International Journal of Mathematical, Engineering and Management Sciences
Vol. 4, No. 1, 191–198, 2019
https://dx.doi.org/10.33889/IJMEMS.2019.4.1-017

Structural Mathematical Changes in Theoretical Music in the Early Renaissance

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(Received June 25, 2018; Accepted November 8, 2018)

Abstract
This paper investigates interrelationships between theories of ratio and theoretical music originating in Antiquity**,** with special attention to the early Renaissance in Europe. It considers evidence from different theories of ratio**,** stressing tendencies in mathematical treatment involving such concepts, which show similarities with music in structure and⁄or terminology and also examines their reflection in music in the period in question. It could be said that from later times and in particular in Euclid's *Elements* Book V, ratios were seen as musical intervals generalized, whose nature was very different from numbers or magnitudes. The change is from operations with ratios related to contiguous musical intervals to theories admitting compounding ratios in general sense with an essentially arithmetic character, manifested for instance in the idea that a ratio is equal to a number. It will be investigated here some attributes of these competing theories of ratios, as well as its close relationships with theoretical music up to the 16th century.

Keywords- Structural changes, Theoretical music, Ratios, Mathematical changes, Mathematics/music relation

1. Introduction

In order to investigate the interrelationships between theories of ratio and theoretical music, his article analyses the existence of competing theories of ratio throughout medieval and modern times, and the historical development of such concepts which would eventually bring about the arithmetization of theories of ratio. Receiving contributions from the Latin and Arabic traditions culminating with the confluence of such traditions, this complex process already began in the Ancient Greece, developed throughout the Middle Ages until the Renaissance, in which it has undergone great acceleration.

In this context, it is worthwhile to emphasize a certain structural instability in the mathematical treatment of the concept of ratio, often present in the treatises approaching to the liberal arts. Characterized by procedures not well demarked in the treatment of ratios – sometimes arithmetical, sometimes geometric/musical – such an instability makes possible to understand some relevant medieval and Renaissance contributions to the process, which lead for shaping theories of the ratio in mathematics.

One must consider the Campanus Latin translation of the Elements from the 13th century, an important contribution to the tension and indefiniteness in the history of theories of ratio, since Campanus insered the concept "denominatio" for ratio to definition 5 of book V concerning proportionality of ratios giving it an arithmetical meaning not presented in the original Euclid. In the context of theory of music, the division of the tone must also be considered since it shaped the conception of ratio throughout the history, contributing also to the discussions concerning arithmetization of ratio**.**

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It discusses some historical settings that led to the development of the theories of ratio in the late Middle Ages and to the emergence of an arithmetical theory of ratio in the medieval Latin tradition. It will be used here the expression "arithmetization of the theory of ratio" to denote changes undergone by ratio mainly in the late Middle Ages and the Renaissance, which lost its geometric character to assume a semantically distinct yet structurally similar arithmetic one. In this sense, ratio lost the meaning of a comparison between two magnitudes of the same nature in order to be identified with number, became defined by a division and was now identified with the quotient of two magnitudes, compounding of ratios turned into a multiplication of ratios, and proportions between ratios an equality of numbers.

Our understanding of medieval mathematics was increased considerably through the analyses of many authors, who studied medieval ratio theory. Among them, it is worth to mention Sylla, Busard, Evans, Folkerts, Hoyrup, Lorch and North, which provided important information on the medieval theory of ratios. Grant researched on Oresme and the concept of fractional exponents, while Molland has concentrated on Bradwardine (Murdoch, 1963). Murdoch wrote a general survey on this subject, as well as a study concerning the introduction of "denomination" into discourse (Murdoch, 1963).

In this context, it is important to take into account the introduction of the term *denominatio* in the Campanus Arabic Latin translation of Euclid"s in the 13th century. Campanus gave an arithmetical meaning to definition 5 in Book V, introducing the concept "*denominatio*" by dealing with proportionality of ratios, which was not contained in the original text. The medieval conception of ratio had been inherited from both the classical Greek geometrical tradition and the later Greek arithmetical tradition, but Campanus, in his translation, did not distinguish the two Greek traditions, and substituted "denomination" by ratio, considering probably equivalent to fractions, anachronistically speaking.

Campanus" Latin translation of Euclid"s *Elements* is generally regarded as the main source for 14th century ratio theory, especially as presented by Bradwardine and Oresme. This theory used an arithmetical vocabulary that did not derive from the geometrical ratio theory expounded in Euclid"s Book V, but rather from a number of different sources3. Oresme used the term "denomination" and represented the ratio of ratios, "*proportio proportionium*", with ratios in exponents, a procedure that allowed for the division of an arbitrary ratio by an arbitrary number, and indirectly conferred to ratio a continuous feature.

In the Middle Ages, a different terminology for ratio dominated, in which this mathematical concept was usually translated in Latin as *proportio*, instead of *ratio*; this was reconsidered since the beginning of the 16th century, with the new translations of the Greek classics, however without immediately displacing its medieval usage. For more detailed discussions about the changes in the terminology of ratio from classical and medieval times to the early modern period, see Wilbur R. Knorr (Knorr, 1994).

Although medieval mathematicians referred to Book V, which deals with ratios with reard to magnitudes, the definition for the equality of ratios given in Campanus" edition of Euclid is not the classical definition 5 in Book V ascribed to Eudoxus, but one which refers to denominations of ratios, which does not appear in Heath"s edition (Euclid, 1956). Unlike Book *V* of the *Elements*, Book VII, in its original form, contains the arithmetical treatment of ratios, which is not applicable to continuous magnitudes, and thus nor to the treatment of incommensurables.

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Basically, the arithmetical theory of ratio manifested in Campanus" version of Euclid"s *Elements*, equipped with the term "*denominatio*" related to the medieval arithmetics, provided the fundament for the late medieval mathematical understanding of ratios.

A crucial question is why Campanus used arithmetical definitions in his translation, when editions containing the original definitions in Book V were available (Murdoch, 1963), a fact that increases the tension between different theories of ratios at that time, leading to an attempt of demarcation between such theories, and eventually to the study of the emergence of an "arithmetical theory of ratios" within the *arithmos* tradition of Euclid.

The complex process of arithmetization of the theories of ratio goes back to Ancient Greece, developing throughout the Middle Ages up the Renaissance, receiving important contributions from the Latin and Arabic traditions, to culminate with the confluence of these traditions – consequently, attended a great intensification of this process in the Renaissance.

Up to the Renaissance, the use of ratios did not have a well-demarcated structure, but sometimes presented arithmetical features, sometimes geometrical features or a combination of both. Such structural differences, which kept up with the concepts of ratio and proportion since Antiquity, corresponded to underlying theoretical treatises not only on mathematics, but also on near disciplines like theoretical music.

2. Two Traditions of Theories of Ratio

In order to comprehend different theories of ratios, it is important to contextualize the idea of compounding ratios, which is crucial for the understanding of the process of arithmetization. This idea is implicitly defined in proposition 23, Book VI of Euclid's *Elements*, which says that "equiangular parallelograms have to one another the ratio compounded of the ratios of their sides" (Heath, 1956). Euclid had to compound two ratios: BC:CG and DC:CE, in order to show it, which he changed to proportional ratios K:L and L:M, respectively, having L in common, before doing the operation. Thus, the compounding ratios according to the classical fashion consisted in taking ratios of the type $a:b$ with $b:c$ to produce $a:c$. This allows for the repetition of this process with $c:d$ and so on. So, considering some ratios to be compounded, the second term of a ratio must equal the first term of the subsequent ratio. Therefore, to compound the ratios $a:b$ with $c:d$, it was necessary to find a magnitude e so that $c:d$ would be proportional to $b:e$, and the resulting compounded ratio would be $a:e$.

The idea of compounding is relevant for the emergence of different structures underlying theories of ratios, an argument corroborated by the fact that Szabo also made use of this concept while raising questions attempting to demonstratre that the pre-Eudoxan theory of proportions developed firstly as an inheritance of the Pythagorean music theory (Szabo, 1978). Compounding ratios are semantically similar o musical intervals, for it is structurally similar to composing contiguous intervals with the monochord. The expression "compose" is used here to express the music process in which 2 musical intervals are taken, so that the highest note of the first is equal to the lowest note of the second, so as to produce a new interval whose lowest note is the lowest note of the first, and its highest note is the highest note of the second. Such an operation can be applied recursively.

For compounding intervals with the monochord, one must begin a given interval from the point one reached in the previous one, which means the common mathematical terms between subsequent ratios in compounding them. Thus, compounding ratios in mathematics corresponds to the composition of musical intervals in music, and vice versa. There is no mathematical sense in defining compounding ratios in such a way, and probably one would not define it so, unless there was firstly a meaning in another context, in the case, music, whereby one understands what is otherwise a purely mathematical phenomenon, as the composing of contiguous intervals.

Another point important for the characterization of the different theories of ratio is that the identification of ratios with fractions relates to the notion of incommensurable magnitudes. If ratios are generally identified with fractions, then ratios between a side and the hypotenuse of an isosceles right-angled triangle become inexpressible. The solution for this situation is either to use approximations or to accept perfect decimal fractions into the domain of numbers. The latter appeared sometime in the late 16th century. However, such situation concerning irrational ratios lacked consensus among medieval, Renaissance and early modern mathematicians, a fact that has been highlighted by recent historians.

As it was already mentioned, the medieval concept of ratio was a heritage from both the Greek classic geometric and the Late Greek arithmetical tradition. The former derived from definition 5 in Book V of Euclid‟s *Elements*, while the latter seems to appear for the first time in the transversal problem of Menelaus (c. 70-130), who compounded ratios without the constraints mentioned before, namely as a multiplication, and then with Theon of Alexandria (c.335-405) (Grosholz, 1987), who inserted interpolations in definition 5 of *Elements* Book VI (Heath, 1956, 120), distorting the original Euclidean sense of compounding ratios, approaching the idea of compounding to the multiplication.

Up to the Renaissance, the treatment of ratios had no clear and well-defined structure. Some traditions had mainly arithmetical features, others geometrical and musical ones, whereas still others incorporated both these tendencies. Sylla discusses the confusion over the geometrical and the arithmetical traditions of ratios, showing how both "strangely mingled" within the context of compounding and multiplying (Sylla, 1984). She categorizes two traditions within the Greek and medieval treatment of ratios, one associated with theoretical mathematics, music, and physics, particularly found in Bradwardine‟s *De proportionibus*; and another associated with practical calculations using ratios and with astronomy (Sylla, 1984). She argues that, "These two traditions may not encompass all ancient and medieval concepts of ratio. Neither were these traditions always separate - in fact, they were often strangely mingled. Nevertheless, they represent two poles of the ways in which ratios and the operations on ratios could be treated" (Sylla, 1984).

Drake further suggests that the theory of ratios in the Middle Ages made use of an elaborate vocabulary that was not originated in Euclid's *Elements* (Drake, 1973). *Alt*hough Bradwardine mentions Euclid‟s Book V, which contains the theory of ratios and proportionality of magnitudes, it is, and the definition presented in the *Treatise* (Crosby, 1955*)* for the equality of ratios is not the Eudoxian definition in Book V, but one making use of "denominations" of ratios.

Such a term, although not present in Euclid, appeared in the definitions of Book VII in the standard medieval Euclid, the one by Campanus. Book VII starts a special treatment of numerical ratios and proportionality independently from the general theory of magnitudes presented in Book V. Bradwardine„s *De proportionibus* used in its fundament the theory of ratios embodied in Book VII, as embellished in the medieval version and supplemented by ancient arithmetical ratio vocabulary. (Crosby, 1955).Thus, Drake suggests that, "Campanus‟ Book VII comprised a complete definitional structure and terminology for the theory of ratio typical mathematics in Middle Ages and established its definitional base without making use of any conceptual need for references to *Book V*" (Crosby, 1955).

The notion of "denominations" has been addressed by several scholars; for example, Murdoch discussed it in the case of Campanus (Murdoch, 1968). However, there still seems to be some disagreement about what it actually means. Strictly speaking, it comes from rhetoric, where it is connected with metonymy, i.e. the substitution of an attribute for the thing named, for example, "crown" for "king". A more mathematical meaning is that of the unit of a quantity, for example, "*metre"* is the denomination of the quantity 5 meters. On the other hand, scholars seem undecided whether "denomination" in medieval mathematics referred to a ratio as a fraction with its lowest terms, or to the quotient concerning such a fraction. If the former, then this is little more than a simplified Boethian ratio terminology, where 2:4 is *dupla* 1:2. If the latter, then decimalization occurred earlier than what has been thought. Interestingly, Molland, who has discussed it in the context of Bradwardine, has noticed that: "Richard of Wallingford, who followed Campanus closely, came near to identifying a ratio with its denomination (Molland, 1968). Nicole Oresme exhibited the denominations of rational ratios by numbers or numbers and fractions, but he also took into account irrational ratios, and there the matter was not so simple".

There seems to be an important source of misunderstanding here. From Euclid"s Book VII, we have unit fractions such as 1/3 and ratios of integers such as 1:3. It seems that medieval mathematicians misread or re-interpreted Euclid in taking the latter as being rational numbers such as 1/3. Further, this may be a source for the new arithmetical theory of ratios, which Sylla discusses.

3. Campanus and Book V of the Elements

In general, historians consider that: "General acquaintance with Euclid in Europe was encouraged especially by a version of the *Elements* made in the 1290s by Campanus of Novara; it was also the first to be printed, in 1482. However, Campanus had elaborated upon the translation made by Adelard of Bath in the 12th century, so that the text was more garbled than its earliest readers realized" (Grattan-Guiness, 1997).

This understanding probably derives from a comment made by Heath on Book V, definition 5 to the effect that Campanus had a confused understanding of Euclid. According to Heath, "From the revival of learning in Europe onwards, the Euclidean definition of proportion was the subject of much criticism. Campanus had failed to understand it, had in fact misinterpreted it all together, and he may have misled others such as Ramus (1515-72)".

Murdoch supports this view by suggesting that, "Campanus, in a general comment to the definitions of Book V of the *Elements*, denies its application to that infinity of irrational proportions, for, he asserts, their denominations are not knowable. Moreover, he adds, Book V does include irrationals in its domain, and hence Euclid was forced to abandon – unlike the arithmetician – the definition of equal proportions by equal denominations".

Molland initially agreed with Heath and Murdoch, and stated that,"This definition [V def. 5], partly as a result of the obscurity of the translation, was not understood in the Middle Ages, and in his version of the *Elements* Campanus flounders hopelessly in search of suitable general criteria of the equality of ratios".

However, when this author reconsidered the matter ten years later, he asserted that, "Campanus's explication of the Eudoxian criterion of equality often seems garbled [...] I am not convinced that he completely misunderstood it, for in his comment on the definition of greater ratio we have the following: The ratio of the first of four quantities to the second is never greater than that of the third to the fourth, unless some equimultiples of the first and the third may be found, such that when they are related to some equimultiples of the second and the fourth, the multiple of the first will be found to exceed the multiple of the second, but the multiple of the third will not exceed the multiple of the fourth. And this can never be found unless the ratio of the first to the second be greater than the ratio of the third to the fourth, as we shall demonstrate below" (Molland, 1978).

This reconsideration raises questions concerning the intentionality of Campanus in displaying definition 5 of Book V in terms of denominations, which further raises questions about the need to arithmetize such concept in this context. And it still remains the question on why Campanus inserted arithmetical interpolations from Jordanus of Nemore into Euclid, when acceptable editions containing Book V, definition 5 were available. Whether Campanus misinterpreted Euclid, or made this purposefully, this fact does raise important questions concerning competing theories of ratios in the Middle Ages and on the emergence of the arithmetization of such theories in the late Middle Ages.

4. Equal Division of the Tone and Theories of Ratio Underlying Theoretical Music

Besides the scholars mentioned above, many theorists involved in the division of the tone indirectly shaped the understanding of ratio throughout the history of the discussions concerning arithmetization of such a concept. The equal division of the tone played an important role in the historical process of arithmetization of ratios. From the point of view of mathematics, dividing equally the tone 8:9 provides incommensurable ratios underlying musical intervals. It means mathematically to find x so that $8:x = x:9$; that results, anachronistically speaking, in irrational numbers, inconceivable in the Pythagorean musical system.

Attempts to divide the tone were made as early as in Antiquity, for instance, by Aristoxenus (4th century B.C.). In contrast with the Pythagoreans, who advocated that musical intervals could be properly measured and expressed only as mathematical ratios, Aristoxenus asserted rather that the ear was the only criterion for musical phenomena (Winnington-Ingram, 1995). In preferring geometry to arithmetic to solve problems involving interelationships between musical pitches, Aristoxenus sustained, also against the Pythagoreans, the possibility of dividing the tones into many equal parts, thinking musical intervals– and indirectly, ratios – as one- dimensional and continuous magnitudes, making possible in this way their division.

This idea provoked many reactions expressed, for example, in the *Sectio Canonis* (Barbera, 1991), which in Antiquity was attributed to Euclid, and afterwards in Boethius's *De institutione musica*, in the early medieval era30, which gave birth to a strong Pythagorean tradition in theory of music throughout the Middle Ages. Following the Platonic- Pythagorean tradition, a great part of theorists of medieval music sustained that the equal division of the tone was not possible, which would lead from the mathematical point of view to incommensurable ratios underlying musical intervals. Gradually, the need to carry out the temperament gave birth to different attempts to divide the tone.

Goldman suggests that Nicholas Cusanus (1401-1464) was the first to assert in *Idiota de Mente* that the construction of the musical half-tone is possible by *geometric division* of the whole-tone and, hence, would be defined by an irrational number (Goldman, As a consequence, Cusanus would have been the first to formulate a concept that set the foundation for the equal temperament proposed in the work of the theorists Faber Stapulensis (1455-1537) and Franchino Gafurius (1451-1524), which was published half a century later (Goldman, 1989).

Nevertheless, one can find in the Byzantine tradition, Michael Psellus (1018-1078), who suggested in his *Liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et] astronomia* (Psellus, 1556), a geometrical division of the tone, whose underlying conception implies an understanding of ratio as a continuous magnitude. Also concerning the division of the tone before Cusanus, Marchetus of Padua (1274? --?) proposed, in his *Lucidarium in Arte Musice Planae,* written in 1317/1318, the division of the tone into five equal parts34, an innovation of extraordinary interest which made Marchettus the first in the Latin tradition to propose such a division, but without any mathematical approach.

At the end of the 15th and beginning of the 16th century, Erasmus Horicius, one of the German humanists gifted in musical matters, wrote his *Musica* (Erasmus, 1500), where he suggested a division of the whole tone. Erasmus stated that any part of any super-particular ratio could be obtained, in particular the half of 8:9, which corresponds to dividing the whole tone equally. Based theoretically on several propositions of geometry, and unusually shaped on axiomatic Euclidean style, his *Musica* considered ratio a continuous quantity, announcing maybe what would emerge as a truly geometric tradition in the use of ratios in the contexts of theory of music during the 16th century. The attempts to dividing the tone led to a conception of ratio as a continuous quantity in theoretical music, and consequently, to the arithmetization of theories of ratios both in musical and mathematical contexts.

5. Concluding Remarks

This survey discussed some facts about the process related with the development of theories of ratios and the arithmetization of ratios in the late Middle Ages and the Renaissance, presenting some evidence for the co-existence of arithmetical and geometrical traditions in the treatment of ratios in this period. It was emphasized the use of ratios in musical contexts as an important factor for the permanence of the classical tradition, while at the same time giving rise, through the problem of the division of the tone, to the use of the arithmetical tradition in this context.

This complexity was due to immensurable factors that polarized sometimes the use of ratios in the classical tradition, sometimes in the arithmetical tradition, a process which was extended practically until the 16th century, when conflicts between these two tendencies resulted in the disappearance of the tradition concerning the compounding ratios, and the consolidation of the arithmetical theory of ratios. During the 1500s, the process of arithmetization accelerated, and in the 17th century, the arithmetical theory of ratio became the dominant one.

Conflict of Interest

The author confirm that this article contents have no conflict of interest.

Acknowledgement

The author would like to express their sincere thanks to the referee and for their valuable suggestions towards to the improvement of the paper.

International Journal of Mathematical, Engineering and Management Sciences
Vol. 4, No. 1, 191–198, 2019
https://dx.doi.org/10.33889/IJMEMS.2019.4.1-017

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