

Fast Calculation Methods for Reliability of Connected- (r,s) -out-of- (m,n) :F Lattice System in Special Cases

Taishin Nakamura*, Hisashi Yamamoto, Xiao Xiao

Faculty of System Design, Tokyo Metropolitan University
6-6, Hino, Tokyo, 191-0065, Japan

*Corresponding author: nakamura-taishin@ed.tmu.ac.jp

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Abstract

A connected- (r,s) -out-of- (m,n) :F lattice system consists of $m \times n$ components arranged as an (m,n) matrix, and fails if and only if the system has an (r,s) sub-matrix where all components fail. Though the previous study has proposed the recursive equation for computing the system reliability, it takes much time to compute the reliability. For one-dimensional systems, a matrix formula was provided based on the existing recursive equation when the system consists of independent and identically distributed components. The numerical experiments showed that the matrix formula was more efficient than the recursive equation. In contrast, for two-dimensional systems, the recursive equation is comparatively complex, so that it is difficult to derive a matrix formula directly from the recursive equation. In this study, we derive general forms of matrices for computing the reliability of the connected- (r,s) -out-of- (m,n) :F lattice system consisting of independent and identically distributed components in the case of $r = m - 1$ and $r = m - 2$. We compare our proposed method with the recursive equation in order to verify the effectiveness of the proposed method using numerical experiments.

Keywords- Connected- (r,s) -out-of- (m,n) :F lattice system, System reliability, Matrix formula.

1. Introduction

Systems of modern societies such as communication systems and electricity supply systems have a significant number of components, and the components have complicated relationships. The failure of systems could make an enormous effect on our society. For example, on March 11, 2011, the nuclear accidents in Japan have resulted in the release of heavy doses of radioactive materials from the Fukushima I Nuclear Power Plant (Kuo and Zhu, 2012). Thus, ensuring the reliability of these systems is a crucial issue and engineers have recently reaffirmed the significance of the system failure analysis.

Consecutive-type reliability systems have been investigated as system reliability models for expressing practical systems (Kuo et al., 2002; Triantafyllou, 2015). A consecutive- k -out-of- n :F system consists of n components which are linearly arranged, and the system fails if and only if at least k consecutive components fail. This system can be used to model computer network systems and pipeline network systems. The two-dimensional version of the consecutive- k -out-of- n :F system was introduced by Salvia and Lasher (1990). After that, Boehme et al. (1992) defined a connected- (r,s) -out-of- (m,n) :F lattice system. The system consists of $m \times n$ components arranged as an (m,n) matrix, and fails if and only if the system has an (r,s) sub-matrix where all components fail (Fig. 1). The system can be applied to supervision systems (Boehme et al., 1992) and lighting systems. Yamamoto and Miyakawa (1995) proposed the recursive equation for computing the system reliability. The recursive equation is based on the event decomposition approach (Kuo et al., 2002), and we can calculate the system reliability when the components are Independent but Non-Identically Distributed (INID).

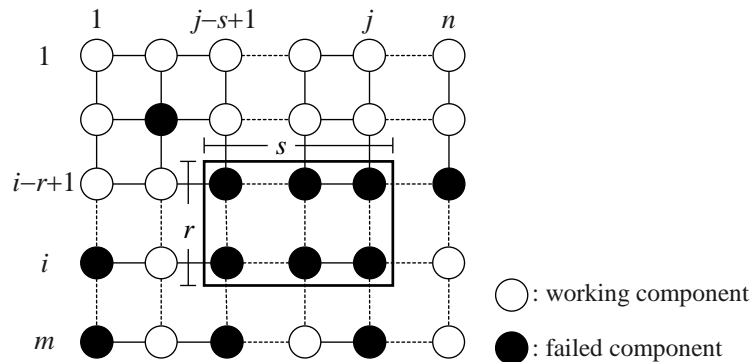


Fig. 1. Example of a connected- (r,s) -out-of- (m,n) :F lattice system failure

Various previous studies have improved methods for computing the reliability of consecutive- k -out-of- n :F system. Hwang (1982) provided a recursive equation for computing the reliability of the system consisting of INID components. After that, Lin (2004) provided a matrix formula for computing the reliability of the system consisting of Independent and Identically Distributed (IID) components based on the recursive equations of Hwang (1982). When we compute the system reliability, it is necessary to calculate M^n , where M means a matrix for computing the system reliability. Then, we can reduce the number of matrix multiplications due to not simply adding one to the multiplication but doubling the exponent by repeating squaring based on the law of exponent. The method is called the fast-matrix-power algorithm. Lin (2004) applied matrix multiplications to the fast-matrix-power algorithm and indicated we can compute the reliability of the system in shorter time.

As an extension to two-dimensional systems, if we obtain matrices for computing the system reliability based on the recursive equation of Yamamoto and Miyakawa (1995), we can compute the reliability of the connected- (r,s) -out-of- (m,n) :F lattice system. However, the recursive equation of Yamamoto and Miyakawa (1995) is complex compared with the recursive equation of Hwang (1982), so that it is difficult to drive a matrix formula directly from the recursive equation. Based on the above, the aim of this paper is to derive general forms of matrices for computing the reliability of the connected- (r,s) -out-of- (m,n) :F lattice system consisting of IID components in the case of $r = m - 1$ and $r = m - 2$, and compute the reliability with the fast-matrix-power algorithm. We compare our proposed method with the recursive equation in order to verify the effectiveness of the proposed method using numerical experiments.

2. Notations

In the following sections, we will assume that

1. Each component and the system are either working or failed.
2. All components are independent.
3. All components have the same unreliability and this value is given.

We define the following notations.

- p, q : component reliability, component unreliability.
- (i, j) : index of component located at i -th row and j -th column for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.
- δ_{ij} : index function which takes 1 when components $(i, j), (i+1, j), \dots, (i+r-1, j)$ fail and 0 otherwise in j -th column for $i = 1, 2, \dots, m-r+1$ and $j = 1, 2, \dots, n$.
- δ : $(m-r+1)$ -dimensional binary vector $(\delta_1, \delta_2, \dots, \delta_{m-r+1})$ where $\delta_i = \delta_{ij}$ for $\forall j$. Note that there must exist at least r 0s or no 0s between the adjacent 1s. For example, assuming $r = 2, m = 4$, a three-dimensional vector $(1, 0, 1)$ does not exist because the first element 1 means that components $(1, \cdot)$ and $(2, \cdot)$ fail and the third element 1 means that components $(3, \cdot)$ and $(4, \cdot)$ fail. On the other hand, the second element 0 means that components either $(2, \cdot)$ or $(3, \cdot)$ works. Consequently, we can find that $(1, 0, 1)$ does not exist due to the contradiction.
- $F(\delta)$: probability that the event represented by $\square\square$ is realized for the components in a certain column.
- \mathbf{g} : $(m-r+1)$ -dimensional vector $(g_1, g_2, \dots, g_{m-r+1})$ where $1 \leq g_i \leq s$ for $i = 1, 2, \dots, m-r+1$.
- $B(g_i; i, j)$: event that all components fail in the $r \times g_i$ grids with component (i, j) at the upper right corner for $i = 1, 2, \dots, m-r+1$ and $j = 1, 2, \dots, n$.
- $R((r, s), (m, n))$: reliability of the connected- (r, s) -out-of- (m, n) :F lattice system.
- $R((r, s), (m, j); \mathbf{g})$: reliability of the connected- (r, s) -out-of- (m, n) :F lattice system given that the event $B(g_i; i, j)$ does not occur for each $i (i = 1, 2, \dots, m-r+1)$ as shown in Fig. 2.

3. Matrix Approach for the Reliability

In this section, we provide matrices to obtain the reliability of the connected- (r, s) -out-of- (m, n) :F lattice systems consisting of IID components. To derive general forms of matrices for arbitrary s , we consider the cases of $r = m-1$ and $r = m-2$.

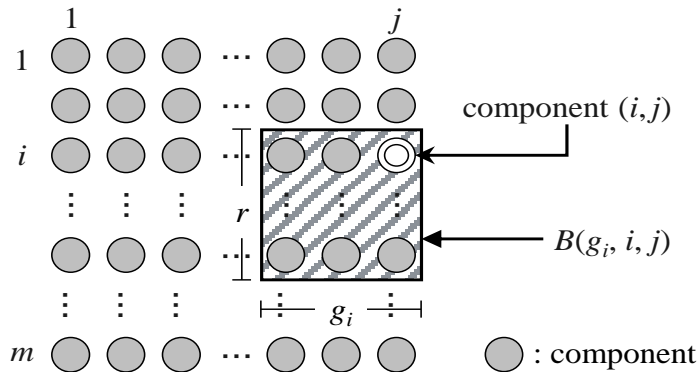


Fig. 2. Illustration of $R((r, s), (m, j); \mathbf{g})$

3.1 Case of $r = m - 1$

In this subsection, we derive a general form of matrix based on the recursive equation (Yamamoto and Miyakawa, 1995). $R((m-1, s), (m, j); (g_1, g_2))$ represents the reliability of connected- $(m-1, s)$ -out-of- (m, j) :F lattice system in which at least one component works in the $r \times g_1$ grids with component $(1, j)$ at the upper right corner and the $r \times g_2$ grids with component $(2, j)$ at the upper right corner. In particular, $R((m-1, s), (m, j); (s, s))$ is the reliability of the system which has no special failure conditions, namely, $R((m-1, s), (m, j))$.

According to Yamamoto and Miyakawa (1995), we derive recursive equations in the case of $r = m - 1$. For simplicity, denote $R((m-1, s), (m, j); (g_1, g_2))$ as $R(j; g_1, g_2)$. In the case of $g_1, g_2 \geq 2$, the reliability $R(j; g_1, g_2)$ can be expressed as

$$R(j; g_1, g_2) = F((0,0)) \cdot R(j-1; s, s) + F((0,1)) \cdot R(j-1; s, g_2-1) + F((1,0)) \cdot R(j-1; g_1-1, s) + F((1,1)) \cdot R(j-1; g_1-1, g_2-1). \quad (1)$$

Equation (1) shows that the system reliability can decompose based on the states of j -th column. Based on Theorem 2 (Yamamoto and Miyakawa, 1995), especially when $g_1 = g_2$, the system reliabilities $R(j-1; s, g_1-1)$ and $R(j-1; g_1-1, s)$ have the same system reliability since both systems have the same conditions symmetrically as shown in Fig. 3 from the assumption that the system consists of IID components. From $R(j-1; s, g_1-1) = R(j-1; g_1-1, s)$ and $F((0,1)) = F((1,0))$ from the assumption that the system consists of IID components, we can rewrite Eq. (1) into the following equation.

$$R(j; g_1, g_1) = F((0,0)) \cdot R(j-1; s, s) + 2F((0,1)) \cdot R(j-1; s, g_1-1) + F((1,1)) \cdot R(j-1; g_1-1, g_1-1). \quad (2)$$

In the case of $g_1 = g_2 = 1$, the reliability $R(j;1,1)$ can be expressed as

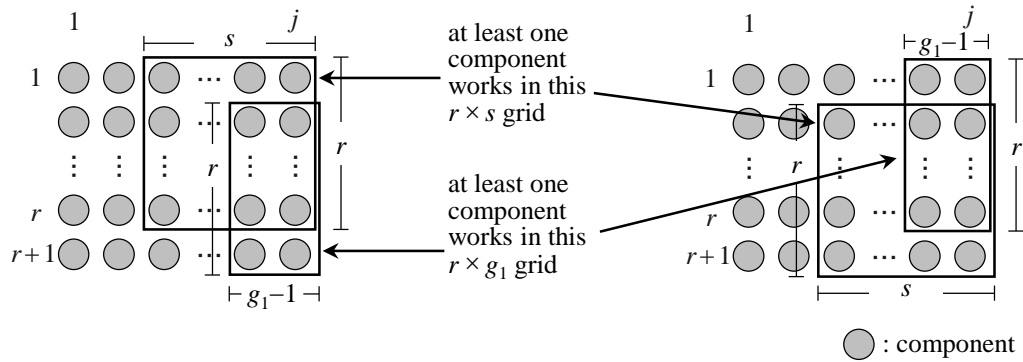


Fig. 3. Example of two systems which have the same conditions symmetrically

$$R(j;1,1) = F((0,0)) \cdot R(j-1;s,s). \quad (3)$$

The system corresponding to $R(j;1,1)$ has the condition that at least one component works in the $r \times 1$ grids with component $(1, j)$ at the upper right corner and the $r \times 1$ grids with component $(2, j)$ at the upper right corner. Hence, when the event represented by $\delta \in \{(0,1), (1,0), (1,1)\}$ is realized for the components in j -th column, the system fails, that is, the reliability is equal to 0. Thus, Eq. (3) holds. In the case of $g_1 = 1$ and $g_2 \geq 2$, the reliability $R(j;1, g_2)$ can be expressed as

$$R(j;1, g_2) = F((0,0)) \cdot R(j-1;s,s) + F((0,1)) \cdot R(j-1;s, g_2-1). \quad (4)$$

For similar reasons to Eq. (3), Eq. (4) holds.

For simplicity, let α, β, γ be occurrence probabilities of column states as follow:

$$\begin{aligned} \alpha &= F((0,0)) = 1 - (2pq^r + q^{r+1}), \\ \beta &= F((0,1)) = F((1,0)) = pq^r, \\ \gamma &= F((1,1)) = q^{r+1}. \end{aligned}$$

From Eqs. (1), (2), (3) and (4), the following matrix representation $M_{1,s}$ is obtained.

$$\begin{pmatrix}
 & (s,s) & (s,s-1) & (s,s-2) & \cdots & (s,1) & (s-1,s-1) & (s-1,s-2) & \cdots & (s-1,1) & \cdots & (1,1) \\
 (s,s) & \alpha & 2\beta & & & O & \gamma & & & O & & \\
 (s,s-1) & \alpha & \beta & \beta & & & & \gamma & & & & \\
 \vdots & \vdots & \vdots & & \ddots & & & & \ddots & & & \\
 (s,2) & \alpha & \beta & O & & \beta & O & & & \gamma & & \\
 (s,1) & \alpha & \beta & & & 0 & & & & 0 & & \\
 \hline
 (s-1,s-1) & \alpha & 0 & 2\beta & & O & & & & & & \\
 \vdots & \alpha & \vdots & \vdots & \ddots & & & & & & & \\
 (s-1,2) & \alpha & 0 & \beta & & \beta & & & & & & \\
 (s-1,1) & \alpha & 0 & \beta & O & 0 & & & & & & \\
 \vdots & \vdots & & & & & & & & & & \\
 \hline
 (1,1) & \alpha & & O & & & & & & & &
 \end{pmatrix} \quad (5)$$

where (g_1, g_2) in a row of Eq. (5) means $R(j; g_1, g_2)$ and (g_1, g_2) in a column of Eq. (5) means $R(j-1; g_1, g_2)$ and O means a zero matrix. For example, the 1st row in Eq. (5) corresponds to Eq. (2). When $g_1 = s$, the system reliability decomposes into s system reliabilities like $R(j; s, s), R(j; s, s-1), \dots, R(j; s, 1)$. Similarly, when $g_1 = s-1$, the system reliability decomposes into $s-1$ system reliabilities. Thus, when dimensions of the matrix $M_{1,s}$ are $k \times k$,

$$k = 1 + 2 + \cdots + (s-1) + s = \frac{1}{2} s(s+1). \quad (6)$$

For $2 \leq s \leq n$, the general form of matrix $M_{1,s}$ is given by

$$M_{1,s} = \begin{pmatrix}
 A_s & B_s & & O \\
 A_{s-1} & & B_{s-1} & \\
 \vdots & & & \ddots \\
 A_2 & & O & B_2 \\
 a & & & 0
 \end{pmatrix}_{k \times k} \quad (7)$$

where, for $l = 2, 3, \dots, s$,

$$A_l = \begin{pmatrix}
 a & & 2b & & O \\
 a & & b & b & \\
 a & O_{l \times (s-l)} & b & & \ddots \\
 \vdots & & \vdots & O & b \\
 a & & b & & 0
 \end{pmatrix}_{l \times s}, \quad B_l = \begin{pmatrix}
 c & & O \\
 & \ddots & \\
 O & & c \\
 & & & 0
 \end{pmatrix}_{l \times (l-1)},$$

and O means a zero matrix.

The initial condition of $R(j-1; g_1, g_2)$ is given as follows:

$$R_{m-1}(0; g_1, g_2) = \begin{cases} 1, & g_1 = g_2 = s, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

We can calculate $R((m-1, s), (m, n))$, namely, the reliability of the connected- $(m \square 1, s)$ -out-of- (m, n) :F lattice system, by using the following equation.

$$R((m-1, s), (m, n)) = \sum_{g_1, g_2: 0 \leq g_1, g_2 \leq s} R(n; g_1, g_2). \quad (9)$$

According to Eqs. (8) and (9), we can give the alternative representation of the reliability $R((m-1, s), (m, n))$ with the matrix of Eq. (5) as follows:

$$R((m-1, s), (m, n)) = \boldsymbol{\pi}_0 (M_{1,s})^n \mathbf{u}^T, \quad (10)$$

where $\boldsymbol{\pi}_0 = (1, 0, \dots, 0)_{1 \times k}$ and $\mathbf{u} = (1, 1, \dots, 1)_{1 \times k}$, and \mathbf{u}^T means the transpose of the vector \mathbf{u} . An element of the $1 \times k$ vector $\boldsymbol{\pi}_0 (M_{1,s})^n$ corresponds to the system reliability $R_{m-1}(n; g_1, g_2)$ under the initial condition given by Eq. (8). The vector \mathbf{u} takes the sum of system reliabilities. The system reliability is obtained by the product of $\boldsymbol{\pi}_0 (M_{1,s})^n$ and \mathbf{u}^T .

3.2 Case of $r = m - 2$

In this subsection, we derive a general form of matrix based on the recursive equation (Yamamoto and Miyakawa, 1995) as with Subsection 2.1. Occurrence probabilities of column states are given by

$$\begin{aligned} a &= F((0,0,0)) = 1 - (2b - c - 2d - e), \\ b &= F((1,0,0)) = F((0,0,1)) = pq^r, \\ c &= F((0,1,0)) = p^2q^r, \\ d &= F((1,1,0)) = F((0,1,1)) = pq^{r+1}, \\ e &= F((1,1,1)) = q^{r+2}. \end{aligned}$$

Let $M_{2,s}$ be a matrix for computing the reliability of the system with $r = m - 2$. The dimension of the matrix $M_{2,s}$ is obtained by

$$h = \frac{1}{6}s(s+1)(2s+1). \quad (11)$$

For $2 \leq s \leq n$, the matrix $M_{2,s}$ is given by

$$M_{2,s} = \begin{pmatrix} A_s & B_s & & O \\ A_{s-1} & & B_{s-1} & \\ \vdots & & & \ddots \\ A_2 & O & & B_2 \\ a & & & 0 \end{pmatrix}_{h \times h}, \quad (12)$$

where, for $l = 2, 3, \dots, s$,

$$A_l = \begin{pmatrix} C_l & & D_l & & \\ \vdots & O_{l^2 \times s(s-l)} & & \ddots & \\ C_l & & & & D_l \\ C_l & & & & O_{l \times s} \end{pmatrix}_{l^2 \times l^2}, \quad B_l = \begin{pmatrix} E_l & O \\ & \ddots \\ O & E_l \\ & & & 0 \end{pmatrix}_{l^2 \times (l-1)^2},$$

$$C_l = \begin{pmatrix} a & 2b & & O \\ a & b & b & \\ a & O_{l \times (s-l)} & b & \ddots \\ \vdots & & \vdots & O & b \\ a & b & & 0 \end{pmatrix}_{l \times s}, \quad D_l = \begin{pmatrix} c & 2d & & O \\ c & d & d & \\ c & O_{l \times (s-l)} & d & \ddots \\ \vdots & & \vdots & O & d \\ c & d & & 0 \end{pmatrix}_{l \times s},$$

$$E_l = \begin{pmatrix} e & & O \\ & \ddots & \\ O & & e \\ & & & 0 \end{pmatrix}_{l \times (l-1)},$$

and O means a zero matrix.

We can calculate $R((m-2, s), (m, n))$, namely, the reliability of the connected- $(m-2, s)$ -out-of- (m, n) :F lattice system as follow:

$$R((m-2, s), (m, n)) = \boldsymbol{\pi}_0 (M_{2,s})^n \mathbf{u}^T, \quad (13)$$

where $\boldsymbol{\pi}_0 = (1, 0, \dots, 0)_{1 \times h}$ and $\mathbf{u} = (1, 1, \dots, 1)_{1 \times h}$, and \mathbf{u}^T means the transpose of the vector \mathbf{u} .

4. Numerical Experiment

All numerical experiments in this paper were conducted by using the computer (Intel Core i7 CPU (2.4GHz), Memory 16.0 GB), and programs were written in C language and compiled by Visual Studio 2010. In the case of $r = m - 1$, first, we generate the matrix for computing the reliability of the connected- $(m-1, s)$ -out-of- (m, n) :F lattice system from Eq. (5). Next, we use the fast-matrix-power algorithm to calculate the power of matrices and compute the system reliability from Eq.(10). This matrix approach is referred to *MA* in this section. As another method, we can compute the system reliability by the recursive equation of Yamamoto and Miyakawa (1995) directly. This algorithm with the recursive equation is referred to *RE* in this section.

Table 1. Comparison of calculating time when $m = 5$, $r = 4$, $s = 12$ and $p = 0.4$

n	10000	40000	160000	640000	2560000	10240000	40960000
MA (sec.)	0.016	0.031	0.031	0.031	0.031	0.046	0.047
RE (sec.)	0.031	0.047	0.141	0.532	2.141	8.548	34.236
System reliability	1.000000	0.999998	0.999994	0.999975	0.999900	0.999600	0.998402

Table 2. Comparison of calculating time when $m = 5$, $r = 4$, $n = 40000$ and $p = 0.3$

s	8	10	12	14	16	18	20
MA (sec.)	0.015	0.015	0.031	0.063	0.141	0.313	0.594
RE (sec.)	0.015	0.031	0.047	0.062	0.063	0.0078	0.078
System reliability	0.523193	0.962683	0.997791	0.999872	0.999993	1.000000	1.000000

We show the results of numerical experiments for systems fixed m , r , s and p in Table 1, and the results for systems fixed m , n , r and p in Table 2. As can be seen from Table 1, our proposed method is efficient, especially for systems with large n . The reason is that by utilizing the fast-matrix-power algorithm (Lin, 2004), the reliability can be obtained in logarithmic time for the parameter n . According to Table 2, the proposed method is efficient to a limited extent, and, when the parameter s is large, we can find that the computing time of the proposed algorithm is longer than that of the recursive equation (Yamamoto and Miyakawa, 1995). The reason is that, as the parameter s is large, the size of the matrix increases.

5. Conclusion

In this paper, we derived general forms of matrices for computing the reliability of the connected- (r,s) -out-of- (m,n) :F lattice system consisting of IID components in the case of $r = m - 1$ and $r = m - 2$ and compute the reliability with the fast-matrix-power algorithm. The results of numerical experiments indicated that our proposed algorithm can compute the reliability faster than the recursive equation of Yamamoto and Miyakawa (1995) when the number of columns is large. With the method obtained in this paper, we can evaluate the reliability of connected- (r,s) -out-of- (m,n) : F lattice system for arbitrary parameters approximately. For example, the proposed method can be used to obtain the upper and lower bounds of the reliability of connected- (r,s) -out-of- (m,n) : F lattice system for arbitrary parameters by combining with Yuge and Yanagi (2004). Yuge and Yanagi (2004) gave the lower bound of the reliability, using the reliability (strictly, lower bound) of the subsystem in the case of $r = m - 1$. The upper bound can also be obtained similarly. By combining our proposed method with Yuge and Yanagi (2004), we can compute the upper and lower bounds of system reliabilities fast.

In this paper, we considered the case that general forms of matrices can be determined uniquely, whereas, since the general forms cannot be determined uniquely in the case of $m-r \geq 3$, it is difficult to generalize this method. Accordingly, we should propose a novel method to derive matrices fast in the future. Furthermore, although we have studied the method to compute the reliability of the system consisting of IID components, we would like to develop a computing method for the system consisting of INID components as a future extension.

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