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# Digital Circuit Design Utilizing Equation Solving over 'Big' Boolean Algebras

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**Abstract**
A task frequently encountered in digital circuit design is the solution of a two-valued Boolean equation of the form $h(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = 1$, where $h$: $B_2^{k+m+n} \to B_2$ and $\boldsymbol{X},\boldsymbol{Y}$, and $\boldsymbol{Z}$ are binary vectors of lengths $k$, $m$, and $n$, representing inputs, intermediary values, and outputs, respectively. The *resultant* of the *suppression* of the variables $\boldsymbol{Y}$ from this equation could be written in the form $g(\boldsymbol{X},\boldsymbol{Z}) = 1$ where $g$: $B_2^{k+n} \to B_2$. Typically, one needs to solve for $\boldsymbol{Z}$ in terms of $\boldsymbol{X}$, and hence it is unavoidable to resort to 'big' Boolean algebras which are finite (atomic) Boolean algebras larger than the two-valued Boolean algebra. This is done by reinterpreting the aforementioned $g(\boldsymbol{X},\boldsymbol{Z})$ as $g(\boldsymbol{Z})$: $B_{2^K}^n \to B_{2^K}$, where $B_{2^K}$ is the free Boolean algebra $FB(X_1, X_2 \ldots \ldots . X_k)$, which has $K = 2^k$ atoms, and $2^K$ elemnets. This paper describes how to unify many digital specifications into a single Boolean equation, suppress unwanted intermediary variables $\boldsymbol{Y}$, and solve the equation $g(\boldsymbol{Z}) = 1$ for outputs $\boldsymbol{Z}$ (in terms of inputs $\boldsymbol{X}$) in the absence of any information about $\boldsymbol{Y}$. The paper uses a novel method for obtaining the parametric general solutions of the 'big' Boolean equation $g(\boldsymbol{Z}) = 1$. The parameters used do not belong to $B_{2^K}$ but they belong to the two-valued Boolean algebra $B_2$, also known as the switching algebra or propositional algebra. To achieve this, we have to use distinct independent parameters for each asserted atom in the Boole-Shannon expansion of $g(\boldsymbol{Z})$. The concepts and methods introduced herein are demonsrated *via* several detailed examples, which cover the most prominent type among basic problems of digital circuit design.

**Keywords-** Digital design, Suppression of variables, 'Big' Boolean algebras, Boolean-equation solving, Parametric solutions.

## 1. Introduction

In a seminal work, done more than half a century ago, Ledley (1959, 1960) posed three 'elementary problems of digital circuit design' inspired by the arrangement in Fig. 1, which entails five quantities $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{s}$ and $\boldsymbol{t}$ that belong to $B_2^k, B_2^m, B_2^n, B_2^l$, and $B_2^l$, respectively. This arrangement includes a 'parent' combinational network $C$ of two (vectorial) inputs $\boldsymbol{X}$ and $\boldsymbol{Y}$ and a vectorial output $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$. Network $C$ consists of two subnetworks $A$ and $B$, where subnetwork $A$ has the single (vectorial) input $\boldsymbol{X}$ and the (vectorial) output $\boldsymbol{Z}(\boldsymbol{X})$, while network $B$ has the two vectorial inputs $\boldsymbol{Z}(\boldsymbol{X})$ and $\boldsymbol{Y}$ and the (vectorial) output $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$, which is exactly the same as the (vectorial) output $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$ of network $C$. The above arrangement involves three vectorial Boolean functions, namely $\boldsymbol{Z}(\boldsymbol{X})$, $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$ and $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$. Three problems arise when one utilizes the information that $\boldsymbol{s}(\boldsymbol{Z}(\boldsymbol{X}),\boldsymbol{Y})$ and $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$ are equal, together with knowledge of two of the three functions $\boldsymbol{Z}(\boldsymbol{X})$, $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$ and $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$ in order to deduce the third function. Therefore, one obtains three distinct problems, *viz.*

1. Type-1 problem: Given $\boldsymbol{Z}(\boldsymbol{X})$ and $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$, find $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$.
2. Type-2 problem: Given $\boldsymbol{Z}(\boldsymbol{X})$ and $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$, find $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$.
3. Type-3 problem: Given $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$ and $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$, find $\boldsymbol{Z}(\boldsymbol{X})$.

Table 1 shows taxonomy of these three problems, together with an outline on how to solve them according to the scheme set by Ledley (1959, 1960) in his seminal work. It is obvious that the Type-1 Problem necessitates only direct substitution and hence does not warrant any further consideration. The Type-2 Problem is, in fact, the inverse problem of logic. This problem was treated extensively by nineteen-century logicians such as Jevons (1872, 1874), Venn (1894) and Poretsky (1898). Though interest in this problem faded away for more than half a century, it witnessed a revival at the hands of pioneers of modern digital design including Ledley (1960), Bell (1968), Cerny and Marin (1974, 1977), Cerny (1976) and Brown (1974, 1975a, 1975b), and it remains an essential element in contemporary digital design practice (Brown, 1990, 2003; Rushdi and Ba-Rukab, 2003; Steinbach and Posthoff, 2003; Baneres et al., 2009; Brown and Vranesic, 2014; Knodel et al., 2014; Rushdi, 2018b; Rushdi and Ahmad, 2018). Ledley devoted his 1959 paper entirely to the Type-3 Problem, and further illustrated it by several examples in his 1960 book, and (admirably) managed to handle it *via* very tedious and definitely outdated techniques. In this paper, we revisit the Type-3 Problem using four major enhancements inspired by certain new techniques, which became available only very recently, namely:

1. Instead of Ledley's representation of Boolean variables *via* (the now outdated) 'black-box' designation numbers and numerical Boolean matrices, we use a transparent algebraic representation all throughout, possibly aided by a pictorial interpretation *via* natural maps (variable-entered Karnaugh maps) (Rushdi, 1987, 2001, 2004, 2012, 2017; Rushdi and Ahmad, 2016; Rushdi and Al-Yahya, 2000, 2001; Rushdi and Amashah, 2010, 2011, 2012; Rushdi and Albarakati, 2014; Rushdi and Ba-Rukab, 2017; Rushdi, 2018a; Ahmad and Rushdi, 2018).

2. We do not follow exactly the scheme outlined in Table 1. We do not separate our requirements into an antecedence requirement ($\boldsymbol{s} \rightarrow \boldsymbol{t}$) and a consequence requirement ($\boldsymbol{t} \rightarrow \boldsymbol{s}$), but instead combine these two requirements into a single requirement ($\boldsymbol{s} \leftrightarrow \boldsymbol{t}$) = ($\boldsymbol{s} \odot \boldsymbol{t}$). Therefore, we end up with a single equation to solve

$$h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = \bigwedge_{i=1} (s_i(\boldsymbol{Z}, \boldsymbol{Y}) \odot \ t_i(\boldsymbol{X}, \boldsymbol{Y})) = 1.$$

3. We do not solve the aforementioned equation to obtain values of independent variables $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ in $B_2^k, B_2^m, B_2^n$, respectively, but instead we solve for the dependent variables $\boldsymbol{Z}$ in terms of the independent variables $\boldsymbol{X}$, and $\boldsymbol{Y}$. This means that instead of viewing the function $h$ above as $h: B_2^{k+m+n} \rightarrow B_2$, we treat it as $h: B^n \rightarrow B$, where $B$ is a 'big' Boolean algebra equivalent to the free Boolean algebra $FB(\boldsymbol{X}, \boldsymbol{Y})$. As such, the roles of $\boldsymbol{X}$ and $\boldsymbol{Y}$ as variables is relegated to those of generators. Since $B$ has $(k + m)$ scalar generators, it has $2^{(k+m)}$ atoms and hence $2^{2^{(k+m)}}$ elements. The task of solving $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = 1$ over such an algebra is facilitated by a two-century effort that culminated recently in a technique for a compact listing of all particular solutions of any 'big' Boolean equation (Rushdi and Ahmad, 2017a, 2017b).

4. We avoid the task of extensive search among the solutions of $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = 1$ for those solutions that interrelate $\boldsymbol{Z}$ and $\boldsymbol{X}$ independently of $\boldsymbol{Y}$. Instead, we employ a technique, recently developed by Brown (2011), that suppresses the undesired variables $\boldsymbol{Y}$ in the equation $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = 1$, replacing that equation by another $g(\boldsymbol{X}, \boldsymbol{Z}) = 1$, where $g$ is not directly interpreted as $g: B_2^{k+n} \rightarrow B_2$ but is interpreted as $g: B_{2^K}^n \rightarrow B_{2^K}$, where $B_{2^K}$ is a 'big' Boolean algebra of $K = 2^k$ atoms and $2^K$ elements constructed as $FB(\boldsymbol{X}) = FB(X_1, X_2 \ldots \ldots . X_k)$.

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According to Brown's technique the equation $g(\boldsymbol{X}, \boldsymbol{Z}) = 1$ is constructed such as to have exactly all the solutions for $\boldsymbol{Z}$ in terms of $\boldsymbol{X}$ that are independent of $\boldsymbol{Y}$.

The organization of the remainder of this paper is as follows. Section 2 reviews our method for handling Ledley's Type-3 Problem of digital design. The section explains how to unify given digital specifications (such as the $\boldsymbol{s} = \boldsymbol{t}$ one) into a single Boolean equation. It also summarizes the method of suppression of variables, derives parametric solutions of any Boolean equation such that they can be recast into a very compact listing of all particular solutions, which enables one to easily locate particular solutions of certain desirable features. Section 3 illustrates the mathematical details of our proposed method by applying it to five examples of Ledley (1959, 1960). This section demonstrates clearly the dramatic advantages of the method including its conceptual clarity, high speed, avoidance of unwarranted cumbersome tasks, and better control on outcomes. These advantages do not pertain only to pedagogical issues, but also suggest some enhancements of contemporary practice of digital design. The paper is concluded in Section 4.

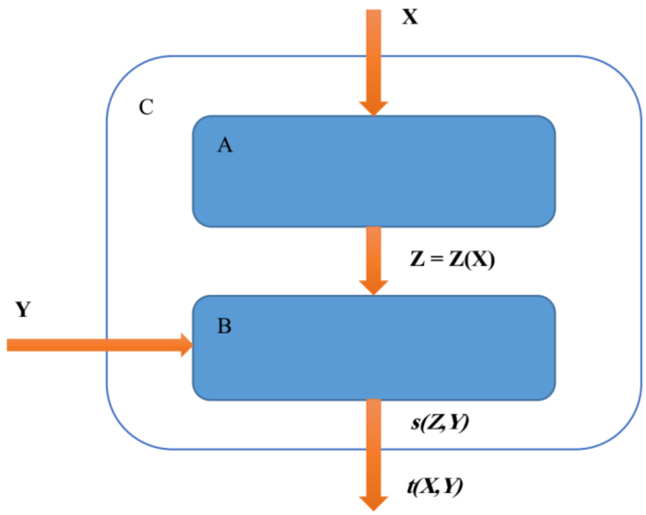


Fig. 1. An outline of digital networks pertaining to the three elementary problems of digital design posed by Ledley (1959, 1960)

Table 1. Taxonomy of Ledley's elementary problems of digital design

| Problem | Given | | Find | How |
|---|---|---|---|---|
| Type 1 | $\boldsymbol{Z}(\boldsymbol{X})$ | $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$ | $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$ | $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{s}(\boldsymbol{Z}(\boldsymbol{X}),\boldsymbol{Y})$, *i.e.*, just direct and straightforward substitution. |
| Type 2 | $\boldsymbol{Z}(\boldsymbol{X})$ | $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$ | $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$ | *Solve for $\boldsymbol{X}(\boldsymbol{Z})$, *i.e.*, perform logical inversion (solve an inverse problem of logic).<br>* $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y}) = \boldsymbol{t}(\boldsymbol{X}(\boldsymbol{Z}),\boldsymbol{Y})$. |
| Type 3 | $\boldsymbol{s}(\boldsymbol{Z},\boldsymbol{Y})$ | $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$ | $\boldsymbol{Z}(\boldsymbol{X})$ | *Find antecedence solutions $\boldsymbol{Z_a}(\boldsymbol{X})$ such that $\boldsymbol{s_a}(\boldsymbol{Z_a}(\boldsymbol{X}),\boldsymbol{Y}) \rightarrow \boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y})$.<br>*Find consequence solutions $\boldsymbol{Z_c}(\boldsymbol{X})$ such that $\boldsymbol{t}(\boldsymbol{X},\boldsymbol{Y}) \rightarrow \boldsymbol{s_c} = \boldsymbol{s}(\boldsymbol{Z_c}(\boldsymbol{X}),\boldsymbol{Y})$.<br>*Find the intersection of the sets of antecedence solutions and consequence solutions, so as to obtain solutions satisfying $((\boldsymbol{s} \rightarrow \boldsymbol{t}) \cap (\boldsymbol{t} \rightarrow \boldsymbol{s})) \leftrightarrow (\boldsymbol{s} \odot \boldsymbol{t}) \leftrightarrow \bigwedge_{i=1}^{n}(s_i \odot t_i) = 1$.<br>*Obtain solutions of this equation that specify values of $\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{Y}$ in $B_2$.<br>*Select solutions that relate $\boldsymbol{Z}$ to $\boldsymbol{X}$ independently of $\boldsymbol{Y}$. |

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## 2. Steps of the Method

### 2.1 Unifying Specifications into a Single Boolean Equation

Without loss of generality, let us assume that the digital system at hand is specified by the system of $l$ Boolean equations (suggested by Fig. 1) of the form

$$\boldsymbol{s} = \boldsymbol{t} \tag{1a}$$

or equivalently

$$s_i = t_i, \qquad 1 \leq i \leq l \tag{1b}$$

where $s_i = s_i(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ and $t_i = t_i(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, and occasionally we might have $t_i = 0$ or $t_i = 1$. The system (1) of $l$ scalar equations reduces to a single Boolean equation of the form

$$h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = 1 \tag{2a}$$

where

$$h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) \equiv \wedge_{i=1}^{l} (s_i \odot t_i) \tag{2b}$$

or of the form

$$r(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = 0 \tag{3a}$$

where

$$r(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = \bar{h}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) \equiv \vee_{i=1}^{l} (s_i \oplus t_i) \tag{3b}$$

The symbols $\wedge$, $\vee$, $\oplus$, and $\odot$ in Equations (2b) and (3b) depict the AND operator, the OR operator, the XOR (Exclusive-OR) operator and the XNOR (coincidence or equivalence) operator, respectively, defined as shown in Table 2. The relation between (2b) and 3(b) is just an expression of the two De' Morgan's laws. Note that the AND and OR operators are dual ones, while the XNOR and XOR operators are both complementary and dual ones.

Table 2. Definition of binary Boolean operators in equations (2b) and (3b)

| $s_i$ | $t_i$ | $s_i \wedge t_i$ | $s_i \vee t_i$ | $s_i \odot t_i$ | $s_i \oplus t_i$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 |

### 2.2 Suppression of Variables

Brown (2011) proved that the *resultant of suppression* of the variables $\boldsymbol{Y}$ from the Boolean equation (3a) (called the parent equation) is the derived Boolean equation

$$f(\boldsymbol{X}, \boldsymbol{Z}) = 0 \tag{4a}$$

where

$$f(\boldsymbol{X}, \boldsymbol{Z}) \equiv \vee_{\mathbf{A} \in \{0,1\}^m} \; r(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{Z}) \tag{4b}$$

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and that the solutions of the derived equation (4a) are exactly those of the parent equation (3a) that do not involve the suppressed variables $\boldsymbol{Y}$.

We will herein utilize the *dual* of the above result, namely that if we use (2a) instead of (3a) as a parent equation, then the resultant of suppression of the variables $\boldsymbol{Y}$ is now the complementary derived Boolean equation

$$g(\boldsymbol{X}, \boldsymbol{Z}) = 1 \tag{5a}$$

where

$$g(\boldsymbol{X}, \boldsymbol{Z}) \equiv \wedge_{\mathbf{A} \in \{0,1\}^m} \ h(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{Z}) \tag{5b}$$

and the solutions of the derived equation (5a) are exactly those of the parent equation (2a) that do not involve the suppressed variables $\boldsymbol{Y}$.

### 2.3 Derivation of Parametric Solutions

We seek solutions of the Boolean equation

$$g(\boldsymbol{X}, \boldsymbol{Z}) = 1 \tag{6}$$

where $g(\boldsymbol{X}, \boldsymbol{Z}): B_2^{k+n} \to B_2$, is a two-valued Boolean function of $k$ two-valued variables $\boldsymbol{X} = [\, X_1\, X_2 \ldots\ X_k\,]^T$ and $n$ two-valued variables $\boldsymbol{Z} = [\, Z_1\, Z_2 \ldots\ Z_n\,]^T$. However, we do not need a listing of binary solutions for $\boldsymbol{X}$ and $\boldsymbol{Z}$, but instead we want to express $\boldsymbol{Z}$ in terms of $\boldsymbol{X}$. This is a prominent case when the use of 'big' Boolean algebras (ones other than the two-valued algebras) is unavoidable. We view $g(\boldsymbol{X}, \boldsymbol{Z})$ as $g(\boldsymbol{X}; \boldsymbol{Z})$ or simply $g(\boldsymbol{Z})$ and rewrite (1) as

$$g(\boldsymbol{Z}) = 1 \tag{6a}$$

where $g(\boldsymbol{Z}): B_{2^K}^n \to B_{2^K}$, and $B_{2^K}$ is the free Boolean algebra $FB(X_1, X_2 \ldots \ldots X_k)$ with $K = 2^k$ atoms and $2^K$ elements. Now we express $g(\boldsymbol{Z})$ by its Minterm Canonical Form (MCF) (Brown, 1990)

$$g(\boldsymbol{Z}) \equiv \vee_{\mathbf{A} \in \{0,1\}^n} \ g(\boldsymbol{A})\, \boldsymbol{Z}^{\boldsymbol{A}} \tag{7}$$

For $\boldsymbol{Z} = [\, Z_1\, Z_2 \ldots\ Z_n\,]^T \in\ B_{2^K}^n,\ \boldsymbol{A} = [\, a_1\, a_2 \ldots\ a_n\,]^T\ \in\ \{0,1\}^n$, the symbol $\boldsymbol{Z}^{\boldsymbol{A}}$ is defined as

$$\boldsymbol{Z}^{\boldsymbol{A}} = {Z_1}^{a_1}\ {Z_2}^{a_2} \ldots\ {Z_n}^{a_n} \tag{8}$$

where ${Z_i}^{a_i}$ takes the value $\bar{Z}_i$ (complemented literal) if $a_i = 0$, and takes the value $Z_i$ (uncomplemented literal) if $a_i = 1$. For $\boldsymbol{A} \in \{0, 1\}^n$, the symbol $\boldsymbol{Z}^{\boldsymbol{A}}$ spans the minterms of $\boldsymbol{Z}$, which are the $2^n$ elementary or primitive products

$$\overline{Z_1}\, \overline{Z_2} \ldots \overline{Z_{n-1}}\, \overline{Z_n},\quad \overline{Z_1}\, \overline{Z_2} \ldots \overline{Z_{n-1}}\, Z_n,\quad \ldots.,\ Z_1\, Z_2 \ldots Z_{n-1}\, Z_n \tag{9}$$

The constant values $g(\boldsymbol{A})$ in equation (7) are elements of $B_{2^K}$ called the discriminants of $g(\boldsymbol{Z})$. These discriminants are the entries of the natural map of $g(\boldsymbol{Z})$ which has an input domain $\{0, 1\}^n \subseteq B_{2^K}^n$. The Boolean algebra $B_{2^K} = FB(X_1, X_2 \ldots \ldots X_k)$, has generators $X_i$ $(1 \leq\ i\ \leq\ k\ )$ which look like variables (In fact, they were originally our input variables before we changed their roles to generators). Therefore, we can accept the name assigned (for historical reasons) to the natural map of $g(\boldsymbol{Z})$, namely the name of the Variable-Entered Karnaugh Map (VEKM). We now observe that the minterms of $\boldsymbol{X}$, which are the $2^k = K$ elementary or primitive products

$$\overline{X_1}\,\overline{X_2} \dots \overline{X_{k-1}}\,\overline{X_k}, \quad \overline{X_1}\,\overline{X_2} \dots \overline{X_{k-1}}\,X_k, \quad \dots, \quad X_1\,X_2 \dots X_{k-1}\,X_k \tag{10}$$

are exactly the atoms of the underlying Boolean algebra. For convenience, we call these atoms $T_i$ $(0 \leq \; i \; \leq \; (K-1))$, and hence $g(\boldsymbol{A})$ can be written as

$$g(\boldsymbol{A}) \; = \vee_{i=0}^{K-1} \; (e_i(\boldsymbol{A}) \wedge T_i \,) \tag{11}$$

where we use the symbol $e_i(\boldsymbol{A})$ to denote an indicator of the event that atom $T_i$ appears in the expression of $g(\boldsymbol{A})$, *i.e.*,

$$e_i(\boldsymbol{A}) = \begin{Bmatrix} 1, & if\; T_i \; \rightarrow \; g(\boldsymbol{A}) \\ 0, & otherwise \end{Bmatrix} = \; g(\boldsymbol{A})/T_i \tag{12}$$

where the symbol $(\boldsymbol{r} \; / \, \boldsymbol{s}) \; = (\boldsymbol{r})_{\boldsymbol{s}=1}$ denotes the Boolean quotient of $\boldsymbol{r}$ by $\boldsymbol{s}$ (Brown, 1990). Equation (12) means that $e_i(\boldsymbol{A})$ indicates whether atom $T_i$ appears in the cell $\boldsymbol{A}$ of the natural map for $g(\boldsymbol{Z})$. Now, we define $n_i$ $(0 \leq \; n_i \; \leq \; 2^n)$ as the total number of actual appearances of $T_i$ in the expression (11) for $g(\boldsymbol{A})$, *i.e.*,

$$n_i \; = \; \sum_{\mathrm{A} \in \{0,1\}^n} \; e_i(\boldsymbol{A}) \tag{13}$$

The total number $N_{unconditional}$ of unconditional particular solutions of (1a) over $B_{2^K}$ (as it is) is given by

$$N_{unconditional} \; = \; \prod_{i=0}^{K-1} \; n_i \tag{14}$$

This number is zero if some $n_i = 0$, *i.e.*, if an atom $T_i$ never makes its way to any expression $g(\boldsymbol{A})$ where $\boldsymbol{A} \in \{0,1\}^n$ (i.e., if $T_i$ does not appear in any cell of the map for $g(\boldsymbol{Z})$). To avoid such a situation, one must insist on the *consistency condition* that any atom $T_i$ such that $n_i = 0$ must be forbidden or nullified. This means that the underlying Boolean algebra loses these atoms and hence collapses to a smaller algebra, *i.e.*, to one of its strict sub algebras. The number of solutions over this new Boolean algebra is

$$N_{conditional} \; = \; \prod_{\substack{i=0 \\ n_i \neq 0}}^{K-1} \; n_i \tag{15}$$

Now we introduce a set of parameters $\boldsymbol{p_i}$ $(0 \leq \; i \; \leq \; (K-1), \; n_i \neq 0)$ to construct an orthonormal set of tags to attach to instances of appearances of the asserted atom $T_i$ in the discriminants $g(\boldsymbol{A})$ (*i.e.*, in the cells $\boldsymbol{A} \in \{0,1\}^n$ of the natural map of $g(\boldsymbol{Z})$). The number of parameters for atom $T_i$ (the length of vector $\boldsymbol{p_i}$) is given by

$$l(\boldsymbol{p}_i)=[\log_2 n_i], \quad 0 \leq \; i \; \leq \; (K-1), n_i \neq 0 \tag{16}$$

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Here, $\lceil x \rceil$ denotes the ceiling of the real number $x$, i.e., the smallest integer greater than or equal to $x$. The parameters $\boldsymbol{p_i}$ can be used to generate a set of $n_i \leq 2^{l(\boldsymbol{p}_i)}$ orthonormal tags $\{t_1, t_2 \dots\ t_{n_i}\}$, such that

$$t_1 \vee t_2 \vee \dots \vee t_{n_i} = 1 \tag{17a}$$

$$t_{j_1} \wedge t_{j_2} = 0 \qquad \forall\, j_1, j_2 \in \{1, 2, \dots, n_i\} \tag{17b}$$

When $n_i = 2^{l(\boldsymbol{p}_i)}$ the set of orthonormal tags can be visualized as the products of cells in a Karnaugh map whose map variables are the underlying parameters. If $2^{l(\boldsymbol{p}_i)-1} < n_i < 2^{l(\boldsymbol{p}_i)}$, some cells of such a map are merged, and the map reduces to a map-like structure.

When each appearance of an atom $T_i$ is tagged by a particular member of its orthonormal set of tags, an auxiliary function $G(\boldsymbol{Z}, \boldsymbol{p_i})$ $(0 \leq\ i\ \leq\ (K-1),\ n_i \neq 0)$ results. The parametric solution is now given by (Brown, 1970, 2003; Rushdi and Amashah, 2011).

$$Z_u = \vee_{\{\mathbf{A} \in \{0,1\}^n | A_u = 1\}}\ G(\boldsymbol{A}, \boldsymbol{p_i}).\ 1 \leq\ u\ \leq n,\ (0 \leq\ i \leq\ (K-1),\ n_i \neq 0) \tag{18}$$

The total number of parameters used in (18) to construct the tags for all atoms is given by

$$E = \sum_{i=1}^{k} l(\boldsymbol{p_i}) = \sum_{i=1}^{k} \lceil \log_2 (n_i) \rceil \tag{19}$$

The conventional method is to select the parameter vectors from a shared pool of parameters so as to minimize the number of parameters used, which then becomes

$$E' = \max_i\ l(\boldsymbol{p}_i) = \max_i \lceil \log_2 n_i \rceil = \left\lceil \log_2\ (\max_i n_i) \right\rceil \tag{20}$$

However, parameters used must then belong to the underlying Boolean algebra (possibly collapsed due to the consistency condition). We now propose to use independent parameters $\boldsymbol{p_i}$ for each atom $T_i$ $(0 \leq\ i\ \leq\ K-1,\ n_i \neq 0)$. The expressions (18) will not be as compact as they are in the conventional case, but the independent parameters $\boldsymbol{p_i}$ now belong to the two-valued Boolean algebra $B_2$ (Brown, 2003; Rushdi and Amashah, 2011), a fact that facilitates the generation of all particular solutions as will be seen shortly in the next subsection.

## 2.4 Listing of All Particular Solutions

The parametric solutions (18) can be used to generate all particular solutions through the use of an expansion tree. Generally, in the conventional method, this expansion tree is a complete tree that entails the assignment of $2^{K'}$ values to each of $E'$ parameters where $K' \leq K$ is the final number of atoms of the underlying Boolean algebra (possibly after some collapse due to the consistency condition). Each parent node has $2^{K'}$ children nodes and the tree has $E'$ levels beyond its root. Therefore, the tree has $(2^{K'})^{E'} = 2^{K'E'}$ leaves. These leaves constitute the whole set of particular solutions, possibly with repetitions. However, to avoid repetitions, we make sure, right from the first expansion level, to combine any sibling nodes that share the same solution value. With this kind of combining, the tree ceases to be a complete one, and its leaves become exactly the particular solutions, *i.e.*, without repetitions. If we further allow combining cousin (same-level) nodes, the tree is replaced by an acyclic graph that lists particular solution compactly (Rushdi, 2012).

In the method proposed herein, a complete version of the expansion tree requires the assignment of binary values $\{0,1\}$ to each of $E$ independent parameters. Since the complete binary tree has $E$ levels beyond its root, it has $2^E$ leaves. With merging of sibling nodes of equal solution values, the tree is no longer complete, and its leaves are just the particular solutions without repetitions. The size of the expansion tree in the proposed method is typically less than that in the conventional method since typically $E < K'E'$ (though $E > E'$). However, the true advantage of the proposed method is that *it allows us to avoid the use of an expansion tree (or an expansion acyclic graph) altogether*. The key to this is the observation that the parametric solution (18) can be rewritten as the weighted sum of the atoms $T_i$ that appear in the discriminants $g(\boldsymbol{A})$ (as expressed in (11)) of the function $g(\boldsymbol{Z})$, *viz.*

$$\boldsymbol{Z} = \bigvee_{\substack{i=0 \\ n_i \neq 0}}^{K-1} (\boldsymbol{Co}(T_i) \wedge T_i) \tag{21}$$

where we call the vector $\boldsymbol{Co}(T_i)$ the 'contribution' of the asserted atom $T_i$ and call the conjunction $(\boldsymbol{Co}(T_i) \wedge T_i)$ the 'total contribution' of that atom. We now note that $\boldsymbol{Co}(T_i)$ {or $\boldsymbol{Co}(T_i) \wedge T_i$} has exactly $n_i$ possible values, which can be conveniently listed via the same Karnaugh-map-like structure used in the representation of the associated tags. Therefore, we interpret (21) as a method of conveniently listing all particular solutions as a disjunction of total contributions of asserted atoms $T_i$, where the total contribution is given in all its $n_i$ possibilities. To obtain a specific particular solution, one has simply to pick up arbitrarily one of the possibilities of the total contribution for every atom, and then add the selected total contributions together. The total number of particular solutions obtained this way agrees with that given by (15).

### 2.5 Picking up a Particular Solution of Specific Features

Equation (21) is of a paramount importance, as it provides a listing of a (possibly huge) number of all particular solutions in a compact space. As such, it allows picking up certain solutions enjoying particular desirable features simply by a quick inspection of the aforementioned listing. This point will be clarified further by way of examples in the next section.

## 3. Examples

In the following, we illustrate the method of Section 2 by revisiting five Type-3 examples handled by Ledley (1959, 1960). In each of these, we recover the results of Ledley in a much faster, more systematic and transparent, and less error-prone way. In one particular case, we resolve and circumvent a certain discrepancy that Ledley's technique fell short of handling completely.

### Example 1

This example was studied before by Ledley (1960) and Brown (2011). For this example, the $\boldsymbol{s}$ and $\boldsymbol{t}$ functions are scalars of the form

$$s = \bar{Z}_1 Z_3 \vee Y_1 Z_1 \bar{Z}_2 \vee Y_2 Z_2 \bar{Z}_3 \tag{22a}$$
$$t = X_1 X_2 \vee \bar{X}_2 X_3 Y_1 \vee X_1 \bar{X}_3 Y_2 \vee X_2 Y_2 \tag{22b}$$

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where
$$s = t. \tag{22c}$$

Equation (22c) can be reduced into an equation similar to (2a) of the form (2b), *viz.*,

$$h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = s \odot t \tag{23}$$

Fig. 2 is an expression of $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ in terms of a variable-entered Karnaugh map (VEKM) of map variables $Y_1$ and $Y_2$. The map has entries in terms of the $\odot$ operator in Fig. 2(a). These terms are rewritten with the $\odot$ operator replaced by its POS form in Fig. 2(b). The $Y$ variables are suppressed via (5b) by ANDing the entries in Fig. 2(b), namely

$$g(\boldsymbol{X}, \boldsymbol{Z}) = (\bar{Z}_1 Z_3 \vee \bar{X}_1 \vee \bar{X}_2)(\bar{Z}_1 Z_3 \vee Z_1 \bar{Z}_2 \vee \bar{X}_1 X_2 \vee \bar{X}_2 \bar{X}_3)(\bar{Z}_1 Z_3 \vee Z_2 \bar{Z}_3 \vee \bar{X}_1 \bar{X}_2 \vee \bar{X}_2 X_3)(\bar{Z}_1 Z_3 \vee Z_1 \bar{Z}_2 \vee Z_2 \bar{Z}_3 \vee \bar{X}_1 \bar{X}_2 \bar{X}_3)(Z_1 \vee \bar{Z}_3 \vee X_1 X_2)(\bar{Z}_1 \bar{Z}_3 \vee Z_1 Z_2 \vee X_1 X_2 \vee \bar{X}_2 X_3)(Z_1 Z_3 \vee \bar{Z}_2 \bar{Z}_3 \vee X_1 \bar{X}_3 \vee X_2)(\bar{Z}_1 \bar{Z}_2 \bar{Z}_3 \vee Z_1 Z_2 Z_3 \vee X_1 \vee X_2 \vee X_3) \tag{24}$$

The function $g(\boldsymbol{X}; \boldsymbol{Z})$ is computed *via* its natural map of map variables $Z_1, Z_2$, and $Z_3$ in Fig. 3a. The final result of the computation is shown in Fig. 3b which is again redrawn as Fig. 3c such that every discriminant (cell entry) of $g(\boldsymbol{X}; \boldsymbol{Z})$ is given as a minterm expansion (disjunction of atoms). Each of the eight atoms of $FB(X_1, X_2, X_3)$ appears twice in Fig. 3c, which means that the consistency condition is the identity

$$0 = 0 \tag{25}$$

and the total number of particular solutions is

$$N = 2^8 = 256 \tag{26}$$

Each atom $T_i$ requires a single parameter $p_i$ to produce a set of orthonormal tags $\{p_i, \bar{p}_i\}$. The corresponding auxiliary function $G(\boldsymbol{X}; \boldsymbol{Z}, \boldsymbol{p})$ is shown in Fig. 4. The parametric solution is then given by either of the following equations

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} T_0 p_0 \vee T_1 \vee T_2 p_2 \vee T_3 p_3 \vee T_4 p_4 \vee T_5 \\ T_0 p_0 \vee T_2 \vee T_3 \vee T_4 \vee T_6 p_6 \vee T_7 p_7 \\ T_0 p_0 \vee T_1 p_1 \vee T_5 p_5 \vee T_6 \vee T_7 \end{bmatrix} \tag{27a}$$

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} \bar{X}_1 \bar{X}_2 \bar{X}_3 p_0 \vee \bar{X}_2 X_3 \vee \bar{X}_1 X_2 \bar{X}_3 p_2 \vee \bar{X}_1 X_2 X_3 p_3 \vee X_1 \bar{X}_2 \bar{X}_3 p_4 \\ \bar{X}_1 \bar{X}_2 \bar{X}_3 p_0 \vee \bar{X}_1 X_2 \vee X_1 \bar{X}_2 \bar{X}_3 \vee X_1 X_2 \bar{X}_3 p_6 \vee X_1 X_2 X_3 p_7 \\ \bar{X}_1 \bar{X}_2 \bar{X}_3 p_0 \vee \bar{X}_1 \bar{X}_2 X_3 p_1 \vee X_1 \bar{X}_2 X_3 p_5 \vee X_1 X_2 \end{bmatrix} \tag{27b}$$

where the symbols $T_i = \boldsymbol{X}^{\boldsymbol{I}}$ denote the eight atoms of $FB(X_1, X_2, X_3)$ and $\boldsymbol{I} = (i_1\ i_2\ i_3)$ expresses the integer $i$ in binary notation ($i = 4i_1 + 2i_2 + i_3$). Fig. 5 displays all 256 particular solutions of this example. Each particular solution is a disjunction of possible contributions of atoms $T_i$ ($0 \le i \le 7$). Here the contribution of each atom is represented by a single-variable (2-cell) Karnaugh map. As an example of identifying one particular solution (without any deliberate effort to achieve compactness), we select the possible contribution in the top cell for each atom. Hence, we obtain

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$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} \bar{X}_2 X_3 \\ \bar{X}_1 X_2 \vee X_1 \bar{X}_2 \bar{X}_3 \\ X_1 X_2 \end{bmatrix} \tag{28}$$

An obvious slight improvement is possible if we use the lower-cell contribution of $T_6 = X_1 X_2 \bar{X}_3$, instead of the upper-cell one, to obtain

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} \bar{X}_2 X_3 \\ \bar{X}_1 X_2 \vee X_1 \bar{X}_3 \\ X_1 X_2 \end{bmatrix} \tag{29}$$

A better solution was found via informal means by Ledley (1960); and re-obtained by branch-and-bound search (Brown, 2011). This solution is

$$\begin{bmatrix} \bar{X}_2 \\ X_2 \vee \bar{X}_3 \\ \bar{X}_1 \bar{X}_2 \vee X_1 X_2 \end{bmatrix} = \begin{bmatrix} \bar{X}_1 \bar{X}_2 \bar{X}_3 \\ \bar{X}_1 \bar{X}_2 \bar{X}_3 \\ \bar{X}_1 \bar{X}_2 \bar{X}_3 \end{bmatrix} \vee \begin{bmatrix} \bar{X}_1 \bar{X}_2 X_3 \\ 0 \\ \bar{X}_1 \bar{X}_2 X_3 \end{bmatrix} \vee \begin{bmatrix} 0 \\ \bar{X}_1 X_2 \bar{X}_3 \\ 0 \end{bmatrix} \vee \begin{bmatrix} 0 \\ \bar{X}_1 X_2 X_3 \\ 0 \end{bmatrix} \vee \begin{bmatrix} X_1 \bar{X}_2 \bar{X}_3 \\ X_1 \bar{X}_2 \bar{X}_3 \\ 0 \end{bmatrix} \vee \begin{bmatrix} X_1 \bar{X}_2 X_3 \\ 0 \\ 0 \end{bmatrix} \vee \begin{bmatrix} 0 \\ X_1 X_2 \bar{X}_3 \\ X_1 X_2 \bar{X}_3 \end{bmatrix} \vee \begin{bmatrix} 0 \\ X_1 X_2 X_3 \\ X_1 X_2 X_3 \end{bmatrix} \tag{30}$$

| | $Y_1$ | |
|---|---|---|
| $\bar{Z}_1 Z_3 \odot X_1 X_2$ | $(\bar{Z}_1 Z_3 \vee Z_1 \bar{Z}_2) \odot (X_1 X_2 \vee \bar{X}_2 X_3)$ | |
| $(\bar{Z}_1 Z_3 \vee Z_2 \bar{Z}_3) \odot (X_1 X_2 \vee X_1 \bar{X}_3 \vee X_2)$ | $(\bar{Z}_1 Z_3 \vee Z_1 \bar{Z}_2 \vee Z_2 \bar{Z}_3) \odot (X_1 X_2 \vee \bar{X}_2 X_3 \vee X_1 \bar{X}_3 \vee X_2)$ | $Y_2$ |

(a) $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ in terms of the $\odot$ operator

| | $Y_1$ | |
|---|---|---|
| $(\bar{Z}_1 Z_3 \vee \bar{X}_1 \vee \bar{X}_2)(Z_1 \vee \bar{Z}_3 \vee X_1 X_2)$ | $(\bar{Z}_1 Z_3 \vee Z_1 \bar{Z}_2 \vee \bar{X}_1 X_2 \vee \bar{X}_2 \bar{X}_3)(\bar{Z}_1 \bar{Z}_3 \vee Z_1 Z_2 \vee X_1 X_2 \vee \bar{X}_2 X_3)$ | |
| $(\bar{Z}_1 Z_3 \vee Z_2 \bar{Z}_3 \vee \bar{X}_1 \bar{X}_2 \vee \bar{X}_2 X_3)(Z_1 Z_3 \vee \bar{Z}_2 \bar{Z}_3 \vee X_1 \bar{X}_3 \vee X_2)$ | $(\bar{Z}_1 Z_3 \vee Z_1 \bar{Z}_2 \vee Z_2 \bar{Z}_3 \vee \bar{X}_1 \bar{X}_2 \bar{X}_3)(\bar{Z}_1 \bar{Z}_2 \bar{Z}_3 \vee Z_1 Z_2 Z_3 \vee X_1 \vee X_2 \vee X_3)$ | $Y_2$ |

(b) $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ with the NOR function replaced by its pos form

Fig. 2. A map expression of the function in Example 1

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| | | $Z_1$ | | |
|---|---|---|---|---|
| $(\bar{X}_1 \vee \bar{X}_2)(\bar{X}_1 X_2 \vee \bar{X}_2\bar{X}_3)(\bar{X}_1\bar{X}_2 \vee \bar{X}_2 X_3)(\bar{X}_1\bar{X}_2\bar{X}_3)$ (1) (1) (1) (1) | $(\bar{X}_1 \vee \bar{X}_2)(\bar{X}_1 X_2 \vee \bar{X}_2\bar{X}_3)$ (1) (1) (1) (1) $(X_1\bar{X}_3 \vee X_2)(X_1 \vee X_2 \vee X_3)$ | $(\bar{X}_1 \vee \bar{X}_2)(\bar{X}_1 X_2 \vee \bar{X}_2\bar{X}_3)$ (1) (1) (1) (1) $(X_1\bar{X}_3 \vee X_2)(X_1 \vee X_2 \vee X_3)$ | $(\bar{X}_1 \vee \bar{X}_2)$ (1) $(\bar{X}_1\bar{X}_2 \vee \bar{X}_2 X_3)$ (1) (1) $(X_1 X_2 \vee \bar{X}_2 X_3)$ (1) $(X_1 \vee X_2 \vee X_3)$ | |
| (1) (1) (1) (1) $(X_1 X_2)(X_1 X_2 \vee \bar{X}_2 X_3)(X_1\bar{X}_3 \vee X_2)(X_1 \vee X_2 \vee X_3)$ | (1) (1) (1) (1) $(X_1 X_2)(X_1 X_2 \vee \bar{X}_2 X_3)(X_1\bar{X}_3 \vee X_2)(X_1 \vee X_2 \vee X_3)$ | $(\bar{X}_1 \vee \bar{X}_2)(\bar{X}_1 X_2 \vee \bar{X}_2\bar{X}_3))(\bar{X}_1\bar{X}_2 \vee \bar{X}_2 X_3)(\bar{X}_1\bar{X}_2\bar{X}_3)$ (1)(1)(1)(1) | $(\bar{X}_1 \vee \bar{X}_2)$ (1) $(\bar{X}_1\bar{X}_2 \vee \bar{X}_2 X_3)$ (1) (1) $(X_1 X_2 \vee \bar{X}_2 X_3)$ (1) $(X_1 \vee X_2 \vee X_3)$ | $Z_3$ |
| | $Z_2$ | | | |

(a) $g(\boldsymbol{X}, \boldsymbol{Z})$

| | | $Z_1$ | | |
|---|---|---|---|---|
| $\bar{X}_1\bar{X}_2\bar{X}_3$ | $\bar{X}_1 X_2 \vee X_1\bar{X}_2\bar{X}_3$ | $\bar{X}_1 X_2 \vee X_1\bar{X}_2\bar{X}_3$ | $\bar{X}_2 X_3$ | |
| $X_1 X_2$ | $X_1 X_2$ | $\bar{X}_1\bar{X}_2\bar{X}_3$ | $\bar{X}_2 X_3$ | $Z_3$ |
| | $Z_2$ | | | |

(b) $g(\boldsymbol{X}, \boldsymbol{Z})$

| | | $Z_1$ | | |
|---|---|---|---|---|
| $\bar{X}_1\bar{X}_2\bar{X}_3$ | $\bar{X}_1 X_2\bar{X}_3 \vee \bar{X}_1 X_2 X_3 \vee X_1\bar{X}_2\bar{X}_3$ | $\bar{X}_1 X_2\bar{X}_3 \vee \bar{X}_1 X_2 X_3 \vee X_1\bar{X}_2\bar{X}_3$ | $X_1\bar{X}_2 X_3 \vee \bar{X}_1\bar{X}_2 X_3$ | |
| $X_1 X_2\bar{X}_3 \vee X_1 X_2 X_3$ | $X_1 X_2\bar{X}_3 \vee X_1 X_2 X_3$ | $\bar{X}_1\bar{X}_2\bar{X}_3$ | $X_1\bar{X}_2 X_3 \vee \bar{X}_1\bar{X}_2 X_3$ | $Z_3$ |
| | $Z_2$ | | | |

(c) $g(\boldsymbol{X}, \boldsymbol{Z})$

Fig. 3. The function $g(\boldsymbol{X}, \boldsymbol{Z})$ in Example 1

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$Z_1$ (over the third and fourth columns)

| | | | |
|---|---|---|---|
| $\bar{X}_1\bar{X}_2\bar{X}_3(\bar{p}_0)$ | $\bar{X}_1X_2\bar{X}_3(\bar{p}_2) \vee \bar{X}_1X_2X_3(\bar{p}_3) \vee X_1\bar{X}_2\bar{X}_3(\bar{p}_4)$ | $\bar{X}_1X_2\bar{X}_3(p_2) \vee \bar{X}_1X_2X_3(p_3) \vee X_1\bar{X}_2\bar{X}_3(p_4)$ | $\bar{X}_1\bar{X}_2X_3(\bar{p}_1) \vee X_1\bar{X}_2X_3(\bar{p}_5)$ |
| $X_1X_2\bar{X}_3(\bar{p}_6) \vee X_1X_2X_3(\bar{p}_7)$ | $X_1X_2\bar{X}_3(p_6) \vee X_1X_2X_3(p_7)$ | $\bar{X}_1\bar{X}_2\bar{X}_3(p_0)$ | $X_1\bar{X}_2X_3(p_5) \vee \bar{X}_1\bar{X}_2X_3(p_1)$ |

$Z_3$ (beside the second row)

$Z_2$ (under the second and third columns)

$g(\boldsymbol{X};\ \boldsymbol{Z}, \boldsymbol{p})$

Fig. 4. The auxiliary function for Example 1

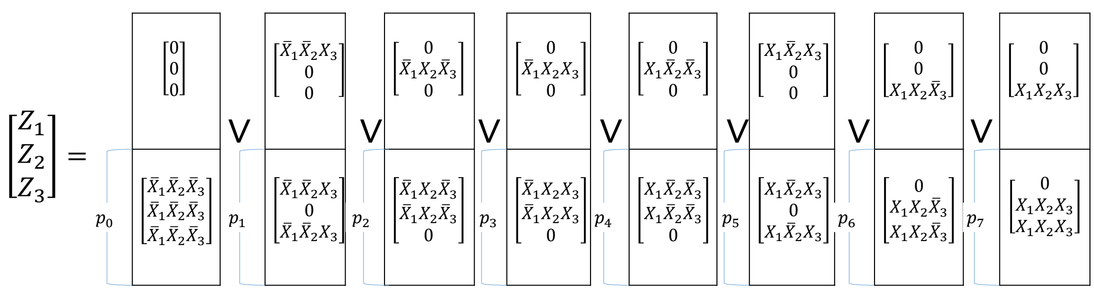

Fig. 5. A display of all 256 particular solutions of Example 1 as a disjunction of possible contributions of atoms $T_i$ ( $0 \leq i \leq 7$)

## Example 2a

Ledley (1959, 1960) described this example as one that illustrates a normally complicated solution. While this example has solutions such that $s = t$, a minor modification of it in Example 2b produces a situation in which no solutions can be produced such that $s = t$. We start with two scalar functions $s$ and $t$, *viz.*

$$s = \bar{Z}_1\bar{Z}_2\bar{Y}_2 \vee Z_1\bar{Z}_2Y_2 \vee \bar{Z}_1Z_2(Y_1\bar{Y}_2 \vee \bar{Y}_1Y_2) \vee Z_1Z_2\bar{Y}_1\bar{Y}_2 \quad (31)$$

$$t = (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3)\bar{Y}_1\bar{Y}_2 \vee (\bar{X}_1\bar{X}_3 \vee X_1X_2)Y_1\bar{Y}_2 \vee (X_1\bar{X}_3 \vee X_2X_3)\bar{Y}_1Y_2 \vee (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3)Y_1Y_2 \quad (32)$$

The function $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ equivalent to $(s \odot t)$ is expressed by the map in Fig. 6. Entries of the cells of this map are ANDed to form the function $g(\boldsymbol{X}, \boldsymbol{Z})$ in which $Y_1$ and $Y_2$ are suppressed. The natural map for $g(\boldsymbol{X}, \boldsymbol{Z})$ is gradually developed in Figs. 7(a)-7(d). The final map in Fig. 7(d) has entries in minterm form. It shows that each of the 8 atoms of $FB(X_1, X_2, X_3)$ appears once. Hence, the number of particular solutions is $1^8 = 1$ and the consistency condition is the identity (0 = 0). Therefore, the map can be used to represent the auxiliary function $G(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{p})$ as well (since neither tagging each atom with 1 nor ORing each entry with $d(0)$ produces any change). The final answer for $Z(\boldsymbol{X})$ is unique and given by:

| | $Y_1$ | |
|---|---|---|
| $(\bar{Z}_1\bar{Z}_2 \vee Z_1Z_2) \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3)$ | $(\bar{Z}_1\bar{Z}_2 \vee \bar{Z}_1Z_2) \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2)$ | |
| $(Z_1\bar{Z}_2 \vee \bar{Z}_1Z_2) \odot (X_1\bar{X}_3 \vee X_2X_3)$ | $Z_1\bar{Z}_2 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3)$ | $Y_2$ |

$h(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z})$

Fig. 6. A VEKM representing a function equivalent to $(s \odot t)$ in Example 2a

| | $Z_1$ | |
|---|---|---|
| $(1 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(1 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(0 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(0 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ | $(0 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(0 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(1 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(1 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ | |
| $(0 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(1 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(1 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(0 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ | $(1 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(0 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(0 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(0 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ | $Z_2$ |

(a) $g(\boldsymbol{X},\boldsymbol{Z})$

| | $Z_1$ | |
|---|---|---|
| $(\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3)$<br>$(\bar{X}_1\bar{X}_3 \vee X_1X_2)$<br>$(\bar{X}_1 \vee X_3)\ (\bar{X}_2 \vee \bar{X}_3)$<br>$(\bar{X}_1 \vee X_2 \vee X_3)\ (X_1 \vee \bar{X}_2 \vee \bar{X}_3)$ | $(X_1 \vee X_3)\ (X_2 \vee \bar{X}_3)$<br>$(X_1 \vee X_3)(\bar{X}_1 \vee \bar{X}_2)$<br>$(X_1\bar{X}_3 \vee X_2X_3)$<br>$(X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3)$ | |
| $(X_1 \vee X_3)\ (X_2 \vee \bar{X}_3)$<br>$(\bar{X}_1\bar{X}_3 \vee X_1X_2)$<br>$(X_1\bar{X}_3 \vee X_2X_3)$<br>$(\bar{X}_1 \vee X_2 \vee X_3)\ (X_1 \vee \bar{X}_2 \vee \bar{X}_3)$ | $(\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3)$<br>$(X_1 \vee X_3)(\bar{X}_1 \vee \bar{X}_2)$<br>$(\bar{X}_1 \vee X_3)\ (\bar{X}_2 \vee \bar{X}_3)$<br>$(\bar{X}_1 \vee X_2 \vee X_3)\ (X_1 \vee \bar{X}_2 \vee \bar{X}_3)$ | $Z_2$ |

(b) $g(\boldsymbol{X},\boldsymbol{Z})$

| | $Z_1$ | |
|---|---|---|
| $\bar{X}_1\bar{X}_3$ | $X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3$ | |
| $X_1X_2$ | $\bar{X}_2X_3$ | $Z_2$ |

(c) $g(\boldsymbol{X},\boldsymbol{Z})$

| | | $Z_1$ | |
|---|---|---|---|
| | $\bar{X}_1 X_2 \bar{X}_3 \vee \bar{X}_1 \bar{X}_2 \bar{X}_3$ | $X_1 \bar{X}_2 \bar{X}_3 \vee \bar{X}_1 X_2 X_3$ | |
| | $X_1 X_2 \bar{X}_3 \vee X_1 X_2 X_3$ | $\bar{X}_1 \bar{X}_2 X_3 \vee X_1 \bar{X}_2 X_3$ | $Z_2$ |

(d) $g(\boldsymbol{X},\boldsymbol{Z})$ or $G(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{p})$

Fig. 7. Gradual development of the natural map for $g(\boldsymbol{X},\boldsymbol{Z})$ obtained by suppressing $\boldsymbol{Y}$ from $h(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z})$ in Fig. 6. The final map in 7(d) also represents the auxiliary function $G(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{p})$

$$\begin{aligned} Z_1 &= X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3 \vee \bar{X}_1\bar{X}_2X_3 \vee X_1\bar{X}_2X_3 \\ &= X_1\bar{X}_2 \vee \bar{X}_1X_3 \end{aligned} \tag{33}$$

$$\begin{aligned} Z_2 &= X_1X_2\bar{X}_3 \vee X_1X_2X_3 \vee \bar{X}_1\bar{X}_2X_3 \vee X_1\bar{X}_2X_3 \\ &= X_1X_2 \vee \bar{X}_2X_3 \end{aligned} \tag{34}$$

## Example 2b

Let us keep $t$ as given by (32) in Example 2a and augment $s$ therein by the term $\bar{Z}_1\bar{Z}_2\bar{Y}_1$ so as to become:

$$s = \bar{Z}_1\bar{Z}_2(\bar{Y}_1 \vee \bar{Y}_2) \vee Z_1\bar{Z}_2Y_2 \vee \bar{Z}_1Z_2(Y_1\bar{Y}_2 \vee \bar{Y}_1Y_2) \vee Z_1Z_2\bar{Y}_1\bar{Y}_2 \tag{35}$$

| | | $Y_1$ | |
|---|---|---|---|
| | $(\bar{Z}_1\bar{Z}_2 \vee Z_1Z_2) \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3)$ | $(\bar{Z}_1\bar{Z}_2 \vee \bar{Z}_1Z_2) \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2)$ | |
| | $(\bar{Z}_1\bar{Z}_2 \vee \bar{Z}_1Z_2 \vee Z_1\bar{Z}_2) \odot (X_1\bar{X}_3 \vee X_2X_3)$ | $Z_1\bar{Z}_2 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3)$ | $Y_2$ |

$h(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z})$

Fig. 8. A VEKM representing a function equivalent to $(s \odot t)$ in Example 2b

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| | | $Z_1$ |
|---|---|---|
| | $(1 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(1 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(1 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(0 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ | $(0 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(0 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(1 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(1 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ |
| $Z_2$ | $(0 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(1 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(1 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(0 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ | $(1 \odot (\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3))$<br>$(0 \odot (\bar{X}_1\bar{X}_3 \vee X_1X_2))$<br>$(0 \odot (X_1\bar{X}_3 \vee X_2X_3))$<br>$(0 \odot (X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3))$ |

(a) $g(\boldsymbol{X}, \boldsymbol{Z})$

| | | $Z_1$ |
|---|---|---|
| | $(\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3)$<br>$(\bar{X}_1\bar{X}_3 \vee X_1X_2)$<br>$(X_1\bar{X}_3 \vee X_2X_3)$<br>$(\bar{X}_1 \vee X_2 \vee X_3)(X_1 \vee \bar{X}_2 \vee \bar{X}_3)$ | $(X_1 \vee X_3)(X_2 \vee \bar{X}_3)$<br>$(X_1 \vee X_3)(\bar{X}_1 \vee \bar{X}_2)$<br>$(X_1\bar{X}_3 \vee X_2X_3)$<br>$(X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3)$ |
| $Z_2$ | $(X_1 \vee X_3)(X_2 \vee \bar{X}_3)$<br>$(\bar{X}_1\bar{X}_3 \vee X_1X_2)$<br>$(X_1\bar{X}_3 \vee X_2X_3)$<br>$(\bar{X}_1 \vee X_2 \vee X_3)(X_1 \vee \bar{X}_2 \vee \bar{X}_3)$ | $(\bar{X}_1\bar{X}_3 \vee \bar{X}_2X_3)$<br>$(X_1 \vee X_3)(\bar{X}_1 \vee \bar{X}_2)$<br>$(\bar{X}_1 \vee X_3)(\bar{X}_2 \vee \bar{X}_3)$<br>$(\bar{X}_1 \vee X_2 \vee X_3)(X_1 \vee \bar{X}_2 \vee \bar{X}_3)$ |

(b) $g(\boldsymbol{X}, \boldsymbol{Z})$

| | | $Z_1$ |
|---|---|---|
| | $\bar{X}_1X_2\bar{X}_3$ | $X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3$ |
| $Z_2$ | $X_1X_2$ | $\bar{X}_2X_3$ |

(c) $g(\boldsymbol{X}, \boldsymbol{Z})$

| | | $Z_1$ |
|---|---|---|
| | $\bar{X}_1X_2\bar{X}_3$ | $X_1\bar{X}_2\bar{X}_3$<br>$\vee \bar{X}_1X_2X_3$ |
| $Z_2$ | $X_1X_2\bar{X}_3$<br>$\vee X_1X_2X_3$ | $\bar{X}_1\bar{X}_2X_3$<br>$\vee X_1\bar{X}_2X_3$ |

(d) $g(\boldsymbol{X}, \boldsymbol{Z})$

Fig. 9. Gradual development of the natural map for $g(\boldsymbol{X}, \boldsymbol{Z})$ obtained by suppressing $\boldsymbol{Y}$ from $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ in Fig. 8

We reproduce Figs. 6 and 7 modified as Figs. 8 and 9. In Fig. 9(d), only 7 atoms still appear (once each) while atom $\bar{X}_1\bar{X}_2\bar{X}_3$ is missing. Therefore, there is a single particular solution under the consistency condition $\bar{X}_1\bar{X}_2\bar{X}_3 = 0$, and the auxiliary function is given by Fig. 10. Hence the conditional solution is:

$$\begin{aligned} Z_1 &= X_1\bar{X}_2\bar{X}_3 \vee \bar{X}_1X_2X_3 \vee \bar{X}_1\bar{X}_2X_3 \vee X_1\bar{X}_2X_3 \vee d(\bar{X}_1\bar{X}_2\bar{X}_3) \\ &= X_1\bar{X}_2 \vee \bar{X}_1X_3 \vee d(\bar{X}_1\bar{X}_2\bar{X}_3), \end{aligned} \tag{36}$$

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$$Z_2 = X_1X_2\bar{X}_3 \vee X_1X_2X_3 \vee \bar{X}_1\bar{X}_2X_3 \vee X_1\bar{X}_2X_3 \vee d(\bar{X}_1\bar{X}_2\bar{X}_3)$$
$$= X_1X_2 \vee \bar{X}_2X_3 \vee d(\bar{X}_1\bar{X}_2\bar{X}_3). \quad (37)$$

| | $Z_1$ | |
|---|---|---|
| $\bar{X}_1X_2\bar{X}_3(1)$ $\vee$ d$(\bar{X}_1\bar{X}_2\bar{X}_3)$ | $X_1\bar{X}_2\bar{X}_3(1)$ $\vee \bar{X}_1X_2X_3(1)$ $\vee$ d$(\bar{X}_1\bar{X}_2\bar{X}_3)$ | |
| $X_1X_2\bar{X}_3(1)$ $\vee X_1X_2X_3(1)$ $\vee$ d$(\bar{X}_1\bar{X}_2\bar{X}_3)$ | $\bar{X}_1\bar{X}_2X_3(1)$ $\vee X_1\bar{X}_2X_3(1)$ $\vee$ d$(\bar{X}_1\bar{X}_2\bar{X}_3)$ | $Z_2$ |

$G(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{p})$

Fig. 10. The auxiliary function for Example 2b

## Example 3

Suppose that both functions $s$ and $t$ are scalars of the form

$$s(\boldsymbol{Z}, \boldsymbol{Y}) = Z_1Y_1 \vee \bar{Z}_1\bar{Y}_1 \quad (38)$$
$$t(\boldsymbol{X}, \boldsymbol{Y}) = \bar{X}_1\bar{X}_2 \vee \bar{X}_1Y_1 \vee \bar{X}_2\bar{Y}_1 \quad (39)$$

Ledley (1960) labeled this example as one of *no solutions*. He said that subject to the condition $X_1 \rightarrow \bar{X}_2$, solutions $Z_1 = X_2$, and $Z_2 = \bar{X}_1$ are antecedent solutions that make $s_a \rightarrow t$ but $s \neq t$. According to our method, we construct the function $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ according to (2b) in Fig. 11, and then suppress $Y_1$ to obtain $g(\boldsymbol{X}, \boldsymbol{Z})$, represented by the natural map in Fig.12(a). Out of the four atoms of $FB(X_1, X_2)$, the two atoms $X_1\bar{X}_2$ and $\bar{X}_1X_2$, make a single appearance in Fig. 12(b), while the two atoms $\bar{X}_1\bar{X}_2$ and $X_1X_2$ make no appearance therein. The number of particular solutions is 1*1 = 1, and the consistency condition is $(\bar{X}_1\bar{X}_2 \vee X_1X_2 = 0)$. We construct the auxiliary function $G(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{p})$ *via* the map in Fig. 13 by tagging each of the asserted atoms $X_1\bar{X}_2$ and $\bar{X}_1X_2$ by 1 and adding each of the nullified atoms don't-care in each cell of the map of $G(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{p})$. The parametric solution of $Z_1$ involves no parameters and constitutes a single particular solution given by

$$Z_1 = \bar{X}_1X_2 \vee d(\bar{X}_1\bar{X}_2 \vee X_1X_2) \quad (40)$$

The expression above for $Z_1$ might be simplified to either $Z_1 = \bar{X}_1$ or $Z_1 = X_2$. These are not alternative solutions as claimed by Ledley (1960) but are the same solution under the above consistency condition which is equivalent to requiring that $\bar{X}_1 = X_2$. Reading of the map in Fig. 13 (or complementing (40)) yields

$$\bar{Z}_1 = X_1\bar{X}_2 \vee d(\bar{X}_1\bar{X}_2 \vee X_1X_2) \quad (41)$$

Substituting (40) and (41) in (38) produces

$$s = (\bar{X}_1X_2Y_1 \vee X_1\bar{X}_2\bar{Y}_1) \vee d(Y_1 \vee \bar{Y}_1)(\bar{X}_1\bar{X}_2 \vee X_1X_2)$$
$$= (\bar{X}_1X_2Y_1 \vee X_1\bar{X}_2\bar{Y}_1) \vee d(\bar{X}_1\bar{X}_2 \vee X_1X_2) \quad (42)$$

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which is exactly the same as $t$ in (39) under the auxiliary condition that $(\bar{X}_1\bar{X}_2 \vee X_1X_2 = 0)$ or equivalently $X_2 = \bar{X}_1$, or $X_1 = \bar{X}_2$. This very simple example reveals many of the shortcoming in Ledley's method, including the inadequacy of splitting solutions into antecedence and consequence ones, and the difficulty of doing without the concept of a consistency condition.

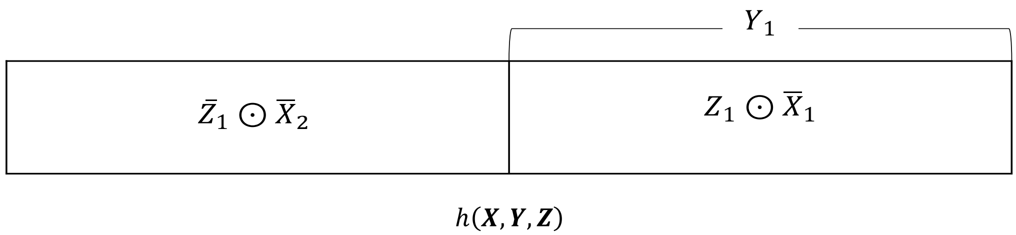


Fig. 11. Map representation of a function $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ whose assertion is equivalent to equality of $s$ and $t$ in Example 3

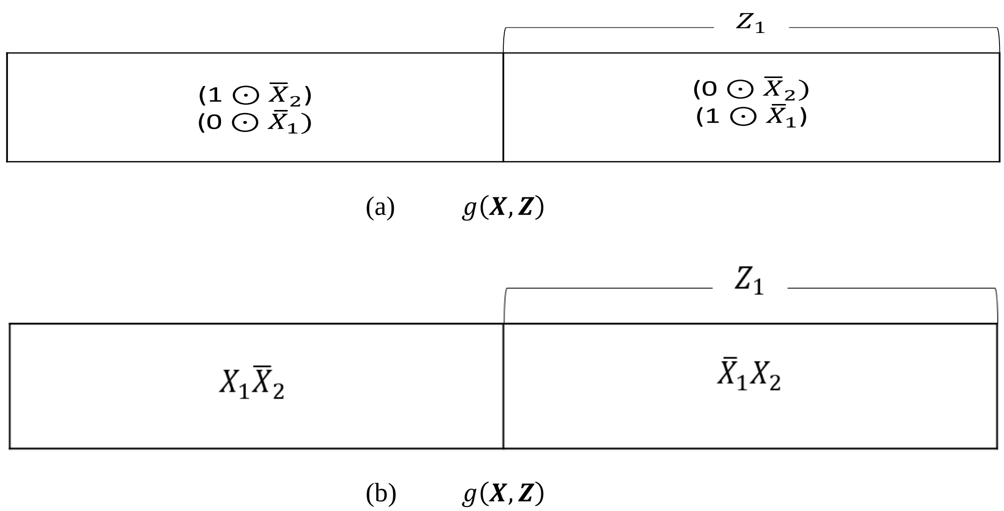


Fig. 12. Resultant of suppression of $Y_1$ in Example 3

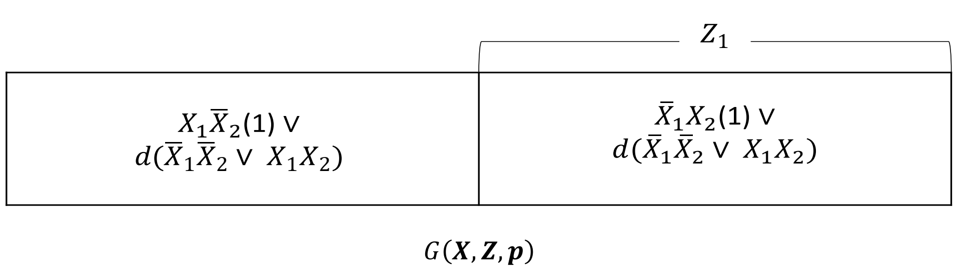


Fig. 13. The auxiliary function for Example 3

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**Example 4**

Ledley (1960) required just a consequence solution $(t \to s)$ rather than a complete one $\{(s \to t) \wedge (t \to s)\}$ for this example, where $s$ and $t$ are given by

$$s = Z_1 \tag{43}$$

$$t(\boldsymbol{X}, \boldsymbol{Y}) = X_1(\bar{X}_2 Y \vee X_2 X_3) \vee \bar{X}_1 \bar{Y}(X_2 \bar{X}_3 \vee \bar{X}_2 X_3) \tag{44}$$

An obvious (albeit trivial and non-genuine) solution for this problem is $Z_1 = 1$ (since 1 is implied by anything). We now use Figs. 14 and 15 to represent the pertinent maps, noting that now the $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ function represents $(t \to s)$ rather than $(s \odot t)$. The final map in Fig. 15(c) shows that all atoms are present, with 5 of them making a single appearance and 3 of them making a double appearance each. These three atoms are $T_0 = \bar{X}_1 \bar{X}_2 \bar{X}_3$ whose instances are tagged by elements of the orthonormal set $\{p_0, \bar{p}_0\}$, $T_3 = \bar{X}_1 X_2 X_3$ whose instances are tagged by elements of the orthonormal set $\{p_3, \bar{p}_3\}$, and $T_6 = X_1 X_2 \bar{X}_3$ whose instances are tagged by elements of the orthonormal set $\{p_6, \bar{p}_6\}$, while the other 5 atoms are each tagged by (1) as shown in Fig. 16. The consistency condition is the identity (0 = 0), and the number of particular solution is $1^5 * 2^3 = 8$. The parametric solution is

$$Z_1 = \bar{X}_1 \bar{X}_2 \bar{X}_3 p_0 \vee \bar{X}_1 X_2 X_3 p_3 \vee X_1 X_2 \bar{X}_3 p_6 \vee \bar{X}_1 \bar{X}_2 X_3 \vee \bar{X}_1 X_2 \bar{X}_3 \vee X_1 \bar{X}_2 \bar{X}_3 \vee X_1 \bar{X}_2 X_3 \vee X_1 X_2 X_3 \tag{45}$$

where each of $p_0, p_3$, and $p_6 \in \{0,1\}$ according to our novel method (Rushdi and Ahmad; 2017a, 2017b). If we employ the conventional method, we use a common value $p$ for each of $p_0, p_3$, and $p_6$, but this $p \in B_{256} = FB(X_1, X_2, X_3)$. Equation (45) can be conventionally written as a display of 8 particular solutions for $Z_1$ as follows.

$$Z_1 = \begin{bmatrix} 0 \\ \bar{X}_1 \bar{X}_2 X_3 \end{bmatrix}_{p_0} \vee \begin{bmatrix} 0 \\ \bar{X}_1 X_2 X_3 \end{bmatrix}_{p_3} \vee \begin{bmatrix} 0 \\ X_1 X_2 \bar{X}_3 \end{bmatrix}_{p_6} \vee \begin{bmatrix} \bar{X}_1 \bar{X}_2 X_3 \vee \\ \bar{X}_1 X_2 \bar{X}_3 \vee \\ X_1 \bar{X}_2 \bar{X}_3 \vee \\ X_1 \bar{X}_2 X_3 \vee \\ X_1 X_2 X_3 \end{bmatrix} \tag{46}$$

These 8 solutions are also displayed in Fig. 17. They include (beside the trivial solution $Z_1 = 1$), the three minimal genuine solutions identified by Ledley (1960).

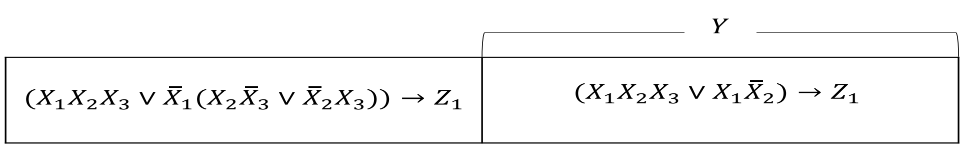


Fig. 14. The function $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = t \to s$ in Example 4

| | $Z_1$ |
|---|---|
| $((X_1X_2X_3 \vee X_1\bar{X}_2) \to 0)$<br>$((X_1X_2X_3 \vee \bar{X}_1(X_2\bar{X}_3 \vee \bar{X}_2X_3)) \to 0)$ | $((X_1X_2X_3 \vee X_1\bar{X}_2) \to 1)$<br>$((X_1X_2X_3 \vee \bar{X}_1(X_2\bar{X}_3 \vee \bar{X}_2X_3)) \to 1)$ |

(a) $g(\boldsymbol{X}, \boldsymbol{Z})$

| | $Z_1$ |
|---|---|
| $(X_2(\bar{X}_1 \vee \bar{X}_3) \vee \bar{X}_1\bar{X}_2)$<br>$(X_1(\bar{X}_2 \vee \bar{X}_3) \vee \bar{X}_1(X_2X_3 \vee \bar{X}_2\bar{X}_3))$ | $(1)(1)$ |

(b) $g(\boldsymbol{X}, \boldsymbol{Z})$

| | $Z_1$ |
|---|---|
| $\bar{X}_1\bar{X}_2\bar{X}_3 \vee$<br>$\bar{X}_1X_2X_3 \vee$<br>$X_1X_2\bar{X}_3$ | $\bar{X}_1\bar{X}_2\bar{X}_3 \vee$<br>$\bar{X}_1X_2X_3 \vee$<br>$X_1X_2\bar{X}_3 \vee$<br>$\bar{X}_1\bar{X}_2X_3 \vee$<br>$\bar{X}_1X_2\bar{X}_3 \vee$<br>$X_1\bar{X}_2\bar{X}_3 \vee$<br>$X_1\bar{X}_2X_3 \vee$<br>$X_1X_2X_3$ |

(c) $g(\boldsymbol{X}, \boldsymbol{Z})$

Fig. 15. Gradual development of the natural map for $g(\boldsymbol{X}, \boldsymbol{Z})$ obtained by suppressing $\boldsymbol{Y}$ from $h(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ in Fig. 14

| | $Z_1$ |
|---|---|
| $\bar{X}_1\bar{X}_2\bar{X}_3\bar{p}_0 \vee$<br>$\bar{X}_1X_2X_3\bar{p}_3 \vee$<br>$X_1X_2\bar{X}_3\bar{p}_6$ | $\bar{X}_1\bar{X}_2\bar{X}_3p_0 \vee$<br>$\bar{X}_1X_2X_3p_3 \vee$<br>$X_1X_2\bar{X}_3p_6 \vee$<br>$\bar{X}_1\bar{X}_2X_3(1) \vee$<br>$\bar{X}_1X_2\bar{X}_3(1) \vee$<br>$X_1\bar{X}_2\bar{X}_3(1) \vee$<br>$X_1\bar{X}_2X_3(1) \vee$<br>$X_1X_2X_3(1)$ |

$G(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{p})$

Fig. 16. The auxiliary function for Example 4

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$p_0$

| | | | |
|---|---|---|---|
| $X_1\bar{X}_2 \vee X_1X_3 \vee \bar{X}_2X_3 \vee \bar{X}_1X_2\bar{X}_3$ | $X_3 \vee X_1\bar{X}_2 \vee \bar{X}_1X_2$ | $\bar{X}_1 \vee \bar{X}_2 \vee X_3$ | $\bar{X}_2 \vee X_1X_3 \vee \bar{X}_1\bar{X}_3$ |
| $X_1 \vee \bar{X}_2X_3 \vee X_2\bar{X}_3$ | $X_1 \vee X_2 \vee X_3$ | $1$ | $X_1 \vee \bar{X}_2 \vee \bar{X}_3$ |

$p_6$

$p_3$

Fig. 17. Listing of all particular solutions of Example 4

## Example 5

This example taken from Ledley (1959) illustrates the handling of genuinely vectorial function $\boldsymbol{s}$ and $\boldsymbol{t}$ (of dimensions $l = 3 > 1$), given by

$$s_1 = (Z_1 \vee \bar{Z}_2)\ \bar{Z}_3 \vee Z_1\bar{Z}_2 \tag{47a}$$

$$s_2 = Z_1\bar{Z}_3 \vee \bar{Z}_1Z_3 \tag{47b}$$

$$s_3 = Z_2\bar{Z}_3 \vee \bar{Z}_2Z_3 \tag{47c}$$

and

$$t_1 = X_1(\bar{X}_2 \vee X_3\bar{X}_4) \vee \bar{X}_2X_3 \tag{48a}$$

$$t_2 = (\bar{X}_1\bar{X}_3 \vee X_1X_3)\ (\bar{X}_2 \vee \bar{X}_4) \tag{48b}$$

$$t_3 = \bar{X}_2(\bar{X}_3 \vee X_1X_4) \vee (X_1 \vee X_2)\ X_3\bar{X}_4 \tag{48c}$$

In (47c), we had to correct a typo in original expression given by Ledley (1959). This example lacks unwarranted $\boldsymbol{Y}$ variables, and hence we directly construct the function $g(\boldsymbol{X}, \boldsymbol{Z})$ such that

$$g(\boldsymbol{X}, \boldsymbol{Z}) = \wedge_{i=1}^{3}(s_i(\boldsymbol{Z}) \odot t_i(\boldsymbol{X})) \tag{49}$$

This function is represented by the map in Fig. 18(b) inspired by the map in Fig. 18(a) and the facts that ($1 \odot t = t$) and ($0 \odot t$) = $\bar{t}$. We compute the entries of the map in Fig. 18(b) *via* the conventional Karnaugh maps in Fig. 19. Our final result for $g(\boldsymbol{X}, \boldsymbol{Z})$ in Fig. 20 shows that there is a single appearance for each of the 16 atoms of $B_{512} = FB(X_1, X_2, X_3, X_4)$. Therefore, the consistency condition is the identity (0 = 0), there is a single particular solution, and the map in Fig. 20 serves as well to represent the auxiliary function $G(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{p})$. The final solution is

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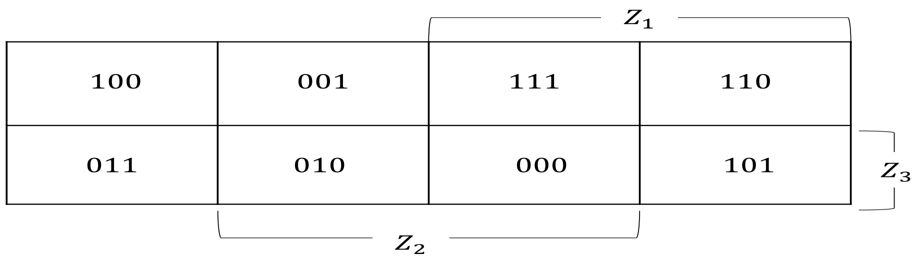
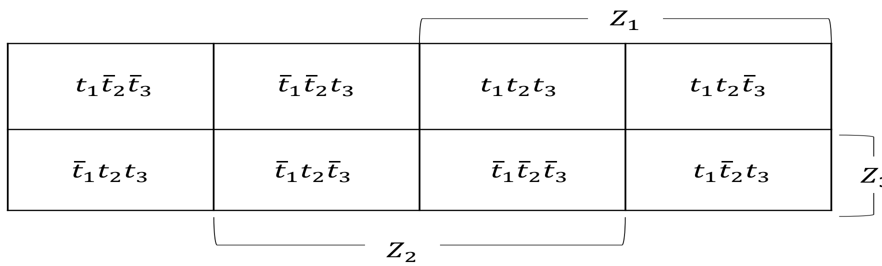
$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} X_1 \vee X_2 X_4 \\ X_2 \vee X_1 X_3 \\ \bar{X}_3 \vee X_2 X_4 \end{bmatrix} \tag{50}$$

in agreement with that found by Ledley (1959).

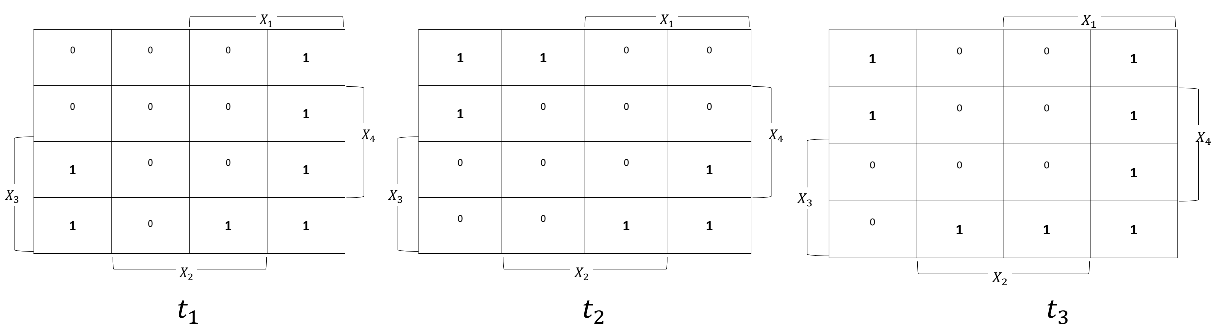



(a) $s_1 s_2 s_3$




(b) $g(\boldsymbol{s}(\boldsymbol{Z}), \boldsymbol{t})$

Fig. 18. Map for the functions $s_1, s_2, s_3$, and $g(\boldsymbol{s}(\boldsymbol{Z}), \boldsymbol{t})$



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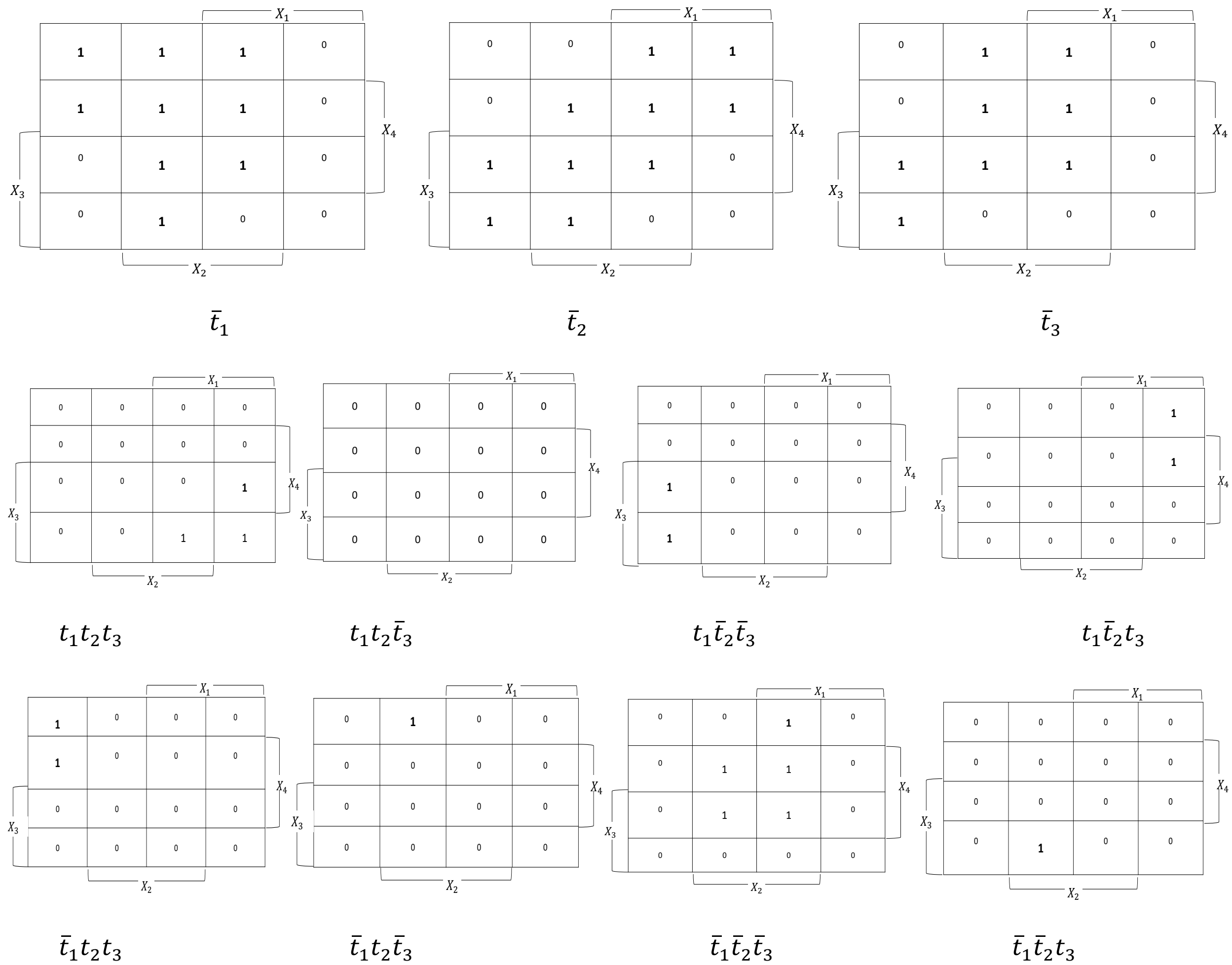


Fig. 19. Conventional Karnaugh maps used in the evaluation of entries in Fig. 18

| | | $Z_1$ | | |
|---|---|---|---|---|
| $\bar{X}_1\bar{X}_2X_3X_4 \vee \bar{X}_1\bar{X}_2X_3\bar{X}_4$ | $\bar{X}_1X_2X_3\bar{X}_4$ | $X_1\bar{X}_2X_3X_4 \vee X_1\bar{X}_2X_3\bar{X}_4 \vee X_1X_2X_3\bar{X}_4$ | $0$ | |
| $\bar{X}_1\bar{X}_2\bar{X}_3\bar{X}_4 \vee \bar{X}_1\bar{X}_2\bar{X}_3X_4$ | $\bar{X}_1X_2\bar{X}_3\bar{X}_4$ | $X_1X_2\bar{X}_3\bar{X}_4 \vee X_1X_2\bar{X}_3X_4 \vee X_1X_2X_3X_4 \vee \bar{X}_1X_2\bar{X}_3X_4 \vee \bar{X}_1X_2X_3X_4$ | $X_1\bar{X}_2\bar{X}_3\bar{X}_4 \vee X_1\bar{X}_2\bar{X}_3X_4$ | $Z_3$ |
| | $Z_2$ | | | |

Fig. 20. Final map for both $g(\boldsymbol{X},\boldsymbol{Z})$ and $G(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{p})$

## 4. Conclusions

This paper introduced dramatic improvements to the techniques used for handling an elementary problem of digital design. These improvements included (a) the use of a transparent algebraic representation all throughout the analysis instead of going to a cryptic numerical representation, (b) the solution of a single equation instead of solving two sets of equations at two sequential stages, (c) the use of powerful techniques of 'big' Boolean algebras to express outputs in terms of inputs, (d) a better conceptual understanding by imposing certain "consistency conditions" that allow complete conditional solutions to emerge instead of resorting to partial solutions or accepting that no solutions exist, and (e) the suppression of unwarranted or intermediary variables, right from the outset instead of making an extensive search as an afterthought. These improvements resulted in much faster (and occasionally corrected) solutions for the examples discussed herein.

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