Fuzzy Reliability of Two-Stage Weighted-*k*-out-of-*n* Systems with Common Components

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Abstract
It seems there is a scope to study how we can find the fuzzy reliability of two-stage weighted-*k*-out-*n*. This paper studies the fuzzy reliability of two-stage weighted-*k*-out-of-*n* model with components in common. Algorithms are developed to calculate the system fuzzy reliability, and generate the minimal cuts and minimal paths of two-stage weighted-*k*-out-of-*n* systems. Fuzzy reliability bounds for systems with *s*-dependent component failures are investigated based on the generated minimal cuts and minimal paths. Two types of two-stage weighted-*k*-out-of-*n* models, the Series Weighted (SW)-*k*-out-of-*n* model, and the Parallel Weighted (PW)-*k*-out-of-*n* model, can be applied to investigate fuzzy reliability issues in network applications, such as the project management, and the shortest path problems. Examples are provided to demonstrate the developed models and algorithms of systems having components following different type of membership functions.

Keywords- fuzzy reliability, two-stage weighted-*k*-out-*n* model, minimal cuts, minimal paths.

1. Introduction
The reliability of an item is the probability that the item will perform a specified function. However, in real world system, the information is not always accurate, hence the estimation of precise values of probability becomes very difficult in many cases. In order to handle the insufficient information, the fuzzy approach is used to evaluate the failure rate status. The basis for this approach is constituted by the fundamental works on fuzzy set theory of Zadeh (1965). The theory of fuzzy reliability was proposed and development by several authors, Kai and Zhang (1991), Utkin and Gurov (1995), Cai (1996). Kai et al. (1991) presented a fuzzy set based approach to failure rate and reliability analysis.

The weighted-*k*-out-of-*n* model was first proposed by Wu and Chen (1994), as an extension of the widely studied *k*-out-of-*n* model, Haung (2000) and Jain and Goel (1985). Similar concepts were also applied in introducing the weighted consecutive-*k* -out-of-*n* models, Wu and Chen (1998). For a weighted-*k*-out-of-*n*:F system, an integer weight $w_i > 0$ is assigned to component i. The system is failed iff the total weight of the failed components is greater than or equal to *k*, a pre-specified threshold value. In other words, the system is functioning iff the total weight of the failed components is less than *k*. Chen and Yang (2005) analyzed the reliability of two-stage weighted-k-out-of-n systems with components in common. In this study, this analysis is extended by involving the fuzzy environment.

Let $R(z,s)$ represents the probability that the total weight of the failed components of a system with s components is less than z, i.e.$R(z,s)$, is the functioning probability of a system with s components, and a threshold equal to z. Then the reliability of a weighted-k-out-of-n:F system is $R(k,n)$. The reliability of the weighted-k-out-of-n:F system when all component failures are mutually s-independent can be obtained based on the following recursive relationship:

$$R(z,s) = \begin{cases} 0 & z \leq 0, s \geq 0 \\ 1 & z > 0, s = 0 \\ (1-\mu_s)R(z-w_s, s-1) + \mu_s R(z, s-1) & \text{otherwise} \end{cases} \tag{1}$$

where w_s denotes the weight of the component s, s =1, 2, 3 …n.

2. Two-Stage Weighted-k-out-of-n Systems with Components in Common
This section introduces the concepts, motivation applications and main properties of the two-stage weighted- k-out-of-n model.

2.1 Two-Stage Weighted-k -out-of-n Model
A two-stage weighted-k-out-of-n system consists of a number of subsystems. Let m be the number of subsystems in a two-stage system. Each subsystem has a (one-stage) weighted- k-out-of-n structure, which is called the second-level structure. The interrelationship between the subsystems follows certain coherent structure, which is called the first-level structure, whose structure function is denoted by$\emptyset_1(\cdot)$. Let x_i denote the functioning state of component i, $x_i = 1$, if component x_i is functioning, $x_i = 0$if it is failed, for i =1, 2, 3……n .Let $\bar{x}_i$be the complement of x_i , i.e.x_i: $\bar{x}_i = 1$, iff component i is failed and $\bar{x}_i = 0$ iff component i is functioning. A two-stage weighted-k-out-of- n: F system is functioning iff $\emptyset_1(y_1, y_2, y_3, …………… y_m) = 1$, where y_i indicates the functioning state of subsystem i. i.e.

$$y_i = \begin{cases} 1 & \text{if} \sum_{j=1}^{n} w_{ij}.\overline{x_j} < k_i \text{subsystem } i \text{ is functioning} \\ 0 & \text{if} \sum_{j=1}^{n} w_{ij}.\overline{x_j} \geq k_i \text{subsystem } i \text{ is failed} \end{cases} \tag{2}$$

Generally, a two-stage weighted-k-out-of-n system can be decomposed into two hierarchical levels: the first (higher) level can be of any coherent structure, whose structure function is $\emptyset_1(\cdot)$; and the second (lower) level has a weighted-k-out-of-n structure. Special cases of two-stage weighted-k-out-of- n systems include SW-k-out-of-n systems, whose first level has a series structure, and PW-k-out-of-n systems, whose first level has a parallel structure. Therefore, an SW-k -out-of-n: F system is failed iff any of the inequalities in (3) are satisfied

Subsystem 1: $w_{11}.\bar{x}_1 + w_{12}.\bar{x}_2 …..+w_{1j}.\bar{x}_1 …..+w_{1n}.\bar{x}_n \geq k_1$
Subsystem 2: $w_{21}.\bar{x}_1 + w_{22}.\bar{x}_2 …..+w_{2j}.\bar{x}_1 …..+w_{2n}.\bar{x}_n \geq k_2$ (3)
Similarly Subsystem m: $w_{m1}.\bar{x}_1 + w_{m2}.\bar{x}_2 …..+w_{3j}.\bar{x}_1 …..w_{mn}.\bar{x}_n \geq k_m$

where$k \equiv [k_1, k_2 …………… k_m]^T$ is a vector of thresholds for each subsystem and i = 1, 2, …n.

A PW-k-out-of-n:F system is failed iff all of the inequalities in (3) are satisfied. If the first-level structure of a system is a k-out-of-m:F structure, then the system is failed iff at least one of the inequalities in (3) are satisfied. The inequality in (3) is referred to as the subsystem failure

condition. When *m* is equal to 1, the two-stage weighted- *k*-out-of-*n* system becomes a (one-stage) weighted-*k* -out-of- *n* system.

➢ Although each subsystem failure condition in (2) and (3) is related to the same set of *n* components, the two-stage weighted-*k*-out-of-n model allows each subsystem to have different subsets of components. A subsystem does not need to contain all *n* components. If component *j* is not in subsystem *i*, for *i* =1, 2, 3......*m*, the weight w_{ij} will be set to zero. Based on this usage of zero weights for nonexistence components in a subsystem, the number of components in each subsystem in the model based on (2) and (3) can be different. The variable *n* in (2) and (3) can be understood as the total number of different components in the entire two-stage weighted-*k*-out-of-*n* system. The actual number of components in a subsystem can be found out by the number of nonzero weights in its failure condition.

➢ In this paper the subsystems are connected in a series structure, a parallel structure, or more complex structures. Usually in fuzzy reliability, it is assumed that there are no components in common between any two subsystems. With this assumption, the failures of the subsystems are s-independent if all components have s-independent failures. And the system reliability can be calculated easily based on the reliability of each individual subsystem. However, in the two-stage weighted-*k*-out-of-*n* model, subsystems typically have some (or all if all the weights w_{ij} are nonzero) components in common. Consequently, the system reliability cannot be evaluated by combining the failure probability of each subsystem in a straightforward way. From the Note 1, the components in common between two subsystems can be found out based on the nonzero weights in each subsystem failure condition.

2.2 Applications of Two-Stage Weighted-*k*-out-of-*n* Model in Network Problems

Consider a directed network (N, A), where N is the set of nodes, and A is the set of directed arcs. In shortest path problems, the system can be considered as functioning iff there exists at least one path from the source node to the destination node with total distance/cost less than a specification (that is, the total distance of the shortest path is within specification). In project management problems, usually the system is considered as functioning iff all the paths from the source (start of a project) to the destination (end of a project) of the project network have total time less than a specified deadline. The distance/time of some arcs can be delayed (increased) by certain amounts upon corresponding arc failures. It is illustrated in the following examples that the two-stage weighted-*k*-out-of-*n* model can be applied to evaluate system reliability in the problems described above, where the shortest path problem fits in a PW-*k*-out-of-*n* model, and the project management problem fits in an SW-*k*-out-of-*n* model.
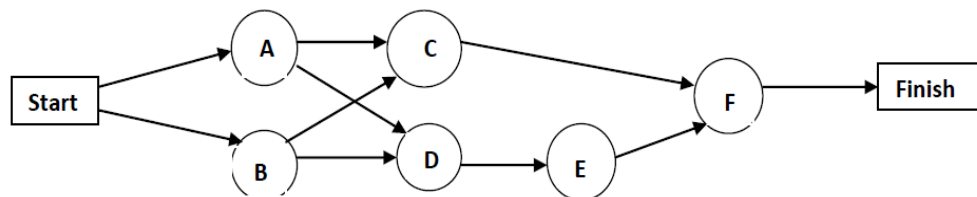

Fig. 1. Network diagram of the problem

Activity	Immediate Predecessors	Estimated Duration (Days)	Possible Delay(Days)
A= train workers	None	<5,6,7>	1
B= purchase raw material	None	<8,9,10>	3
C= produce product 1	A, B	<7,8,9>	3
D= produce product 2	A, B	<6,7,8>	2
E= Test product 1	D	<9, 10,11>	2
F= Assemble product 1 and 2	C,E	<11, 12,13>	4

Table 1. Time duration for the activities

(*i*) *Project Management Example*

Consider a project management example from Winston (1994) about the assembly of two products (products 1 and 2) to make a new product. This project consists of six activities as illustrated in Table 1. The project network is illustrated in Fig. 2.
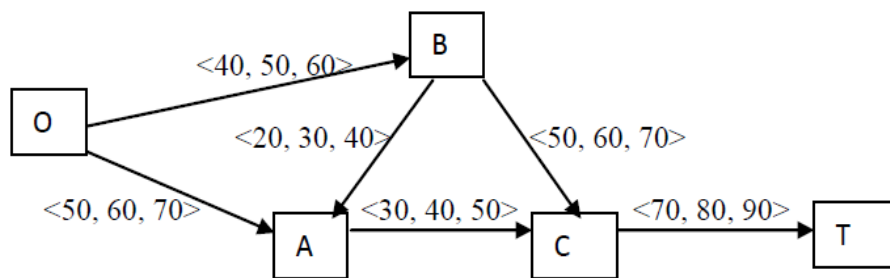

Fig. 2. Diagram of the traffic

There are four paths through this project network:

1. Start → A → C → F → Finish
(Total duration without delay =<23,26,29>)
2. Start →A → D → E → F → Finish
(Total duration without delay = <31, 35, 39>)
3. Start → B → C → F → Finish
(Total duration without delay = <26, 29, 32>)
4. Start → B → D → E → F → Finish
(Total duration without delay =<34, 38, 42>)

The critical path, which is the longest path through the project network, is the one with total duration of <34, 38, 42> days. Therefore, the earliest finish time of the whole project, if there are no delays, is <34, 38, 42> days. Suppose each activity is subjected to delays with a probability (1-p). The delay time for each activity is also listed in Table 1. Let x_i denote the functioning state (delayed=0; not delayed=1) of activity i, i=A, B, C, D, E, F. Assume that the project will fail iff the actual completion time of at least one of the four paths from the Start node to the Finish node in Fig. 1 is greater than or equal to <40, 41, 42> days. That is, the total duration without delay for the activities on the second path:

Start →A → D → E → F → Finish is <31, 35, 39>

Therefore a delay of <40, 41, 42> - <31, 35, 39> = <1, 6, 11> or more in completion time of this path will delay the whole project. The total delay due to possible failures of activities A, D, E, and F on this path is

$$1 \cdot \bar{x}_A + 2 \cdot \bar{x}_D + 2 \cdot \bar{x}_E + 4 \cdot \bar{x}_F = 1.\bar{x}_A + 0.\bar{x}_B + 0.\bar{x}_C + 2.\bar{x}_D + 2.\bar{x}_E + 4.\bar{x}_F$$

where the coefficients/weights of $\bar{x}_A$, $\bar{x}_D$, $\bar{x}_E$, and $\bar{x}_F$ are the delay times of activities A, D, E, and F respectively, as shown in Table 1. The weights of $\bar{x}_B$ and $\bar{x}_C$ for the second path are set to zero because they are not in this path. As a result, the completion time of the second path is larger than deadline iff

$$1.\bar{x}_A + 0.\bar{x}_B + 0.\bar{x}_C + 2.\bar{x}_D + 2.\bar{x}_E + 4.\bar{x}_F \geq < 1, 6, 11 >$$

Similarly the project will fail to meet the deadline iff *at least one* of the following failure conditions is satisfied

$$1.\bar{x}_A + 0.\bar{x}_B + 3.\bar{x}_C + 0.\bar{x}_D + 0.\bar{x}_E + 4.\bar{x}_F \geq < 11, 15, 19 >$$
$$1.\bar{x}_A + 0.\bar{x}_B + 0.\bar{x}_C + 2.\bar{x}_D + 2.\bar{x}_E + 4.\bar{x}_F \geq < 1, 6, 11 >$$
$$0.\bar{x}_A + 3.\bar{x}_B + 3.\bar{x}_C + 0.\bar{x}_D + 0.\bar{x}_E + 4.\bar{x}_F \geq < 8, 12, 16 > \tag{4}$$
$$0.\bar{x}_A + 3.\bar{x}_B + 0.\bar{x}_C + 2.\bar{x}_D + 2.\bar{x}_E + 4.\bar{x}_F \geq < 0, 3, 6 >$$

The failure conditions of (4) are exactly the failure conditions of an SW-*k*-out-of-*n*:F system by considering each activity as a component, and each path as a subsystem. Because the maximal total weights in the left hand side of the first and the third failure conditions in (4) are less than the right hand side. These two failure conditions can be removed to simplify the computation. Consequently, we can focus on the following two failure conditions, where activity is obviously irrelevant, and does not need to be included:

$$1.\bar{x}_A + 0.\bar{x}_B + 2.\bar{x}_D + 2.\bar{x}_E + 4.\bar{x}_F \geq < 1, 6, 11 > \tag{5}$$
$$0.\bar{x}_A + 3.\bar{x}_B + +2.\bar{x}_D + 2.\bar{x}_E + 4.\bar{x}_F \geq < 0, 3, 6 >$$

Hence this project management problem is formulated as a SW-weighted-[<1,6,11>, <0,3,6>]^T-out-of-5:F system.

(*ii*)*Example of a Shortest Path Problem*
Consider a shortest path problem of taking a trip from one town (the origin), to another (the destination). The corresponding traffic network is illustrated in Fig. 2. The nodes denote towns (O = Origin and T = Destination). The numbers on each arc represent the time (in minutes) to travel from one town to the next and each arc can allow one-way traffic only. There are 3 paths from the origin to destination

1. OB→ BC → CT   (total travel time without delay = <160, 190, 210> minutes)
<40,50,60>
<20,30,40><50,60,70>
<50,60,70><30,40,50><70,80,90>
2. OB→ BA → AC → CT   (total travel time without delay = <160,200,240> minutes)

3. OA→ AC → CT   (total travel time without delay = <150,180,210> minutes)

The shortest path (the 3rd path above) has a total travel time of <150,180,210> minutes. Suppose each arc has a probability (1-*p*) to be delayed by road constructions. Assume, if there is a road construction underway on an arc, the delay in travel time is as given in Table 2.

Arc	Delay (minutes)	Arc	Delay (minutes)
OB	10	BC	12
OA	12	BA	6

Table 2. Delay time for paths

We are interested in the possibility that there exists at least one path with total travel time less than <190,195,210> minutes. Each arc in the shortest path problem can be considered as a component. If there is no path with travel time within <190, 195, 210>minutes (considered as system failure), all of the following conditions must be satisfied (the second path needs not to be considered because its travel time is greater than <190,195,210> minutes no matter whether a delay occurs or not):

$$10. \bar{x}_{OB} + 0. \bar{x}_{OA} + 0. \bar{x}_{BA} + 0. \bar{x}_{AC} + 12. \bar{x}_{BC} + 16. \bar{x}_{CT} \geq < 0, 5, 40 > \tag{6}$$
$$0. \bar{x}_{OB} + 12. \bar{x}_{OA} + 0. \bar{x}_{BA} + 8. \bar{x}_{AC} + 0. \bar{x}_{BC} + 16. \bar{x}_{C} \geq < 0, 15, 50 >$$

It can be seen that the failure conditions in (6) are the same as the failure conditions of a PW-[< $0, 5, 40 >, < 0, 15, 50 >]^{T}$-out-of-6:F system.

2.3 Main Properties of Two-Stage Weighted-*k* -out-of-*n* Systems
Similar to the *k*-out-of-*n* and one-stage weighted-*k*-out-of-*n* systems a one-to-one correspondence between F systems and G systems exists for the two-stage weighted-*k*-out-of-*n* systems. A two-stage weighted-*k*-out-of-*n*:G system is functioning iff $\emptyset_1(y') = 1$ where

$$y' = [y'_1 \dots \dots \dots \dots y'_m]^T \text{with } y'_i \equiv \begin{cases} 0, & \text{if} \sum_{j=1}^{n} w_{ij}.x_j < k_i \\ 1, & \text{if} \sum_{j=1}^{n} w_{ij}.x_j \geq k_i \end{cases} \tag{7}$$

That is,y'_i indicates the functioning state of subsystem *i*, where subsystem *i* has a weighted-k_i-out-of- *n*:G structure. The relationship between the F systems and G systems is as following:

- ➢ The two-stage weighted-*k*-out-of-*n*:F (or G) system is equivalent to the two-stage weighted-*(W-k+1)*-out-of-*n*:G (or F) system with the same first level structure function $\emptyset_1(\cdot)$ where $\mathbf{1} = [\mathbf{1} \dots \dots \dots \dots \mathbf{1}]^T$. The dual structure $\emptyset^D(x)$ to a given structure $\emptyset(x)$ is defined as
  $\emptyset^D(x) = 1 - \emptyset(1 - x)$.

- ➢ The two-stage weighted-*k*-out-of-*n*:F system with the first-level structure function $\emptyset_1(\cdot)$ and the two-stage weighted-*k*-out-of-*n*:G system with the first-level structure function $\emptyset^D(\cdot)$, are duals of each other.
  The following result can be used to facilitate minimal path generation of a two-stage weighted-*k* -out-of- *n*:F system.

> ➢ The minimal pathof a two-stage weighted-k-out-of-n:F system with the first level structure function $\emptyset_1(\cdot)$can be obtained by generating minimal cuts for the two-stage weighted-(W-k+1)-out-of-n:F system with the first level structure function $\emptyset_1^D(\cdot)$.

3. Algorithms for System Reliability Evaluation and Generation of Minimal Cuts and Minimal Paths

3.1 Reliability Evaluation for Two-StageWeighted-k-out-of-n:F Systems

For a two-stage weighted-k-out-of-n:F system with first-level structure function $\emptyset_1(\cdot)$ assuming s–independent component failures, the system reliability $R(k, n)$ can be calculated based on the following recursive relationship:

$$R(z,s) = \begin{cases} 0 & \text{if } \emptyset_1(\beta_0) = 0 \\ 1 & \text{if } \emptyset_1(\beta_1) = 1 \\ (1 - p_s)R(z - w_s, s - 1) + p_s R(z, s-1) & \text{otherwise} \end{cases} \tag{8}$$

Where$\beta_0 = [\beta_{01} \ldots \ldots \ldots \ldots \ldots . \beta_{0m}]^T$with$\beta_{0i} = \begin{cases} 1, & \text{if } z_i > 0 \\ 0, & \text{if } z_i \le 0 \end{cases}$

and $\beta_0 = [\beta_{11} \ldots \ldots \ldots \ldots \ldots . \beta_{1m}]^T$with$\beta_{0i} = \begin{cases} 1, & \text{if} \sum_{j=1}^{s} w_{ij} < z_i \\ 0, & \text{if} \sum_{j=1}^{s} w_{ij} \ge z_i \end{cases}$

3.2 Minimal Cuts, and Minimal Paths of Two-Stage Weighted-k-out-of-n:F Systems

Minimal paths and minimal cuts are useful toward the representation of coherent systems. The generation of the minimal cuts and paths for two-stage weighted-k-out-of-n systems is based on the generation of minimal cuts and paths of one-stage systems. One important application of minimal paths and minimal cuts is to evaluate bounds of system reliability when component failures are s-dependent.

(*i*) *Generation of Minimal Cuts and Paths for One-Stage Systems*
To generate the minimal cut and minimal path of one-stage weighted-k -out-of-n systems, we first assume that component weights have been sorted such that $w_1 \le w_2 \ldots \ldots \ldots \ldots . \le w_n$. Let $MCS(z, s)$denote the cut polynomial of a one-stage weighted-z-out-of-i:F system. A cut polynomial is a logic function formed by writing terms for each minimal cut, separated by a plus sign (logical OR). The recursive relationship to generate all the minimal cuts is as follows

$$MCS(z,s) = \begin{cases} 0 & \text{if} \sum_{j=1}^{s} w_{ij} < z_i \\ 1 & \text{if } z \le 0 \\ MCS(z, s-1) + MCS(z - w_s, s-1).\bar{x}_s & \text{otherwise} \end{cases} \tag{9}$$

The operations on cut polynomials in (9) are Boolean operations. It is worth to note that only additions and multiplications are used for the Boolean operations in (9). The sorted component weights guarantee that every term in the resulted cut polynomial represents a minimal cut without replications.

Let $MPS(z, s)$ denote the path polynomial of a one-stage weighted-z-out-of-s:F system. A path polynomial is a logic function formed by writing terms for each minimal path, separated by a plus

sign (logical OR). The recursive relationship shown in (10) can be used to derive the set of minimal paths of a one-stage weighted-k-out-of-n:F system.

$$MPS(z,s) = \begin{cases} 0 & \text{if} \sum_{i=1}^{s} w_i < z_i \\ 1 & \text{if } z \leq 0 \\ MPS(z,s-1).x_s + MPS(z-w_s,s-1) & \text{otherwise} \end{cases} \qquad (10)$$

The operations on path polynomials in (10) are Boolean operations. The relationship in (10) can be verified by (9), as described below:

$$
\begin{aligned}
MPS(z,s) = \overline{MCS(z,s)} \quad &= \overline{MCS(z,s-1) + MCS(z-w_s,s-1)\bar{x}_s} \\
&= \overline{MCS(z,s-1)}.\overline{(MCS(z-w_s,s-1) + x_s)} \\
&= \overline{MCS(z,s-1)}.\overline{MCS(z-w_s,s-1)} + \overline{MCS(z-w_s,s-1)}.\bar{x}_s \\
&= MPS(z,s-1).x_s + MPS(z-w_s,s-1)\,MPS(z,s-1) \\
&= MPS(z,s-1).x_s + MPS(z-w_s,s-1)
\end{aligned}
$$

In addition, the recursive relationship can be used to generate minimal cuts and minimal paths without requiring sorting the component weights as given in (11) and (12). An upper bound z_U and a lower bound z_L are added to $MCS(z_L,s)$ and $MPS(z_U,s)$ respectively, to guarantee that the obtained cut sets and path sets are minimal without sorting the component weights. The minimal cuts of a one-stage weighted-k-out-of-n:F system can be obtained by evaluating $MCS(k,\infty,n)$; and its minimal paths can be obtained by evaluating $MPS(0,k,n)$.

$$MCS(z_L,z_U,s)$$
$$= \begin{cases} 0 & \text{if}(s = 0 \text{ and } 0 < z_L)\text{or}z_U \leq 0 \\ 1 & \text{if } s = 0 \text{ and}z_L \leq 0 < z_U \\ MCS(z_L,z_U,s-1) + MCS(z_L - w_s, \min(z_L,z_U - w_s),s-1).\bar{x}_s & \text{otherwise} \end{cases}$$

$$(11)$$

$$MPS(z_L,z_U,s)$$
$$= \begin{cases} 0 & \text{if}(s = 0 \text{ and } 0 < z_L)\text{or}z_U \leq 0 \\ 1 & \text{if } s = 0 \text{ and}z_L \leq 0 < z_U \\ MPS(\max((z_L,z_U - w_s),z_U,s-1).x_s(z_L,z_U,s-1) + MPS(z_L - w_s, z_U - w_s,s-1) & \text{otherwise} \end{cases}$$
$$(12)$$

3.3 Reliability Bounds for Dependent Component Failures Based on Minimal Paths and Cuts

In a great many situations with failure dependency, (nonnegative) association can be assumed for component states. Association means that the random variables $x_1, x_2, \ldots \ldots x_n$, which represent the functioning states of the components in a system, have nonnegative covariance. In other words, given a component is failed, the failure possibilities of some other components tend to be higher. Consequently, the probability that a set of components are in a specific state (functioning or failed) is then greater than or equal to the multiplication of the probabilities for each component to be in that state. The (nonnegative) association assumption seems to cover most cases of s-dependent failures. Examples of negative failure dependence are not often observed in reliability

applications. With (nonnegative) association assumed, the min-max bound can be used to bound the system reliability:

$$\min_{1 \leq i \leq r} \prod_{j \epsilon MP_i} p_j \leq R(k,n) \leq \min_{1 \leq i \leq l}\{1 - \prod_{j \epsilon MC_i}(1-p_j)\} \qquad (13)$$

Where MC_i denote all the minimal cuts of the system, and MP_i denote all the minimal paths. When we cannot assume nonnegative association/covariance among component states, a lower bound of system reliability can be derived for two-stage weighted-k-out-of-n systems using a method similar to that used by Cheng and Mon (1993).

$$1 - \min_{1 \leq i \leq r}\left(|MP_i| - \sum_{j \epsilon MP_i} p_j\right) \leq R(k,n) \qquad (14)$$

4. Examples
In this section, the problem introduced in Section II-B-1 on project management, and that in Section II-B-2 on shortest path problem, will be investigated based on the algorithms developed in Section III.

4.1 Project Management Example
In the Section II-B-1, the project management example is formulated as a SW-weighted-$[<1,6,11>,<0,3,6>]^T$-out-of-5:F system with failure conditions specified in (5). Using (8), the system reliability can be evaluated as:

$R([<1,6,11>,<0,3,6>]^T,1) = 1$ (because $\beta_1 = [1,1]^T$ and $\emptyset_1(\beta_1) = 1.1 = 1$)

$R([<1,6,11>,<0,0,0>]^T,1) = 0$ (because $\beta_0 = [1,0]^T$ and $\emptyset_1(\beta_0) = 1.0 = 0$)

$R([<1,6,11>,<0,3,6>]^T,2) = (1-p)R([<1,6,11>,<0,0,0>]^T,1)$
$$+p.R([<1,6,11>,<0,3,6>]^T,1)$$
$$= (1-p).0 + p.1 = p$$
$$R([<2,4,6>,<0,0,0>]^T,1) = 0$$
$$R([<2,4,6>,<0,1,2>]^T,1) = 1$$
$$R([<2,4,6>,<0,1,2>]^T,2) = (1-p)R([<2,4,6>,<0,0,0>]^T,1)$$
$$+p.R([<2,4,6>,<0,1,2>]^T,1)$$
$$= (1-p).0 + p.1 = p$$
$$R([<1,6,11>,<0,3,6>]^T,3) = (1-p)R([<2,4,6>,<0,1,2>]^T,2)$$
$$+p.R([<1,6,11>,<0,3,6>]^T,2)$$
$$= (1-p).p + p.p = p$$
$$R([<0,2,4>,<0,0,0>]^T,2) = 0$$
$$R([<2,4,6>,<0,1,2>]^T,3) = (1-p)R([<0,2,4>,<0,0,0>]^T,2)$$
$$+pR([<2,4,6>,<0,1,2>]^T,2)$$
$$= (1-p).0 + p.p = p^2$$
$$R([<1,6,11>,<0,3,6>]^T,4) = (1-p).R([<2,4,6>,<0,1,2>]^T,3)$$
$$+p.R([<2,4,6>,<0,1,2>]^T,2)$$
$$= 2(1-p).p^2 + p.p^2 = 2p^2 - p^3$$
$$R([<0,2,4>,<0,0,0>]^T,4) = 0$$
$$R([<1,6,11>,<0,3,6>]^T,5) = (1-p)R([<0,2,4>,<0,0,0>]^T,4)$$
$$+p \cdot R([<1,6,11>,<0,3,6>]^T,4)$$
$$= 2(1-p) \cdot 0 + p \cdot 2p^2 - p^3 = 2p^3 - p^4$$

The system reliability, the probability that the project can still meet the deadline with the risk of delays, is

$$R([< 1,6,11 >, < 0,3,6 >]^T, 5) = 2p^3 - p^4 \tag{15}$$

4.2 Example of Shortest Path Problem

In Section II-B-2, the shortest path example is formulated as a PW- $[< 0,5,40 >, < 0,15,50 >]^T$-out-of-6:F system with subsystem failure conditions specified in (6). Based on either (9) or (11), the set of minimal cuts for each subsystem can be derived as

$$MCS(k_1 =< 0,5,40 >, n = 6) = \bar{x}_{OB} + \bar{x}_{BC} + \bar{x}_{CT}$$
$$MCS(k_1 =< 0,15,50 >, n = 6) = \bar{x}_{CT} + \bar{x}_{OA}\bar{x}_{AC}$$

From Result 4, the set of minimal cuts for the whole system can be obtained as

$$MCS([< 0,5,40 >, < 0,15,50 >]^T, 6)$$
$$= S_C(\overline{MCS(k_1 =< 0,5,40 >, 6)}, \overline{MCS(k_2 =< 0,15,50 >, 6)})$$
$$= S_C(MCS(k_1 =< 0,5,40 >, 6) \cdot MCS(k_2 =< 0,15,50 >, 6))$$
$$= (\bar{x}_{OB} + \bar{x}_{BC} + \bar{x}_{CT})(\bar{x}_{CT} + \bar{x}_{OA}\bar{x}_{AC}) = (\bar{x}_{CT} + \bar{x}_{OB}\bar{x}_{OA}\bar{x}_{AC} + \bar{x}_{BC}\bar{x}_{OA}\bar{x}_{AC})$$

Based on either (10) or (12), the set of minimal paths for each subsystem can be derived as
$$MPS(k_1 =< 0,5,40 >, n = 6) = x_{OB}x_{BC}x_{AC}$$
$$MPS(k_1 =< 0,15,50 >, n = 6) = x_{CT}x_{OA} + x_{CT}x_{AC}$$

From Result 4, the set of minimal paths for the whole system is
$$MPS([< 0,5,40 >, < 0,15,50 >]^T, 6)$$
$$= S_P(MPS(k_1 =< 0,5,40 >, 6), MPS(k_2 =< 0,15,50 >, 6))$$
$$= MPS(k_1 =< 0,5,40 >, 6) + MPS(k_2 =< 0,15,50 >, 6)$$
$$= x_{OB} \cdot x_{BC} \cdot x_{CT} + x_{CT} \cdot x_{OA} + x_{CT} \cdot x_{AC}$$

Using the minimal paths, and the inclusion-exclusion method, the system reliability can be evaluated as

$$R([< 0,5,40 >, < 0,15,50 >]^T, 6)$$
$$= \Pr(MP_1 \cup MP_2 \cup MP_3)$$
$$= \Pr(MP_1) + Pr(MP_2) + Pr(MP_3) - \Pr(MP_1 \cap MP_2) - \Pr(MP_1 \cap MP_3)$$
$$- \Pr(MP_2 \cap MP_3) + \Pr(MP_1 \cap MP_2 \cap MP_3)$$
$$= 2 \cdot p^2 - 2 \cdot p^4 + p^5$$

5. Conclusion

The system reliability, the probability that the project can meet the deadline with the risk of delays, for the project management problem is evaluated as

$$R([< 1,6,11 >, < 0,3,6 >]^T, 5) = 2p^3 - p^4$$

The system reliability for shortest path problem is evaluated as

$$R([< 0,5,40 >, < 0,15,50 >]^T, 6) = 2 \cdot p^2 - 2 \cdot p^4 + p^5$$

The bounds of system reliability when activity delays are s-dependent with (nonnegative) association can be obtained based on (13) as

$$p^2 \leq \quad R([< 0,5,40 >, < 0,15,50 >]^T, 6) \leq p \qquad (16)$$

When nonnegative association between activity delays cannot be assumed, the lower bound of system reliability based on (14) is

$$1 - 2(1 - p) \leq \quad R([< 0,5,40 >, < 0,15,50 >]^T, 6) \qquad (17)$$

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