International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

Controlling Pest by Integrated Pest Management: A Dynamical Approach

Vandana Kumari
Department of Mathematics,
Amity Institute of Applied Science,
Amity University, Sector-125, Noida, U.P., India.
E-mail: vandana_rakesh11@yahoo.com

Sudipa Chauhan
Department of Mathematics,
Amity Institute of Applied Science,
Amity University,Sector-125, Noida, U.P., India.
Corresponding author: sudipachauhan@gmail.com

Joydip Dhar
Mathematical Modelling and Simulation Laboratory,
Atal Bihari Vajpayee Indian Institute of Information Technology and Management,
Gwalior, M.P., India.
E-mail: jdhar@iiitm.ac.in

(Received August 28, 2019; Accepted January 28, 2020)

Abstract
Integrated Pest Management technique is used to formulate a mathematical model by using biological and chemical control impulsively. The uniform boundedness and the existence of pest extinction and nontrivial equilibrium points is discussed. Further, local stability of pest extinction equilibrium point is studied and it has been derived that if $T \leq T_{max}$, the pest extinction equilibrium point is locally stable and for $T > T_{max}$, the system is permanent. It has also been obtained that how delay helps in eradicating pest population more quickly. Finally, analytic results have been validated numerically.

Keywords- Plant-pest-natural enemy, Boundedness, Local stability, Permanence.

1. Introduction

Plants as we all know conflict between and pests has been a root cause of concern in our ecology from almost two decades. Rescuing crops from predator pests such as insects has become a tedious task for farmers. With the advent in science and technology, effective measures have been discovered to deal with predator pest effectively like introducing natural enemies and chemical pesticides in relevent environment. It is a well known fact that excessive use of chemical pesticide such as organochlorine (DDT and toxaphene) is hazardous both for animals and human being as studied by authors (James, 1997). Therefore, Integrated pest management came into scenario in which selective pesticides control pests as natural predators when regulation through biological means fails. Many biological food web models to control pests have been discussed by many scholars (Changguo et al., 2009; Liu et al., 2013; Jatav et al., 2014; Song et al., 2014) where they took assumptions of either impulsive release of natural enemies or chemical pesticides. Authors (Jatav and Dhar, 2014) studied a model in which they formulated a mathematical model and obtained a threshold value below which pests gets eradicated. Later, many more IPM approach inclined models were proposed where impulsive control strategies for pest eradication were introduced and to name a few are (Tang et al., 2005; Akman et al., 2015; El-Shafie, 2018; Paez Chavez et al., 2018). They studied various prospect of IPM method and its application. Scholars (Zhang et al., 2004) did comparison between IPM method and classical method for pest control and obtained that IPM strategy is better than any classical method to control pests. Recently, Yu et al. (2019) introduced IPM method for predator–prey model with Allee effect and stochastic effect respectively where they obtained thresholds based on biological and chemical control. However, in all the papers discussed above no-one discussed significanlty about delays, in particularly gestation delay which in a real situation always exist.

Hence, keeping in mind the above alma matter, we have formulated our model in reference to the previous models and studied the dynamics of the new system with delay. The highlight of the paper is that how delay parameter helps in reducing the pest population more quickly in comparison to the system without delay. The results would be extremely beneficial for those crops where pest population are growing exponentially due to favourable habitable condition. A relevent biological example to our model is as follows:

Australian herb is always at the verge of being attacked by green Lacewing Larvae, which is a well known pest. Encapsulating biological controls like mealy bugs followed by chemical control such as chlorothalonil has shown remarkable results which advocates our approach of hybrid technique. The organisation of the paper is as follows: In Section 2, 3 model formulation and preliminary lemmas are discussed. In Section 4, local stability of pest extinction is achieved followed by permanence in Section 5. Finally, in the last two sections numerical simulation is done for validation of analtical results with conclusion.

2. Mathematical Model

We have proposed our mathematical model by the following set of differential equations:

$$\left.\begin{aligned}
\frac{dp}{dt} &= p(r-p) - a_1pq\\
\frac{dq}{dt} &= a_1b_1pq - a_2q(t-\tau)r_2(t-\tau)e^{-d_1\tau} - Dq\\
\frac{dr_1}{dt} &= a_2b_2q(t-\tau)r_2(t-\tau)e^{-d_1\tau} - (D_3+\mu_0)r_1\\
\frac{dr_2}{dt} &= \mu_0r_1 - D_3r_2
\end{aligned}\right\} t \neq nT \qquad (1)$$

$$\left.\begin{aligned}
p(t^+) &= p\\
q(t^+) &= (1-\delta)q\\
r_1(t^+) &= r_1 + \mu_1\\
r_2(t^+) &= r_2 + \mu_2
\end{aligned}\right\} = nT \qquad (2)$$

The model completes with the following initial conditions:

$p(\theta) = \phi_1(\theta),\ q(\theta) = \phi_2(\theta),\ r_1 = \psi_1(\theta),\ r_2 = \psi_2(\theta),\ \phi_i(0) > 0,\ \psi_i(0) > 0,\ \theta \in [-\tau, 0]$, $(i = 1,2)$, where $(\phi_1, \phi_2, \psi_1, \psi_2) \in C([-\tau, 0], \mathbb{R}_+^4$, the Banach space of continuous functions mapping on the interval $[-\tau, 0]$into $\mathbb{R}_+^4$. The graphical representation of the model is

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

as follows in Figure 1. Negative and positive sign represents outgoing and incoming rates.

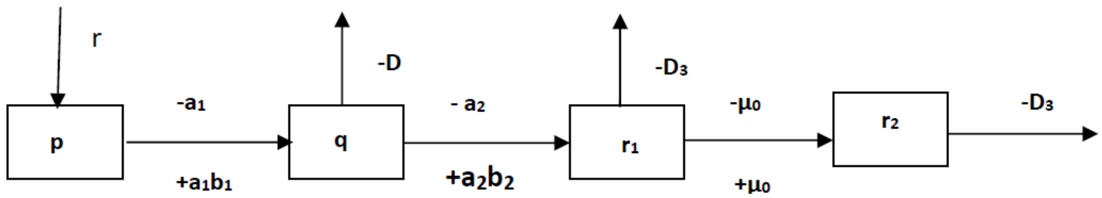

Figure 1. Graphical representation of model

The parameters/variables used in the model are explained in detail in Table 1 mentioned below and for convenience t is removed from the variables throughout the paper.

Table. 1 Meaning of parameters /variables

Parameters/Variables	**Meaning**
$r_1(t)$	Immature natural enemy
r	growth rate of plant population
r_2(t)	Mature natural enemy
τ	Time delay
$p(t)$	Plant population
a_1	Rate at which plant population is decreasing to pest population
b_1	Growth rate of pest population
D	Mortality Rate
a_2	Rate at which pest population is decreasing
b_2	Rate at immature natural enemy population
μ_0	Mortality rate of immature natural enemy
D_3	Mortality rate of mature natural enemy
T	Period of impulse
μ_1	Amount of pulse release of immature natural enemy
μ_2	Amount of pulse release of mature natural enemy
$0 \le \delta < 1$	harvesting rate of pest through chemical pesticide
$q(t)$	Pest population

3. Preliminary Lemmas

In this section, we have given a few Lemmas, which will be useful for our main result.

Lemma 3.1 Let us consider the system

$$w'(t) = b - cw(t), t \neq nT, \tag{3}$$

$$w(t^+) = w(t) + \mu, t = nT, n = 1,2,3 \dots. \tag{4}$$

Then the system has a positive periodic solution $\widetilde{w}(t)$and for any solution $w(t)$ of the system (3),we have,

$$|w(t) - \widetilde{w}(t)| \to 0,$$

for $t \to \infty$, where, for

$$t \in (nt, (n+1)T], \widetilde{w}(t) = \frac{b}{c} + \frac{\mu exp(-c(t-nT))}{1-exp(-cT)} \quad \text{with } \widetilde{w}(0^+) = \frac{b}{c} + \frac{\mu}{1-exp(-cT)}.$$

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

The boundedness is given lemma 3.2.

Lemma 3.2 There exists a constant $M > 0$ *s.t* $p(t) \leq M, q(t) \leq M, r_1(t) \leq M, r_2(t) \leq M,$ *for* $(1-2)$ with *t* being sufficiently large where

$$M = \frac{M_0}{\widetilde{D}} + \frac{(\mu_1 + \mu_2)exp(\overline{D}t)}{exp(\overline{D}t) - 1} > 0.$$

Now, we will discuss the pest extinction case and our impulsive system $(1-2)$ reduces to:

$$\left.\begin{aligned} \frac{dr_1(t)}{dt} &= -(D_3 + \mu_0)r_1(t) \\ \frac{dr_2(t)}{dt} &= \mu_0 r_1(t) - D_3 r_2(t) \end{aligned}\right\} t \neq nT, \tag{5}$$

$$\left.\begin{aligned} r_1(t^+) &= r_1 + \mu_1 \\ r_2(t^+) &= r_2 + \mu_2 \end{aligned}\right\} t = nT, \tag{6}$$

For the system $(5-6)$, we integrate it over the interval $(nT, (n+1)T]$, and by means of stroboscopic mapping we get, $r_1((n+1)T^+) = exp(-(D_3 + \mu_0)T)\ r_1(nT^+) + \mu_1$

Thus the corresponding periodic solution of $(5-6)$ in $t \in (nT, (n+1)T]$ is,

$$\tilde{r}_1(t) = \frac{\mu_1 exp(-(D_3 + \mu_0)(t - nT))}{1 - exp(-(D_3 + \mu_0)T)}$$

with

$$\tilde{r}_1(0^+) = \frac{\mu_1}{1 - exp(-(D_3 + \mu_0)T)}$$

and is stable globally. Substituting $\tilde{r}_1(t)$ into $(5-6)$, we obtain the following subsystem:

$$\left.\begin{aligned} \frac{dr_2(t)}{dt} &= \mu_0 \tilde{r}_1(t) - D_3 r_2(t), t \neq nT \\ r_2(t^+) &= r_2 + \mu_2, t = nT \end{aligned}\right\} \tag{7}$$

Further, integrating (7) in the interval $(nT, (n+1)T]$, we get,

$$\tilde{r}_2(t) = \frac{-\mu_1 exp(-(D_3 + \mu_0)(t - nT))}{1 - exp(-(D_3 + \mu_0)T)} + \frac{(\mu_1 + \mu_2)exp(-D_3(t - nT))}{1 - exp(-D_3 T)},$$

with initial value

$$\tilde{r}_2(0^+) = \frac{-\mu_1}{1 - exp(-(D_3 + \mu_0)T)} + \frac{(\mu_1 + \mu_2)}{1 - exp(-D_3 T)},$$

which is stable globally.

Moreover, due to the absence of pest, the subsystem of $(1-2)$ can also be considered as follows:

$$\frac{dp(t)}{dt} = p(r-p) \tag{8}$$

With $p=0$ as unstable equilibrium and $p=r$ as globally stable. Therefore, the two periodic solutions of $(1-2)$ are $(0,0,\tilde{r}_1,\tilde{r}_2)$ and $(r,0,\tilde{r}_1,\tilde{r}_2)$.

4. Local Stability of Pest Extinction Case

This section will discuss the local stability analysis of the equilibrium point with pest population.

Theorem 4.1 *Let* (p,q,r_1,r_2) *be a solution of* $(1-2)$, *Then*

(i) $(0,0,\tilde{r}_1,\tilde{r}_2)$ is unstable.

(ii) $(r,0,\tilde{r}_1,\tilde{r}_2)$ is locally asymptotically stable iff $T \le T_{max}$, where

$$T_{max} = \frac{1}{(a_1 b_1 - d)}\{log\frac{1}{(1-\delta)} + e^{-d_1\tau}a_2(\frac{D_3\mu_2 + \mu_0(\mu_1+\mu_2)}{D_3(D_3+\mu_0)})\}, a_1 b_1 > d \tag{9}$$

Proof: (i) Here, we define,

$$p = \phi_1 \ , q = \phi_2 \ , r_1 = \tilde{r}_1 + \phi_3, r_2 = \tilde{r}_2 + \phi_4$$

where, $\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)$ are perturbation in $p, q,\ r_1,\ r_2$ then the system's linearized form becomes:

$$\left.\begin{aligned} \frac{d\phi_1(t)}{dt} &= -r\phi_1(t) \\ \frac{d\phi_2(t)}{dt} &= -(D + a_2\tilde{r}_2(t)e^{-d_1\tau})\phi_2(t) \\ \frac{d\phi_3(t)}{dt} &= a_2 b_2 \phi_2(t)\tilde{r}_2(t)e^{-d_1\tau} - (D_3+\mu_0)\phi_3(t) \\ \frac{d\phi_4(t)}{dt} &= \mu_0\phi_3(t) - D_3\phi_4(t) \end{aligned}\right\} t \neq nT \tag{10}$$

$$\left.\begin{aligned} \phi_1(t^+) &= \phi_1(t) \\ \phi_2(t^+) &= (1-\delta)\phi_2(t) \\ \phi_3(t^+) &= \phi_3(t) + \mu_1 \\ \phi_4(t^+) &= \phi_4(t) + \mu_2 \end{aligned}\right\} = nT \tag{11}$$

Let $\phi(t)$ be the fundamental matrix of $(10-11)$, then $\phi(t)$ must satisfy,

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

$$\frac{d\phi(t)}{dt} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -\left(D + a_2\tilde{r}_2(t)e^{-d_1\tau}\right) & 0 & 0 \\ 0 & a_2b_2\tilde{r}_2(t)e^{-d_1\tau} & -(D_3+\mu_0) & 0 \\ 0 & 0 & \mu_0 & -D_3 \end{bmatrix}\phi(t) = A\phi(t) \tag{12}$$

Thus, the monodromy matrix of $(10-11)$ is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\phi(t)$$

From (12), we get $\phi(t) = \phi(0)exp\left(\int_0^T Adt\right)$, where $\phi(0)$ is an identity matrix and hence the eigen values corresponding to matrix M are as follows:

$$\lambda_3 = exp\ \left(-(D_3+\mu_0)\right)T < 1, \qquad \lambda_4 = exp(-D_3T) < 1, \qquad \lambda_1 = exp(rT) > 1,$$
$$\lambda_2 = (1-\delta)exp\int_0^T \left(-\left(D + a_2\tilde{r}_2(t)e^{-d_1\tau}\right)\right)dt < 1.$$

Therefore, according to the Floquet theory (Bainov and Sineonov, 1993) the pest eradication periodic solution is unstable as $|\lambda_1| > 1$.

Remark 1: The effect of delay can be easily seen in the value of T_{max} which helps in reducing its value.

(ii) The local stability of $(r, 0, \tilde{r}_1(t), \tilde{r}_2(t))$ is proved in the similar fashion. We define $p = r + \phi_1(t), q = \phi_2(t), r_1 = \tilde{r}_1(t) + \phi_3(t), r_2 = \tilde{r}_2(t) + \phi_4(t)$ and the system $(1-2)'s$ linearized form is as follows:

$$\left.\begin{aligned} \frac{d\phi_1(t)}{dt} &= -r\phi_1(t) - a_1\phi_2 \\ \frac{d\phi_2(t)}{dt} &= (a_1b_1 - D - a_2\tilde{r}_2(t)e^{-d_1\tau})\phi_2(t) \\ \frac{d\phi_3(t)}{dt} &= a_2b_2\phi_2(t)\tilde{r}_2(t)e^{-d_1\tau} - (D_3+\mu_0)\phi_3(t) \\ \frac{d\phi_4(t)}{dt} &= \mu_0\phi_3(t) - D_3\phi_4(t) \end{aligned}\right\} t \neq nT \tag{13}$$

$$\left.\begin{aligned} \phi_1(t^+) &= \phi_1(t), \\ \phi_2(t^+) &= (1-\delta)\phi_2(t) \\ \phi_3(t^+) &= \phi_3(t) + \mu_1 \\ \phi_4(t^+) &= \phi_4(t) + \mu_2 \end{aligned}\right\} t = nT \tag{14}$$

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

Let $\phi(t)$ be the fundamental matrix of $(13-14)$, then $\phi(t)$ must satisfy

$$\frac{d\phi(t)}{dt}=\begin{bmatrix}-r & -a_1 & 0 & 0\\ 0 & a_1b_1-D-a_2\tilde{r}_2(t)e^{-d_1\tau} & 0 & 0\\ 0 & a_2b_2\tilde{r}_2(t)e^{-d_1\tau} & -(D_3+\mu_0) & 0\\ 0 & 0 & \mu_0 & -D_3\end{bmatrix}\phi(t)$$

$$\frac{d\phi(t)}{dt}=A\phi(t) \tag{15}$$

Thus, the monodromy matrix of $(13-14)$ is

$$M=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1-\delta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\phi(t).$$

From (15), we get $\phi(t)=\phi(0)exp(\int_0^T Adt)$, where $\phi(0)$ is an identity matrix. Then the characteristic values obtained for M are as follows:

$$\lambda_1=exp(-rT)<1,\qquad \lambda_2=(1-\delta)exp\int_0^T\left(a_1b_1-D-a_2\tilde{r}_2(t)e^{-d_1\tau}\right)<1,$$

$$\lambda_3\quad =exp((-(D_3+\mu_0)-\lambda)T)<1,\qquad \lambda_4=exp(-D_3T)<1.$$

Therefore, pest eradication periodic solution of $(1-2)$ is locally asymptotically stable as per Floquet theory (Bainov and Sineonov, 1993) if and only if $|\lambda_2|\leq 1$ which implies $T\leq T_{max}$. Hence, the theorem is proved.

5. Permanence

In this section, we will discuss permanence of system $(1-2)$.

Theorem 5.1 The system (1-2) is permanent if $T>T_{max}$.
Proof. Suppose $(p,q,r_1,r_{2)}$ is the solution of the system $(1-2)$, t being removed for convenience, We have already proved that $p(t)\leq M$, $q(t)\leq M$, $r_1(t)\leq M$ and $r_2(t)\leq M$ $\forall$ t. From, $(1-2)$ we have $\frac{dp}{dt}\geq p(r-a_1M-p)$ which implies that $p(t)>r-a_1M\triangleq m_1$ for all large t. For small $\epsilon_4>0$, we choose $m_1=1-\epsilon>0$ and also define,

$$m_2=\frac{-\mu_1exp(-(D_3+\mu_0)(t-nT))}{1-exp(-(D_3+\mu_0)T)}-\epsilon_4>0,$$

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

$$m_3 = \frac{-\mu_1 exp(-(D_3+\mu_0)(t-nT))}{1-exp(-(D_3+\mu_0)T)} + \frac{(\mu_1+\mu_2)exp(-D_3(t-nT))}{1-exp(-D_3T)} - \frac{\epsilon_4\mu_0}{D_3} - \epsilon_4 > 0.$$

Now, the system $(1-2)$ can be rewritten as:

$$\left.\begin{aligned} \frac{dr_1(t)}{dt} &= -(D_3+\mu_0)r_1(t) \\ \frac{dr_2(t)}{dt} &= \mu_0 r_1(t) - D_3 r_2(t) \end{aligned}\right\} t \neq nT, \tag{16}$$

$$\left.\begin{aligned} r_1(t^+) &= r_1 + \mu_1 \\ r_2(t^+) &= r_2 + \mu_2 \end{aligned}\right\} t = nT. \tag{17}$$

The system $(16-17)$ is same as $(5-6)$, using same technique, we can easily find that $r_1(t) > m_2$ and $r_2(t) > m_3$ $\forall$ t. Hence, for proving the permanence we have only have to prove $m_4 > 0$, such that $q(t) \geq m_4 \forall$ t which will be done in two steps.

Step 1: Let $q(t) \geq m_4$ is false $\exists$ a $t_1 \in (0,\infty)$ s.t $q(t) < m_4$ $\forall$ $t > t_1$. Using this supposition, we get subsystem of $(1-2)$:

$$\frac{dr_1(t)}{dt} \leq a_2 b_2 M m_4 e^{-d_1\tau} - (D_3+\mu_0)r_1, t \neq nT$$
$$r_1(t^+) = r_1(t) + \mu_1, t = nT, n = 1,2,3 \ldots \ldots$$

Let us assume the comparison system:

$$\left.\begin{aligned} \frac{d\overline{w}_1(t)}{dt} &\leq a_2 b_2 M m_4 e^{-d_1\tau} - (D_3+\mu_0)\overline{w}_1(t), t \neq nT \\ \overline{w}_1(t^+) &= \overline{w}_1(t) + \mu_1, t = nT, n = 1,2,3.\ldots \end{aligned}\right\} \tag{18}$$

Using lemma 3.1, equation (18) has periodic solution

$$(t)\tilde{\overline{w}}_1 = \frac{a_2 b_2 m_4 M exp(-d_1\tau)}{D_3+\mu_0} + \frac{\mu_1 exp(-(D_3+\mu_0)(t-nT))}{1-exp(-(D_3+\mu_0)T)}$$

which is globally asymptotically stable. Then, $\exists$ an $\epsilon_5 > 0$ s.t

$$r_1(t) \leq \tilde{\overline{w}}_1(t) < \frac{a_2 b_2 m_4 M exp(-d_1\tau)}{D_3+\mu_0} + \frac{\mu_1 \exp\left(-(D_3+\mu_0)(t-nT)\right)}{1-\exp(-(D_3+\mu_0)T)} + \epsilon_5 > 0.$$

For sufficiently large t. Thus we find the following subsystem of $(1-2)$:

$$\left.\begin{array}{l}\dfrac{dr_2(t)}{dt}= \quad \mu_0\left(\dfrac{a_2b_2m_4Mexp(-d_1\tau)}{D_3+\mu_0}+\dfrac{\mu_1\exp\left(-(D_3+\mu_0)(t-nT)\right)}{1-\exp(-(D_3+\mu_0)T)}+\epsilon_5\right)-D_3r_2, t\neq nT\\ r_2(t^+)=r_2+\mu_2, \qquad t=nT, \qquad n=1,2,3\ldots\ldots\end{array}\right\} \tag{19}$$

Consider the comparison system (19) as follows:

$$\left.\begin{array}{l}\dfrac{d\overline{w}\quad\;\; _2(t)}{dt}=\mu_0(\dfrac{a_2b_2m_4Mexp(-d_1\tau)}{D_3+\mu_0}+\dfrac{\mu_1exp(-(D_3+\mu_0)(t-nT))}{1-exp(-(D_3+\mu_0)T)}+\epsilon_5)(t)-D_3\overline{w}_2(t), t\neq nT\\ \overline{w}_2(t^+)=\overline{w}_2(t)+\mu_2, t=nT, n=1,2,3\ldots\ldots\end{array}\right\} \tag{20}$$

Similarly, system (20) also has a periodic solution

$$r_2(t)<\tilde{\overline{w}}_2(t)<\frac{-\mu_1exp(-(D_3+\mu_0)(t-nT))}{1-exp(-(D_3+\mu_0)T)}+\frac{(\mu_1+\mu_2)exp(-D_3(t-nT))}{1-exp(-D_3T)}$$
$$+\frac{\mu_0}{D_3}(\frac{a_2b_2m_4Mexp(-d_1\tau)}{(D_3+\mu_0)}+\epsilon_5)$$

$$\tilde{\overline{w}}_2(t)<\frac{-\mu_1\exp\left(-(D_3+\mu_0)(t-nT)\right)}{1-\exp(-(D_3+\mu_0)T)}+\frac{(\mu_1+\mu_2)\exp\left(-D_3(t-nT)\right)}{1-\exp(-D_3T)}$$
$$+\frac{\mu_0}{D_3}\left(\frac{a_2b_2m_4Mexp(-d_1\tau)}{(D_3+\mu_0)}\epsilon_5\right) \tag{21}$$

which is globally asymptotically stable and $\exists$ an $\epsilon_6>0$ s.t

$$r_2(t)<\tilde{\overline{w}}_2(t)<\frac{-\mu_1exp(-(D_3+\mu_0)(t-nT))}{1-exp(-(D_3+\mu_0)T)}+\frac{(\mu_1+\mu_2)exp(-D_3(t-nT))}{1-exp(-D_3T)}$$
$$+\frac{\mu_0}{D_3}\left(\frac{a_2b_2m_4Mexp(-d_1\tau)}{(D_3+\mu_0)}+\epsilon_5\right)+\epsilon_6.$$

It shows that $\exists$ a $T_1>0$ s.t for $nT<t\leq(n+1)T,$ we are having the following subsystem of $(1-2)$:

$$\left.\begin{array}{l}\dfrac{dq(t)}{dt}\geq[a_1b_1m_1-a_2(\tilde{\overline{w}}_2(t)+\epsilon_6)e^{-d_1\tau}-D]q, t\neq nT\\ q(t^+)=(1-\delta)q(t), t=nT, and, t>T_1\end{array}\right\} \tag{22}$$

Integrating the system, (22) on $(nT,(n+1)T], n\geq N_1$(here,$N_1$is the nonnegative integer and $N_1T\geq T_1$), then we obtain that,

$$q((n+1)T) \geq (1-\delta)q(nT^+)exp(\int_{nT}^{(n+1)T} (a_1 b_1 m_1 - a_2(\tilde{\bar{r}}_2(t) - \epsilon_6)e^{-d_1\tau} - D)dt)$$
$$= q(nT^+)\bar{\sigma}$$

where, $\bar{\sigma}(1-\delta)q(nT^+)exp(\int_{nT}^{(n+1)T} (a_1 b_1 m_1 - a_2(\tilde{\bar{r}}_2(t) - \epsilon_6)e^{-d_1\tau} - D)dt) > 1$, as, $T > T_{max}$, therefore, for $\epsilon_5 > 0$, we obtain that,

$$(a_1 b_1 m_1 - a_2 \epsilon_6 exp(-d_1\tau) - D)T - \frac{a_2\mu_0 exp(-d_1\tau)}{D_3}(\frac{b_2 m_4 M exp(-d_1\tau)}{(D_3+\mu_0)} - \epsilon_5) -$$
$$a_2(\frac{\mu_1}{(D_3+\mu_0)}\frac{(\mu_1+\mu_2)\exp(-d_1\tau)}{D_3}) - log(\frac{1}{1-\delta}) > 1.$$

Thus, $q((N_1+k)T) \geq q(N_1T^+)\bar{\sigma}^k \to \infty$ as $k \to \infty$, which violoates our assumption $q(t) < m_4$, for every $t > t_2$. Hence there exists a $t_2 > t_1$ s.t $q(t_2) \geq m_4$.

Step 2: If $q(t) \geq m_4 \;\; \forall \;\; t \geq t_2$, then our aim will be fulfilled. On the contrary let us assume that $q(t) < m_4$ for some $t > t_2$. Let $t^* = \inf\{t | q(t) < m_4, t > t_2\}$, then there will be two cases:

Case 1: Let $t^* = n_1 T$, $n_1 \in Z^+$. In this case $q(t) \geq m_4$ for $t \in [t_2, t^*)$ and $(1-\delta)m_4 \leq q(t^{*+} = (1-\delta)q(t^*) < m_4)$. Let $T_2 = n_2 T + n_3 T$, where $n_2 = n_2' + n_2'', n_2', n_2''$ and n_3 satisfy these inequalities:

$$n_2'T > -\frac{1}{D_3+\mu_0} ln\frac{\epsilon_5}{M+\mu_1},$$
$$n_2''T > -\frac{1}{D_3+\mu_0} ln\frac{\epsilon_6}{M+\mu_2},$$
$$(1-\delta)^{n_2+n_3}\exp(\eta n_2 T)\exp(n_3\sigma) > 1,$$

$\eta = a_1 b_1 m_1 - a_2 \epsilon_6 exp(-d_1\tau) - D < 0$. Now, we claim that $\exists$ a time $t_2' \in (t^*, t^* + T_2)$ such that $q(t_2') \geq m_4$, if it is not true, then $q(t_2') < m_4, t_2' \in (t^*, t^* + T_2)$. If the system (18) is taken with initial value $\bar{w}_1(t^{*+}) = r_1(t^{*+})$, then from lemma (3.1) for $t \in (nT, (n+1)T]$,
we have

$$\bar{w}_1(t) = (\bar{w}_1(t^{*+}) - \frac{a_2 b_2 m_4 M exp(-d_1\tau)}{D_3+\mu_0} + \frac{\mu_1}{1-exp(-(D_3+\mu_0)T)})exp(-(D_3+\mu_0)(t-t^*)) + \tilde{\bar{w}}_1(t),$$

for $n_1 \leq n \leq n_1 + n_2 + n_3$ which shows that $|\bar{w}_1(t) - \tilde{\bar{w}}_1(t)| \leq (M+\mu_1)exp(-(D_3+\mu_0)(t-n_1T)) < \epsilon_5$, and $r_1(t) \leq \bar{w}_1(t) < \tilde{\bar{w}}_1(t) + \epsilon_5$ for $t^* + n_2'T \leq t \leq t^* + T_2$.

Now, from the system (18) with initial values $\bar{w}_2(t^* + n_2'T) = q_2(t^* + n_2'T) \geq 0$ and again from lemma (3.1), we have $|\bar{w}_1(t) - \tilde{\bar{w}}_1(t)| < (M+\mu_2)exp(D_3(t-(n_1+N_2')T)) < \epsilon_6$, and $r_2(t) \leq \bar{w}_2(t) < \tilde{\bar{w}}_2(t) + \epsilon_6$ for $t^* + n_2'T + n_2''T \leq t \leq t^* + T_2$, which shows that

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

system (22) holds for $[t^* + n_2T, t^* + T_2]$.

Integrating equation (22) on $[t^* + n_2T, t^* + T_2]$, we have

$$q((n_1 + n_2 + n_3)T) \geq q((n_1 + n_2)T)(1 - \delta)^{n_3} exp(n_3\sigma) \tag{23}$$

In addition from the system $(1 - 2)$, we have

$$\left.\begin{aligned} \frac{dq(t)}{dt} &= \left(a_1b_1m_1 - a_2Me^{-d_1\tau} - D\right)\ q(t) = \eta q(t), t \neq nT \\ q(t^+) &= (1 - \delta)q, t = nT, n = 1,2,3\dots. \end{aligned}\right\} \tag{24}$$

On integrating (24) in the interval $[T^*, (n_1 + n_2)T]$, it is obtained that

$$q((n_1 + n_2)T) \geq m_4(1 - \delta)^{n_2} exp(\eta n_2 T) \tag{25}$$

Now substitute (25) into (24), we get that

$$q((n_1 + n_2 + n_3)T) \geq m_4(1 - \delta)^{n_2+n_3} exp(n_3\sigma)exp(\eta n_2 T) > m_4 \tag{26}$$

which contradicts to our supposition, so there exists a time $t_2' \in [t^*, t^* + T_2]$ such that $q_2' \geq m_4$. Let $\hat{t} = \inf\{t | t \geq t^*, q(t) \geq m_4\}$,since $0 < \delta < 1$, $q(nT^+) = (1 - \delta)q(nT) < q(nT)$ and $q(t) < m_4, t \in (t^*, \hat{t})$. Thus, $q(\hat{t}) = m_4$.

Suppose $t \in (t^* + (l - 1)T, T^* + lT]$ (l is a positive integer) and $l \leq n_2 + n_3$, from the system (24), we have

$$q(t) \geq q(t^* + (l - 1)T)exp(\eta(t - t^* - (l - 1))T)$$
$$q(t) \geq q(nT^+)exp(\eta T(l - 1))(1 - \delta)^{l-1} exp(\eta T)$$
$$q(t) \geq m_4(1 - \delta)^l exp(l\eta T)$$
$$q(t) \geq m_4(1 - \delta)^{(}n_2 + n_3)exp((n_2 + n_3)\eta T) \triangleq \bar{m}_4$$

for $t > \hat{t}$. The same argument can be continued since $q(\hat{t}) \geq m_4$. Hence $q(t) \geq \bar{m}_4 \forall t > t_2$.

Case 2: If $t^* \neq nT$, then $q(t^*) = m_4$ and $q(t) \geq m_4, t \in [t_2, t^*]$. Suppose $t^* \in (n_1'T, (n_1' + 1)T]$, we are having two subcases for $t \in [t^*, (n_1' + 1)T]$ as given below:

Case a: $q(t) \leq m_4, t \epsilon [t^*, (n_1' + 1)T]$, we claim that there exists a $t_3 \varepsilon [(n_1' + 1)T, (n_1' + 1)T + T_2]$ s.t $q(t_3) > m_4$. Otherwise, integrating system (24) on the interval $[(n_1' + 1 + n_2)T, (n_1' + 1 + n_2 + n_3)T]$, we have, $q((n_1' + 1 + n_2 + n_3)T) \geq q((n_1' + 1 + n_2)T)(1 - \delta)^{n_3} exp(n_3\sigma)$

Since $q(t) \leq m_4, t \in [t^*, (n_1' + 1)T]$, therefore, (13) holds on $[t^*, (n_1' + n_2 + n_3)T]$.

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

Thus,

$$q((n_1'+1+n_2)T) = q(t^*)(1-\delta)^{n_2} exp(\eta(n_1'+1+n_2)T - t^*)$$
$$q((n_1'+1+n_2)T) \geq m_4(1-\delta)^{n_2} exp(\eta n_2 T)$$

and

$$q((n_1'+1+n_2+n_3)T) \geq m_4(1-\delta)^{n_2+n_3} exp(\eta n_2 T) exp(n_3\sigma) > m_4$$

which negates the assumption. Let $\check{t} = \inf\{t | q \geq m_4, t > t^*\}$, then $q(\check{t}) = m_4$ and $q < m_4, t \in (t^*, \check{t})$. Choose $t \in (n_1'T + (l'-1)T, n_1'T + l'T] \subset (t^*, \check{t}), l'$ is a positive integer and $l' < 1 + n_2 + n_3$, we have

$$q(t) \geq q((n_1'+l'-1)T^+) exp(\eta(t-(n_1'+l'-1)T))$$
$$q(t) \geq (1-\delta)^{l'-1} q(t^*) exp(\eta(t-t^*))$$
$$q(t) \geq m_4(1-\delta)^{n_2+n_3} exp(\eta(n_2+n_3+1)T).$$

Hence, $q \geq \bar{m}_4$ for $t \in (t^*, \check{t})$. For $t > \check{t}$, we can proceed in the same manner since $q(\check{t}) \geq m_4$.

Case b: If $\exists$ a $t \in (t^*, (n_1'+1)T)$ s.t $q(t) \geq m_4$. Let $\check{t} = \inf(t | q(t) \geq m_4, t > t^*)$, then $q(t) < m_4$ for $t \in [t^*, \bar{t})$ and $q(\bar{t}) = m_4$. For $t \in [t^*, \bar{t})$ (24) holds. On integrating (24) on $t^*, \check{t}$, we obtain

$$q \geq q(t^*) \geq exp(\eta(t-t^*) \geq m_4 exp(\eta T) > \bar{m}_4$$

Since, $q(\hat{t}) \geq m_4$ for $t > \hat{t}$, we can proceed in the same manner. Hence, we have $q(t) \geq \bar{m}_4$ for all $t > t_2$. Therefore we can conclude that $q(t) \geq \bar{m}_4$ for all $t \geq t_2$ in both cases.

6. Numerical Section

For the intended process, we have taken data per week in view of the short term life cycle of the insect population under investigation. Our aim is to validate the analytical results numerically. We have considered numerical values for the following set of parameters in reference to (Jatav and Dhar, 2014) as mentioned in Table 2.

Table 2. Parametric values

Parameters	μ_0	r	a_1	b_1	d_1	τ	a_2	b_2	D	D_3
Values	50	1	1	0.1	0.3	0.2	0.3	0.5	0.03	25

Using the above parametric values, we obtained the threshold value T_{max} for the parameters per week as 0.8. It is proved that $(r, 0, \tilde{r}_1(t), \tilde{r}_1(t))$ is locally asymptotically stable if $T = 0.5 < T_{max}$ as stated above in the theorem 4.1 (Figure 2-5). Further, it is also verified that the system $(A-B)$ is permanent if $T = 4 > T_{max}$(Figure 6-9) which is inline with theorem 5.1. It is also shown that if there is no biological control, that is, $\mu_1 = 0$ and $\mu_2 = 0, \mu_1 = 0$ and $\mu_2 > 0$ or $\mu_1 > 0$ and $\mu_2 = 0$, then both plants and pest population survives.This concludes, that solely using chemical pesticide cannot eradicate pest population (Figure 10-14).

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

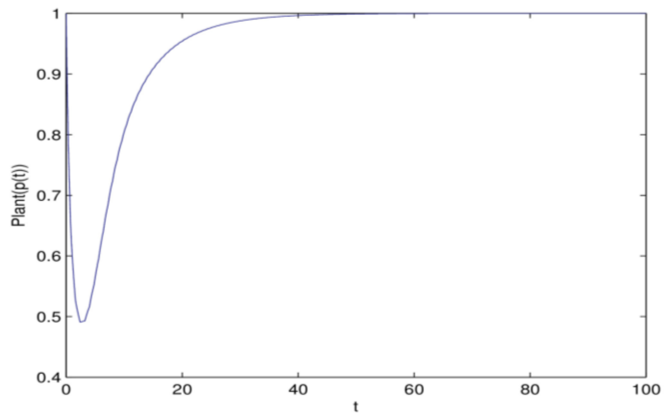

Figure 2. Plant population $(P(t))$ existing

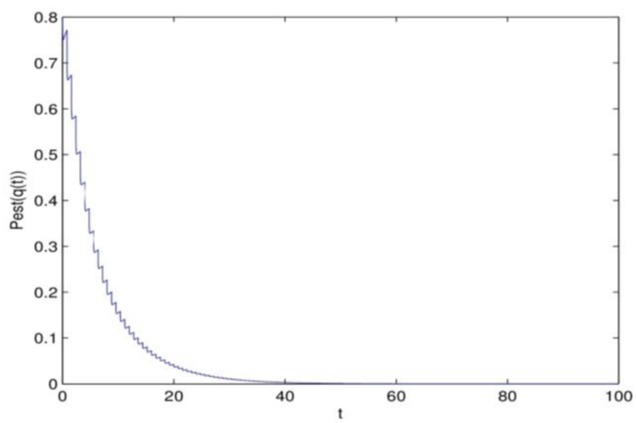

Figure 3. Pest population $(q(t))$ vanishes

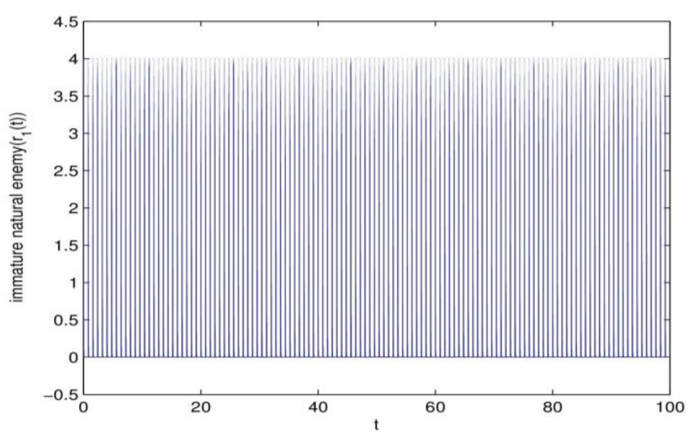

Figure 4. Periodic behaviour of $r_1(t)$

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

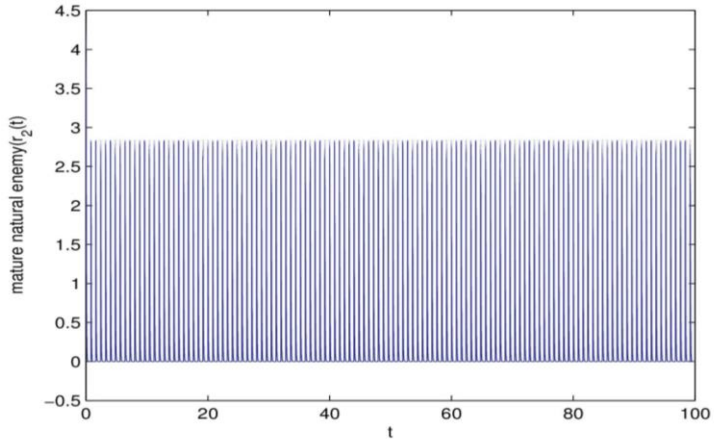

Figure 5. Periodic behaviour of $(r_2(t))$

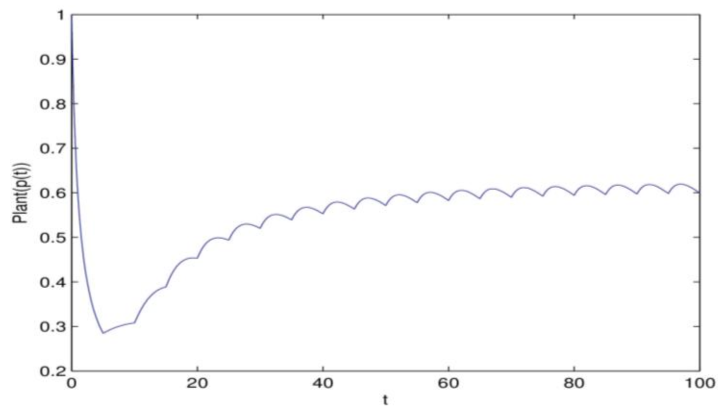

Figure 6. Plant population $(p(t))$ exists

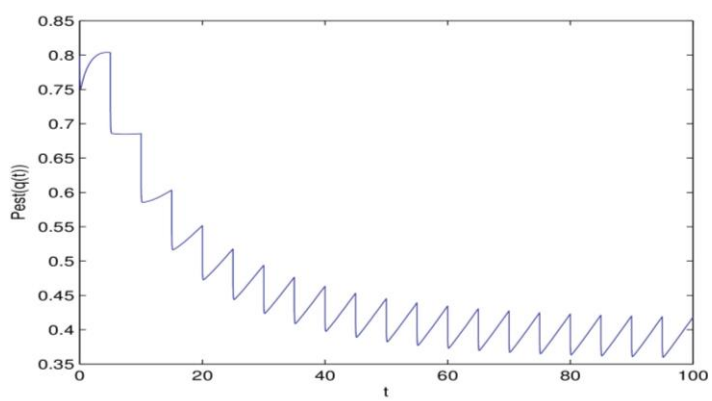

Figure 7. Pest population $(q(t))$ survives

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

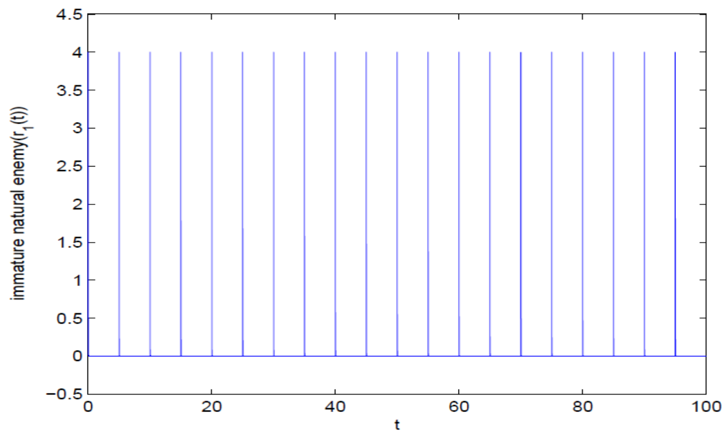

Figure 8. Immature natural enemies $(r_1(t))$ exists

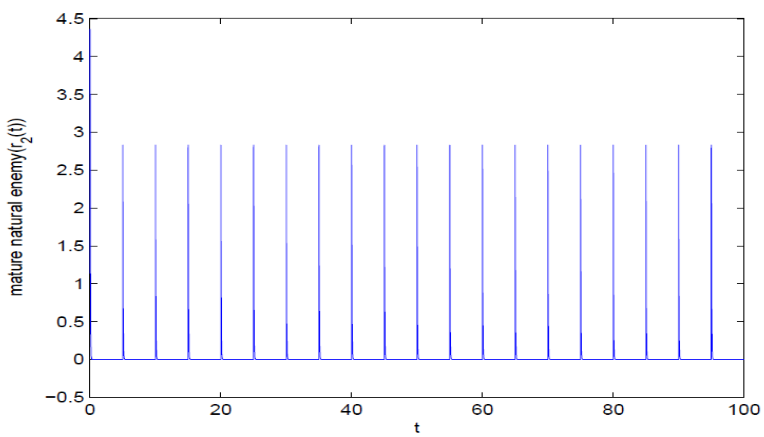

Figure 9. Behaviour of mature natural enemies $\left(r_2(t)\right)$

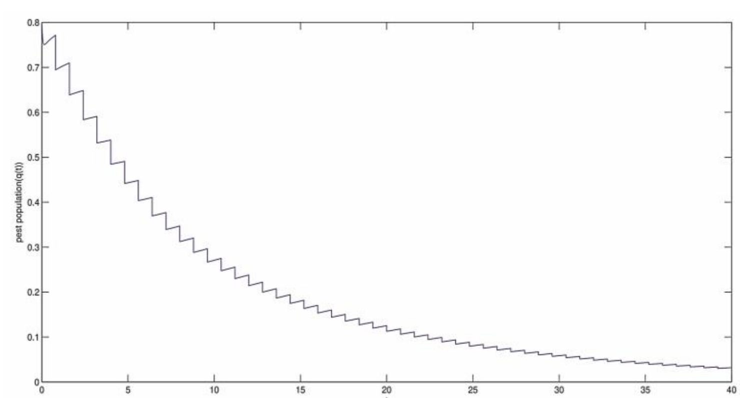

Figure 10. Existence of the pest population$(q(t))$ for $\mu_1, \mu_2 = 0$

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 4, 769-786, 2020
https://doi.org/10.33889/IJMEMS.2020.5.4.061

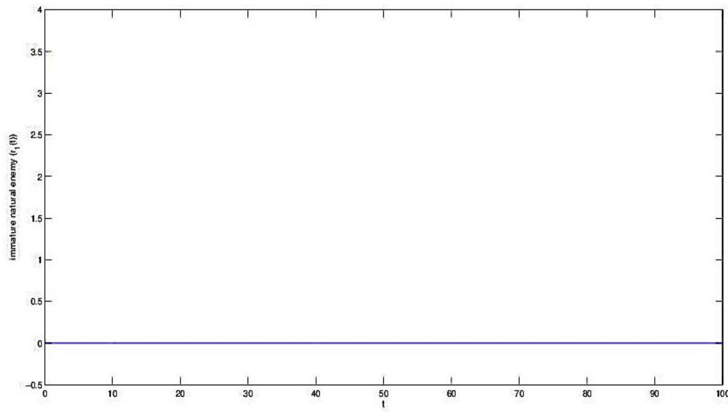

Figure 11. Immature natural enemy $(r_1(t))$ vanishes for $\mu_1, \mu_2 = 0$

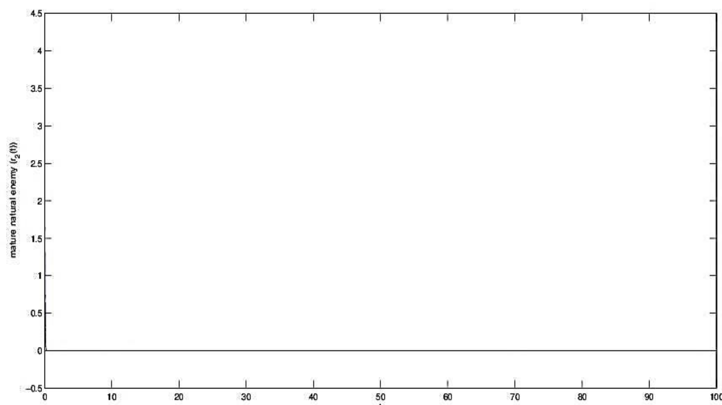

Figure 12. Mature natural enemy $(r_2(t))$ for $\mu_1, \mu_2 = 0$

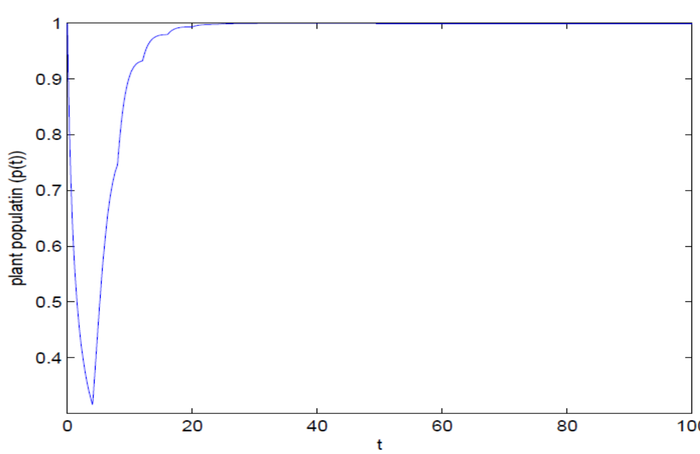

Figure 13. Plant population $(p(t))$ is stable for $\mu_1 = 100, \mu_2 = 50$

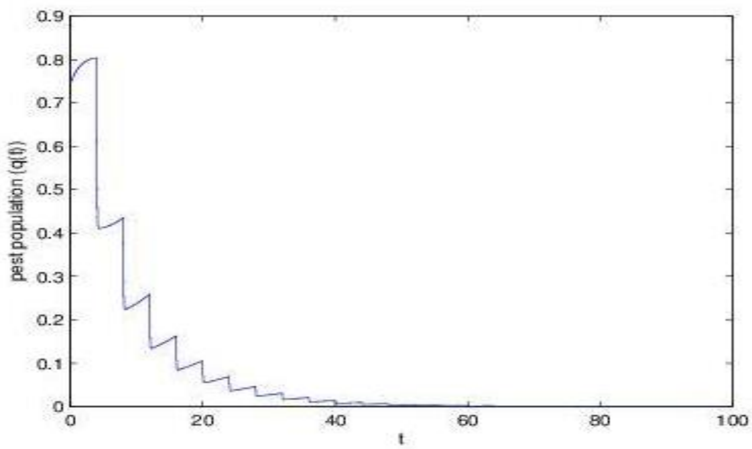

Figure 14. Pest population $(q(t))$ declines for $\mu_1 = 100, \mu_2 = 50$

7. Conclusion

In this paper, we have examine the effects of hybrid approach to control the pests by release of natural enemies and pesticides impulsively. It is evident that pest population can become extinct when large amount of the natural enemies are released impulsively. Thus, integrated pest management reduces pest quickly rather than using any one of the methods. Hence, in this paper, we have shown that by incorporating delay in the pests, we are able to control the pest population but to a lower threshold value which in a way is helpful as it is leading to early reduction in the pest which is not only economic but it also prevents pest resistance to crops. Incorporating delay lowered the threshold level from to $T_{max} = 7$ to $T_{max} = 0.8$ for the same set of parameters as in (Jatav & Dhar, 2014). Thus, we can conclude that various control measures should be applied collectively for the eradication of pest. Such a practice improves economy as it is cost effective and synonymous with sustainable development.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgements

The first author would like to express her sincere thanks to her guide, co-guide for their constant guidance and support and special thanks to all the reviewers and editor.

References

Akman, O., Comar, T.D., & Hrozencik, D. (2015). On impulsive integrated pest management models with stochastic effects. *Frontiers in Neuroscience*, *9*,119. doi: 10.3389/fnins.2015.00119.

Bainov, D.D., & Simeonov, P. (1993). Impulsive differential equations: periodic solutions and application. *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Chapman and Hall/CRC Press, ISBN 9780582096394.

Changguo, L., Yongzhen, P., & Xuehui, J. (2009). Dynamic behavior of a multiple species prey-predator system with impulsive chemical and biological control. In *2009 Fifth International Conference on Natural Computation* (Vol. 5, pp. 477-481). IEEE. Tianjin, China.

El-Shafie, H. (2018). Integrated insect pest management. *Pests - Insects, Management, Control [Working Title]*.

James, D.G. (1997). Imidacloprid increases egg production in Amblyseius victoriensis (Acari: Phytoseiidae). *Experimental & Applied Acarology*, *21*(2), 75-82.

Jatav, K.S., & Dhar, J. (2014). Hybrid approach for pest control with impulsive releasing of natural enemies and chemical pesticides: a plant–pest–natural enemy model. *Nonlinear Analysis: Hybrid Systems*, *12*, 79-92.

Jatav, K.S., Dhar, J., & Nagar, A.K. (2014). Mathematical study of stage-structured pests control through impulsively released natural enemies with discrete and distributed delays. *Applied Mathematics and Computation*, *238*, 511-526.

Liu, B., Xu, L., & Kang, B. (2013). Dynamics of a stage structured pest control model in a polluted environment with pulse pollution input. *Journal of Applied Mathematics*, *2013*. doi.org/10.1155/2013/678762.

Páez Chávez, J., Jungmann, D., & Siegmund, S. (2018). A comparative study of integrated pest management strategies based on impulsive control. *Journal of Biological Dynamics*, *12*(1), 318-341.

Song, Y., Wang, X., & Jiang, W. (2014). The pest management model with impulsive control. *Applied Mechanics and Materials*, *519*, 1299-1304. doi.org/10.4028/www.scientific.net/amm.

Tang, S., Xiao, Y., Chen, L., & Cheke, R.A. (2005). Integrated pest management models and their dynamical behaviour. *Bulletin of Mathematical Biology*, *67*(1), 115-135.

Yu, T., Tian, Y., Guo, H., & Song, X. (2019). Dynamical analysis of an integrated pest management predator–prey model with weak Allee effect. *Journal of Biological Dynamics*, *13*(1), 218-244.

Zhang, Y., Liu, B., & Chen, L. (2004). Dynamical behavior of Volterra model with mutual interference concerning IPM. *ESAIM: Mathematical Modelling and Numerical Analysis*, *38*(1), 143-155.