\sigma(n)}/n) = 0$, by (E1) and (E9). As $\lim_{n\to\infty} \sigma(n) = \infty$ and $\sigma(n) > \gamma n$ for large n, due to (E11), we obtain

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$$\liminf_{n\to\infty}(y_n/n) = 0. \tag{20}$$

As $w_n > 0$ and $\Delta w_n > 0$, we can find $M_0 > 0$ such that $w_n > M_0$ for $n \geq n_3 \geq n_2$. For any $0 < \epsilon$, from (15) it follows due to (E3) and (E4) that $z_n \geq w_n - \epsilon$ for large n. It implies, by Remark 2.8 that $\exists$ M_1, with $0 < M_1 < M_0$, and $y_n - p_n^{\{1\}} y_{n-m_1} - p_n^{\{2\}} y_{n-m_2} > M_1$ for $n \geq n_4 > n_3$. Using (R4) with $i = 1$ or (R9) we have $p_n^{\{1\}} > 1$. Hence one may obtain

$$y_n > y_{n-r} + M_1, n \geq n_4, \tag{21}$$

where $r = m_1$.

Let, $N_0 > n_4$, $M = \min\{y_n : N_0 \leq n \leq N_0 + r\}$ and $0 < \beta < \min \left\{\frac{M}{(N_0+r)}, \frac{M_1}{2r}\right\}$, define, for $n \geq N_0$, $A(n) = M_1 - \beta r$.

Thus $A(n) > 0$ for $n \geq N_0$. Since $y_n \geq M$ for $N_0 \leq n \leq N_0 + r$ and $\beta(N_0 + r) < M$, then $y_n > \beta n$ for $N_0 \leq n \leq N_0 + r$ and $N_0 + r \leq n \leq N_0 + 2r$ implies $N_0 \leq n - r \leq N_0 + r$. Using (21), we obtain, for $N_0 + r \leq n \leq N_0 + 2r$, $y_n > y_{n-r} + M_1 \geq \beta(n - r) + M_1 > \beta n$,

then $\beta n < A(n) + \beta n = M_1 + \beta(n - r)$. Using a simple induction we prove $y_n > \beta n$ for $n \geq N_0$. Hence $\liminf_{n\to\infty}[y_n/n] \geq \beta > 0$, a contradiction to (20). Thus, λ *is not equal to* ∞. Further, if λ *is not equal to* $-\infty$ then $\lambda \in R$. Then easily, we conclude that $\Delta w_n > 0$ and $\lim_{n\to\infty} \Delta w_n = 0$. The proof for the case $y_n < 0$, eventually is similar. Therefore the lemma is proved.

3. Sufficient Conditions

In this section, it is investigated to find, sufficent conditions for all the non oscillatory solutions of (2), tending to zero.

Theorem 3.1 *Let any one of the conditions* (R1) *or* (R2) *hold for k=2. Consider* $p_n^{\{j\}}$ *to satisfy* $\liminf_{n\to\infty}|p_n^j| > 0$ *for (R2) . If* (E1), (E3), (E9) *and* (E11) *hold, then any solution of (2) which does not oscillate, tends to zero as* $n \to \infty$.

Proof: Suppose $y = y_n$ be a solution of (2) for $n \geq N_1$ which is non-oscillatory. Then $y_n > 0$ or $y_n < 0$. Suppose $y_n > 0$ eventually. $\exists$ a +ve integer $n = n_0$ such that $y_n > 0, y_{n-m_1} > 0, y_{n-m_2} > 0$ and $y_{\sigma(n)} > 0$ for $n \geq n_0 \geq N_1$. For $n \geq n_0$, we set z_n and w_n as in (14) and (15) respectively, to obtain (16). Hence $w_n, \Delta w_n$ are monotonic and of one sign for $n \geq n_1 \geq n_0$. Then $\lim_{n\to\infty} w_n = \lambda$, $-\infty \leq \lambda \leq +\infty$. We claim y_n is bounded. Otherwise, y_n is unbounded. Hence $\exists$ a sub-sequence $\{y_{n_k}\}$ such that

$$n_k \to \infty, \qquad y_{n_k} \to \infty \text{ as } \quad k \to \infty, \quad \text{and } y(n_k) = \max\{y_n : n_1 \leq n \leq n_k\}. \tag{22}$$

We may choose n_k large enough so that for $n_k - r \geq n_1, \sigma(n_k) \geq n_1$, where $r = \max\{m_1, m_2\}$. Then by (E3), for $\epsilon > 0$, we can find a +ve integer n_2 such that, for $k \geq n_2 \geq n_1$ implies $|F_{n_k}| < \gamma$, for some constant $\gamma > 0$. Hence for $k \geq n_2$, if (R1) holds, then we have $w_{n_k} \geq$

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$y_{n_k}(1-b)-\gamma.$

Similarly, if (R2) holds, then for $k \geq n_2$, we have $w_{n_k} \geq y_{n_k} - \gamma$.

Taking $k \to \infty$, for either case (R1) or (R2), we find $\lim_{n\to\infty} w_n = \infty$, because of the monotonic nature of w_n. Hence $w_n > 0, \Delta w_n > 0$ for $n \geq n_2 \geq n_1$ and $\Delta^2 w_n \ngeq 0$ and is in –ve. As $m = 2$, $\exists$ a +ve integer $p = 1$. by Lemma 2.4. Then $\lim_{n\to\infty} \Delta w_n = l$ (finite) exists. Application of Lemma 2.7 to (16), results (18). Consequently (19) follows. Because of (E2), the inequality (19) yields $\liminf_{n\to\infty} G\left(y_{\sigma(n)}\right) = 0$ for $n \geq n_3$. Then we claim $\liminf_{n\to\infty} y_{\sigma(n)} = 0$. Otherwise, there exists $n_4 \geq n_3$ and $\gamma > 0$ such that $n \geq n_4$ implies $y_{\sigma(n)} > \gamma$. By (E1) and (E9), we obtain $G(y_{\sigma(n)}) > \gamma\delta > 0$, for $n \geq n_4$, contradiction Therefore, $\liminf_{n\to\infty} y_{\sigma(n)} = 0$ As $\lim_{n\to\infty} \sigma(n) = \infty$, it follows that $\liminf_{n\to\infty} y_n = 0$. Since $w_n > 0$ and $\Delta w_n > 0$, we choose $B > 0$, such that $w_n > B$ for $n \geq n_4 \geq n_3$. Then we claim,

$$\liminf_{n\to\infty} \frac{y_n}{w_n} = 0. \tag{23}$$

Otherwise, there exists $a > 0$ such that eventually $y_n > a w_n > aB$ which implies $\liminf_{n\to\infty} y_n \geq aB > 0$, a contradiction to $\liminf_{n\to\infty} y_{\sigma(n)} = 0$. Set, for $n \geq n_4$,

$$a_n^{\{1\}} = p_n^{\{1\}} \frac{w_{n-m_1}}{w_n} \quad \text{and} \quad a_n^{\{2\}} = p_n^{\{2\}} \frac{w_{n-m_2}}{w_n}.$$

It is clear from (E3) and $\lim_{n\to\infty} w_n = \infty$, that $\lim_{n\to\infty} \frac{F_n}{w_n} = 0$.

Then we have

$$\begin{aligned} 1 &= \lim_{n\to\infty} [\frac{w_n}{w_n}] \\ &= \lim_{n\to\infty} [\frac{y_n - p_n^{\{1\}} y_{n-m_1} - p_n^{\{2\}} y_{n-m_2} - F_n}{w_n}] \\ &= \lim_{n\to\infty} [\frac{y_n}{w_n} - \frac{a_n^{\{1\}} y_{n-m_1}}{w_{n-m_1}} - \frac{a_n^{\{2\}} y_{n-m_2}}{w_{n-m_2}} - \frac{F_n}{w_n}] \\ &= \lim_{n\to\infty} [\frac{y_n}{w_n} - \frac{a_n^{\{1\}} y_{n-m_1}}{w_{n-m_1}} - \frac{a_n^{\{2\}} y_{n-m_2}}{w_{n-m_2}}]. \end{aligned} \tag{24}$$

Since $\{w_n\}$ is a increasing sequence, then $\frac{w_{n-m_j}}{w_n} < 1$ for $j = 1,2$. For $j = 1,2$ if $p_n^{\{j\}}$ is defined as in (R1) then $0 \leq a_n^{\{j\}} < p_n^{\{j\}} \leq b < 1$. However, if $p_n^{\{j\}}$ is defined as in (R2) then $0 \geq a_n^{\{j\}} \geq p_n^{\{j\}} \geq -b_j > -1$. Hence it is clear that for $j = 1,2$, if $p_n^{\{j\}}$ satisfies (R1) or (R2) then $a_n^{\{j\}}$ also satisfies the corresponding conditions (R1) or (R2) accordingly. If (R1) holds then Lemma 2.12(a) yields, due to (23), that $\lim_{n\to\infty} [\frac{y_n}{w_n} - \frac{a_n^{\{1\}} y_{n-m_1}}{w_{n-m_1}} - \frac{a_n^{\{2\}} y_{n-m_2}}{w_{n-m_2}}] \leq 0$, a contradiction to (24). Again if (R2) holds then by Lemma 2.15 $\lim_{n\to\infty} [\frac{y_n}{w_n} - \frac{a_n^{\{1\}} y_{n-m_1}}{w_{n-m_1}} - \frac{a_n^{\{2\}} y_{n-m_2}}{w_{n-m_2}}] = 0$, a contradiction to (24).

Hence $\{y_n\}$ is bounded. Then z_n and w_n are bounded. By (E3), (E4) and monotonic nature of w_n we obtain $\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n = \lambda$ (finite). We claim $\liminf_{n\to\infty} y_n = 0$. Apply Lemma 2.7 to (16), to get

$$w_n = \lambda - \sum_{i=n}^{\infty} (i - n + 1) v_i G(y_{\sigma(i)}), \tag{25}$$

for $n \geq n_1$, where n_1 is some large +ve integer. Therefore,

$$\sum_{i=n}^{\infty} (i - n + 1) v_i G(y_{\sigma(i)}) < \infty, \qquad n \geq n_1. \tag{26}$$

Use Lemma 2.9 and Remark 2.10 in the inequality (26), to get

$$\sum_{i=n}^{\infty} i v_i G(y_{\sigma(i)}) < \infty, \qquad n \geq n_1. \tag{27}$$

The inequality (27), due to (E7) yields $\liminf_{n\to\infty} G(y_{\sigma(n)}) = 0$. Since $\lim_{n\to\infty} \sigma(n) = \infty$, it can be easily shown that $\liminf_{n\to\infty} G(y_n) = 0$. This implies due to (E1) and (E9) that $\liminf_{n\to\infty} y_n = 0$. From Lemma 2.12, it follows that $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} y_n = 0$.

Next, if $y_n<0$, is a solution of (2) for large n , then we put $x_n = -y_n$ to obtain $x_n > 0$ and then (2) reduces to

$$\Delta^2 (x_n - p_n^{\{1\}} x_{n-m_1} - p_n^{\{2\}} x_{n-m_2}) + v_n \tilde{G}(x_{\sigma(n)}) = \tilde{f}_n, \tag{28}$$

where

$$\tilde{f}_n = -f_n, \tilde{G}(v) = -G(-v). \tag{29}$$

Further,

$$\tilde{F}_n = -F_n \qquad \text{implies} \qquad \Delta^2(\tilde{F}_n) = \tilde{f}_n. \tag{30}$$

Taking the above facts into consideration, the following conditions can be verified to hold.

- $x\tilde{G}(x) > 0$ for $x \neq 0$.
- For $u > 0$ there exists $\delta > 0$ such that $\tilde{G}(u) \geq \delta u$. For $u < 0$ there exists $\delta > 0$ such that $\tilde{G}(u) \leq \delta u$.
- $\exists$ a sequence $\{\tilde{F}_n\}$ which is bounded, $\Delta^2(\tilde{F}_n) = \tilde{f}_n$ and $\lim_{n\to\infty} \tilde{F}_n = 0$.

Rest of the proof follows on similar lines as above, hence the proof is complete.

From the above theorem the following corollary follows.

Corollary 3.2 *Solution of (2) which are unbounded, oscillate under the assumptions of Theorem 3.1.*

Remark 3.3 Corollary 3.2 extends (Rath and Behera, 2018) [Theorem 1, Theorem 2] to second order .

Theorem 3.4 *Consider* $p_n^{\{j\}}$ *to satisfy one of the conditions* (R1)–(R4) *for k=2. If* (E1), (E3),(E4) *and* (E7) *hold good , then non oscillatory bounded solutions of (2) tend to zero as* $n \to \infty$.

Proof: Suppose $y = y_n$ be a solution of (2) which is bounded for $n \geq N_1$. If it fails to oscillate then eventually $y_n > 0$ or $y_n < 0$. $\exists$ a +ve integer n_0 such that $y_n > 0, y_{n-m_1} > 0, y_{n-m_2} > 0, y_{\sigma(n)} > 0$ for $n \geq n_0 \geq N_1$. Set z_n and w_n as in (14), and (15) respectively, to obtain (16). Then $w_n, \Delta w_n$ are monotonic and of one sign for $n \geq n_1 \geq n_0$. Since y_n is bounded, z_n and w_n are bounded. Using (E3), (E4) and monotonic bahaviour of w_n, we get $\lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = \lambda$. It exists finitely. Now apply Lemma 2.7 to (16), to get (25) and (26) for $n \geq n_2 > n_1$, where $n_2 > 0$ is some large integer. By using Lemma 2.9 and Remark 2.10 in the equation (26), we get (27). The inequality (27), due to (E7) yields $\liminf_{n\to\infty} G(y_{\sigma(n)}) = 0$. Since $\lim_{n\to\infty} \sigma(n) = \infty$, it can be easily shown that $\liminf_{n\to\infty} G(y_n) = 0$. This implies due to (E1) that $\liminf_{n\to\infty} y_n = 0$. From Lemma 2.12, it follows that $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} y_n = 0$. If y_n is eventually –ve , then as in the proof of the theorem 3.1, we may move with $x_n = -y_n$ (x_n is a positive solution of (28)) to prove $\lim_{n\to\infty} x_n = 0$, hence, the theorem is complete.

Remark 3.5 All type of G, be it linear, sublinear or super linear, are accomodated in theorem 3.4 . It improves, extends, generalize the sufficient part of the theorem due to (Parhi and Tripathy, 2003) [Theorem 2.8].

Theorem 3.6 *Suppose that* (R6) *holds. Assume that* $\sigma(n - m_j) = \sigma(n) - m_j$ *for* $j = 1,2$. *Let* (E1), (E3)–(E6), (E9)–(E11) *hold. Then non oscillatory solutions of (2) tend to zero as* $n \to \infty$.

Proof: Consider an eventually +ve solution $y = \{y_n\}$ of (2) for $n \geq n_0 \geq N_1$. Then set z_n, and w_n as in (14) and (15) respectively to get (16) for $n > n_1 \geq n_0$. Hence $w_n, \Delta w_n$ are monotonic and of one sign for $n \geq n_1$. Then (17) holds by (E3), (E4), which implies

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n = \lambda, \qquad \lambda \in [-\infty, \infty].$$

If λ *is* $-ve$, then $z_n < 0$, for very large n, which is a contradiction. If λ is equal to 0, then $y_n \leq z_n$, implies $\lim_{n\to\infty} y_n = 0$. If $\lambda > 0$, then $w_n > 0$ for $n \geq n_2$. As m=2 , by Lemma 2.4, we have $p = 1$,and this implies $w_n > 0, \Delta w_n > 0$. Hence $\lim_{n\to\infty} \Delta w_n = l$ exists. Note that, $\lambda \in (0, \infty) \Rightarrow p = 0$, a contrdiction. Hence $\lambda = \infty$. Application of Lemma 2.7 to (16), yields (18) and consequently (19) holds. Using Lemma 2.9 and Remark 2.10, we get,

$$\sum_{i=N_2}^{\infty} v_i G\left(y_{\sigma(i)}\right) < \infty, \; N_2 \geq n_2. \tag{31}$$

Putting $i = j - m_1$, one may get

$$\sum_{j=N_2+m_1}^{\infty} v_{j-m_1} G(y_{\sigma(j-m_1)}) < \infty.$$

As $0 \leq -p_n^{\{j\}} \leq b_j, j = 1,2$ then by (E0) and (E1), one has $G(-p_{\sigma(j)}) \leq c$. Since $v_j^* \leq v_{j-m_i}$ for $i = 1,2$ then using Lemma 2.9 and $\sigma(j - m_i) = \sigma(j) - m_i$ for $i = 1,2$, it follows from the above inequality that

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$$\sum_{j=N_3}^{\infty} v_j^* G(-p_{\sigma(j)}^{\{1\}}) G(y_{\sigma(j)-m_1}) < \infty.$$

Then using (E13), one obtains

$$\sum_{j=N_3}^{\infty} v_j^* G(-p_{\sigma(j)}^{\{1\}} y_{\sigma(j)-m_1}) < \infty. \tag{32}$$

Following the line of argument that (32) is obtained from (31), one also finds

$$\sum_{j=N_3}^{\infty} v_j^* G(-p_{\sigma(j)}^{\{2\}} y_{\sigma(j)-m_2}) < \infty. \tag{33}$$

From (31) and the fact that $v_n \geq v_n^*$, it follows that

$$\sum_{j=N_3}^{\infty} v_j^* G(y_{\sigma(j)}) < \infty. \tag{34}$$

Further, use of (E13), (32), (33) and (34), yields

$$\beta \sum_{i=N_3}^{\infty} v_i^* G(z_{\sigma(i)}) < \infty. \tag{35}$$

Since $m = 2$ then by Lemma 2.4 we have $p = 1$, hence there exists $A > 0$ such that $w_n > A$ for $n \geq N_4 \geq N_3$. For any $\epsilon > 0$, using (E3), (E4), we obtain $z_n \geq w_n - \epsilon$, for $n \geq N_5 \geq N_4$. Thus, due to Remark 2.8, we can find $0 < B < A$ such that

$$z_n > B \qquad \text{for } n \geq N_6 \geq N_5. \tag{36}$$

By (E11), we have $\sigma(n)/n > b > 0$ for $n \geq N_7 \geq N_6$. Subsequent use of (36) , (E5) and (E9) implies

$$\sum_{i=N_7}^{\infty} v_i^* G(z_{\sigma(i)}) \quad \geq B\delta \sum_{i=N_8}^{\infty} v_i^* = \infty,$$

which is a contradiction because of (35). Therefore, the proof is complete for the case $y_n > 0$.

If $y_n < 0$, for some large n, then one may go ahead as in the proof of theorem 3.1, by the substitution $x_n = -y_n$ and note that, $x_n > 0$, is a solution of (28) with (29) and (30). One may further observe that , $G = \tilde{G}$ by (E10). Taking note of the above facts and following the proof for the case when y_n is $+ve$, as above, one may prove that $\lim_{n\to\infty} x_n = 0$, which yields $\lim_{n\to\infty} y_n = 0$ and this proves the theorem.

Note that the above result even holds, for (R10) instead of (R6).

Remark 3.7 Theorem 3.6 extends, improves and generalizes the sufficiency part of the Theorem 2.6 of (Parhi and Tripathy, 2003).

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Remark 3.8 The function $G(u) = (\beta + |u|^{\mu})|u|^{\delta}$ {sgn}(u), for $\delta > 0, \mu > 0, \delta + \mu \geq 1, \beta \geq 1$ satisfies (E1) (E6), (E9) and (E10) which could be proved by using the well known inequality (Hilderbrandt, 1963, p.292)

$$u^p + v^p \geq \begin{cases} (u+v)^p, & 0 \leq p < 1, \\ 2^{1-p}(u+v)^p, & p \geq 1. \end{cases}$$

Remark 3.9 The condition $\sum_{n=N_1}^{\infty} v_n^* = \infty$, implies (E2). (37)

Theorem 3.10 *Suppose that* (R6) *holds. Assume that* $\sigma(n - m_j) = \sigma(n) - m_j$ *for* $j = 1,2$. *Let* (E1)– (E4), (E6), (E9)– (E11) *hold and* v_n *is monotonic. Then any solution of (2) tends to zero as* $n \to \infty$ *or oscillates* .

Proof: This proof follows from the proof of theorem 3.6 by the following consideration. We claim if v_n is monotonic then both (E2) and (E5) are equivalent. Obviously, if v_n is non increasing then $v_n^* = v_n$. As a result, the equivalence of (E2) and (E5) is evident. Further, if v_n is non decreasing, then assume that (E2) holds good. Then $v_n^* = v_{n-r}$, where $r = \max\{m_1, m_2\}$. Hence $\sum_{n=N_1}^{\infty} v_n^* = \sum_{n=N_1}^{\infty} v_{n-r} = \sum_{j=N_1-r}^{\infty} v_j = \infty$ by Lemma 2.9. Hence (E5) holds. Thus, (E2) and (E5) are equivalent, when v_n is monotonic and the proof is complete.

Theorem 3.11 *Consider the second order NDDE*

$$\Delta^2(y_n - \sum_{j=1}^{k} p_n^{\{j\}} y_{n-m_j}) + v_n G(y_{\sigma(n)}) = 0. \tag{38}$$

Suppose that $p_n^{\{j\}}$ *satisfies the condition* (R4) *with* $k = 2$. *Let* (E1),(E3), (E4), (E7),(E9) *and* (E11) *hold. Then*

(i) all non oscillatory solutions y_n *of (38), which are bounded, tend to zero, as* $n \to \infty$,
(ii) all non oscillatory solutions y_n *of (38) ,which are unbounded satisfies*
$\lim_{n\to\infty}|(y_{n-m_1} + y_{n-m_2})| = \infty$. *or* $\liminf_{n\to\infty}|y_n| = 0$.

Proof. Let y_n be a eventually positive solution of (38) in some interval $[n_1, \infty)$. Then defining z_n as in (14) we obtain

$$\Delta^2 z_n = -v_n G(y_{\sigma(n)}) \leq 0. \tag{39}$$

From this, it follows that $z_n, \Delta z_n$ are monotonic and of constant sign on some interval $[n_1 \ ,\infty)$. Let us prove (A) and assume y_n to be bounded. Then applying Lemma 2.17 with $f_n \equiv 0$, we have $\lim_{n\to\infty} z_n = \lambda$. Since y_n is bounded, $\lambda = -\infty$ is not possible. Hence λ is finite. Then apply Lemma 2.7 to (39), to get

$$z_n = \lambda - \sum_{i=n}^{\infty} (i - n + 1) v_i G(y_{\sigma(i)}). \tag{40}$$

Consequently (26) and (27) hold. The inequality (27), because of (E7) implies $\liminf_{n\to\infty} G(y_{\sigma(n)}) = 0$. $\lim_{n\to\infty}\sigma(n) = \infty$, it can be easily shown that $\liminf_{n\to\infty} G(y_n) = 0$. This implies due to (E1) and continuity of G that $\liminf_{n\to\infty} y_n = 0$. Then applying Lemma 2.12,

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we obtain $\lim_{n\to\infty} y_n = 0$. Next let us proceed to prove (B) and consider y_n to be positive solution of (38) which is unbounded in some interval $(n_1 \ \infty)$. Then by Lemma 2.17 it follows either $\lim_{n\to\infty} z_n = \lambda$ (finite) or $\lim_{n\to\infty} z_n = -\infty$. If the latter holds then Since $p_n^{\{j\}}$ for $j = 1,2$ are bounded, there exists a positive scalar b such that $0 < p_n^{\{j\}} < b$. From (14) it follows that

$$z_n = y_n - p_n^{\{1\}} y_{n-m_1} - p_n^{\{2\}} y_{n-m_2} \geq -b y_{n-m_1} - b y_{n-m_2},$$

This implies $y_{n-m_1} + y_{n-m_2} \geq \frac{z_n}{-b} \to +\infty$ as $n \to \infty$. So, $\lim_{n\to\infty}(y_{n-m_1} + y_{n-m_2}) = +\infty$.

If the former holds then proceeding as in part (a) of the proof one may obtain $\liminf_{n\to\infty} y_n = 0$. For the case when $y_n < 0$ for $n \geq n_0$, the proof is similar. Thus, the theorem is proved.

Theorem 3.12 *Let* (R4) *with* $k = 2, \ hold$. *Suppose* (E1), (E3) ,(E4), (E8), (E9) *and* (E11) *hold good.*

Then

(i) non oscillatory bounded solutions y_n of (2), tend to zero as $n \to \infty$,
(ii) non oscillatory unbounded solutions y_n of (2), satisfy $\lim_{n\to\infty}(y_{n-m_1} + y_{n-m_2}) = +\infty$.

Proof*:* Clearly, (E8) implies (E7). Then proof of (i) follows from, proof of Theorem 3.4 for case (R4). Now to prove (ii), assume $y = \{y_n\}$ be a +ve solution of (2) which is unbounded. By virtue of Lemma 2.17, one is to get $\lim_{n\to\infty} w_n = \lambda$ (finite) or $\lim_{n\to\infty} w_n = -\infty$. In this situation we claim $\lim_{n\to\infty} w_n = \lambda$, cannot hold. Else, apply Lemma 2.7 to (16) to obtain (25) and (26) and then use Lemma 2.9 and remark 2.10 to show that (27) holds. As $y_{\sigma(n)}$ is unbounded, we find a sub-sequence $\{\sigma(n_j)\}$ of $\{\sigma(n)\}$ such that $y_{\sigma(n_j)} > \zeta > 0$, for $j > n_1$. Hence using (E8) and (E9), we have

$$\sum_{j=n_1}^{\infty} (n_j) v_{n_j} G(y_{\sigma(n_j)}) > \zeta\delta \sum_{j=n_1}^{\infty} (n_j) v_{n_j} = \infty,$$

a contradiction to (27). Thus $\lim_{n\to\infty} w_n = -\infty$. We observe that (17) holds because of (E3), (E4). Hence $\lim_{n\to\infty} z_n = -\infty$. Since $p_n^{\{j\}}$ for $j = 1,2$ are bounded then there exists a positive scalar b such that $0 < p_n^{\{j\}} < b$. From (14) it follows that

$$z_n = y_n - p_n^{\{1\}} y_{n-m_1} - p_n^{\{2\}} y_{n-m_2} \geq -b y_{n-m_1} - b y_{n-m_2}.$$

This implies $y_{n-m_1} + y_{n-m_2} \geq \frac{z_n}{-b} \to +\infty$ as $n \to \infty$. So, $\lim_{n\to\infty}(y_{n-m_1} + y_{n-m_2}) = +\infty$.

For the case, $when$ y_n is $-$ve for large n, the proof is similar and this is the end of the proof.

Before, this article gets closed , some examples are given to illustrate the outcomes.

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Example 3.13 Consider the NDDE

$$\Delta^2(y_n + \frac{1}{4}y_{n-1} + \frac{1}{8}y_{n-2}) + n^{-2}y_{n-3}^{\alpha} = \frac{1}{2^{n+1}} + \frac{2^{3\alpha}}{2^{\alpha n}n^2} \quad (41)$$

where $n \geq 3$ $G(x) = x^{\alpha}, \alpha$ is +ve and is the quotient of two odd integers. Here, $p_n^{\{j\}}$ satisfies (R2) and $v_n = n^{-2}, f_n = \frac{1}{2^{n+1}} + \frac{2^{3\alpha}}{2^{\alpha n}n^2}$. Easily, we can verify that, $\sum_{n=n_0}^{\infty} \mathrm{n} f_n < \infty$ and all the conditions of Theorem 3.1 are satisfied by the equation (41). Hence $y_n = 2^{-n}$ is a solution of (41), tending to zero as $n \to \infty$. Here G could be linear , super linear, or sublinear,

Example 3.14 Consider the NDDE

$$\Delta^2\left(y_n - \frac{1}{4}y_{n-1} - \frac{1}{8}y_{n-2}\right) + n^{-1}y_{n-3}^{\alpha} = \frac{2^{3\alpha}}{2^{\alpha n}\mathrm{n}} \quad (42)$$

where $n \geq 3, G(x) = x^{\alpha}, \alpha > 1$ is the quotient of two integers, which are odd. Here, $p_n^{\{j\}}$ satisfies (R1) and $v_n = n^{-1}, f_n = \frac{2^{3\alpha}}{2^{\alpha n}\mathrm{n}}$. Easily, we can verify that, $\sum_{n=n_0}^{\infty} \mathrm{n} f_n < \infty$ and all the conditions of Theorem 3.4 are satisfied by the equation (42). Therefore the solution $y_n = 2^{-n}$ of (42), tends to zero as $n \to \infty$.

4. Conclusion

This paper, investigates to establish that the condition (E2) or (E7) is sufficient, for every solution of (2) to be oscillatory or tending to zero. Theorem 3.11 is obtained under (E7), which is less restrictive than (E2). The condition "G is non decreasing," which is very often used for non linear neutral equations, is relaxed in this work. As a result, the theorems 3.11 and 3.12 extend and generalize the sufficiency part of the theorems 2.8 and 2.7 of (Parhi and Tripathy, 2003) respectively. Further, the results extend (Rath and Behera, 2018) to 2nd order NDDE. At the end, the following open problems are proposed to the reader, which might be helpful for further research.

Problem 4.1 It would be interesting to prove theorem 3.12, with (R5) instead of (R4) and under the hypothesis, (E8) or, a condition weaker than (E8).

Problem 4.2 If v_n changes sign, under the consideration of $G(x) = x$ or $G(x) \neq x$, then one should investigate to find the sufficient conditions for the qualitative behaviour of (1) or that of an equation of order $m > 2$.

Conflict of Interest

The authors declare that this publication is not subject to conflict of interest.

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