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Studying Multi-Stage Diffusion Dynamics using Epidemic Modeling Framework

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Abstract
Buying process has always carried a two-fold perspective with itself. On one hand, it is important for individuals and on other hand it is equally important for the firms to deliver the perfect need and want to the customer. Amongst this entire process, awareness along with positive motivation towards the product; plays an equally significant role in strategizing the plans for any company. Plenty of models have been proposed and many would be in the pipeline that have talked about the connectivity of these processes and their impact on the final adoption. In the current work, these processes have been studied through the analogy taken from epidemic modelling framework. Furthermore, an approximation method; Range Kutta of 4^{th} order has been utilized to come to a near approximate solution to the otherwise available non-closed form solution. The proposed modelling framework is validated on real-life data sets and the results depict the existence and presence of various stages under consideration.

Keywords- Buying behaviour, Epidemiology, Innovation adoption, Runge-Kutta method, SIR-model.

1. Introduction

In today's hectic and technological environment, all living and non-living things around the world have to struggle to maintain their existence. There is an adage that change is the law of nature and if you don't change with nature then your existence will be finished. The golden toad (Bufoperigones), Polar Bear, Adelie Penguin, North Atlantic Cod etc. are some species which did not change themselves according to nature so these species become either extinct or are on the verge of extinction (Crump et al.,1992; Lynch et al., 2014; Myers and Worm, 2005). A similar thing happens in market, if a company does not change its product according to the demand of potential customers, the product gets eliminated from the market (Bumgardner et al., 2011). Motorola, an American telephone company that launched the first handphone in 1973, failed after a few years because this company didn't change the software of the phone according to

the demand of customers (Nair et al., 2014). That is why companies always try to upgrade products according to the needs of customers to stay in the market. Practical implementation of a new idea or some improvement/deterioration in the already invented product can be defined as innovation (Kanagal, 2015). Innovation takes place mainly in two ways: First by inventing something and second by improving quality, reliability etc. in the existing offering (Anand et al., 2014).

The success of any innovation depends on how the information about its existence of innovation is spread in the social system and how potential customers react towards it (Ebadi and Utterback,1984). That's why it is very important and imperative to understand the diffusion of innovation. Diffusion of innovation is the study of the spread of innovation in the social system (Singh et al., 2012; Singhal et al., 2019a). There are four fundamental elements in the diffusion of the innovation process namely as time, innovation, the channel of communication and the social system (Anand et al., 2016; Aggrawal et al., 2022). The study of the diffusion of innovation via the different channels of communication in the social system can be defined as the diffusion of innovation (Anand et al., 2018). Researchers have done a lot of intensive studies on different domains like social media, economics, sociology and marketing (Palloni, 2001; Hall, 2004; Anand et al., 2021). In today's ever-evolving and technological world, it has become comparatively easier to propagate innovation in any type of market. Before the invention of the internet, companies used to spend a large part of the budget for this work and used to spend good time on promotion and advertising (Berger and Bechwati, 2001). But since the inception of internet, it now takes very less time to reach to people and acquaint them of the presence of innovation (Subramanian et al., 2016). Internet has thus greatly reduced the expenditure on the promotion and advertising of companies (Alyoubi and Yamin, 2021; Zhai et al., 2021). An adopter of innovation does not adopt the innovation immediately after becoming aware of it. Potential customers first extract the information from any external or internal source and form his/her own vision for the innovation, after this vision he/she embraces the innovation. So, adoption process can be defined by the actions taken by the potential customer before the final acceptance of the product (Beckinsale et al., 2011).

The adoption of a product involves a multi-stage process in which the activities of potential customers are studied from the time they become aware of the product's existence in the market until they make a decision to finally purchase it. The adoption process incorporates, knowledge, persuasion, decision, implementation, and conformation (Aggarwal et al., 2019). The Field of the adoption process is not new in the market but as known that there are always many changes that can occur in the market. Fifty - sixty years ago technology in the market was very less compared to the present day but with the passage of time, there has been a change in technology, due to which there has been a lot of change in thinking and understanding of the people. That's why the process of Customer acceptance of the product cannot be fully understood using the old principles. In the field of the Adoption process of a product in the market, various researchers have done their research (Ryan, 1948; Kapur et al., 2019; Singh et al., 2021). Some researchers defined adoption as a five-stage process which includes awareness, interest, evaluation, trial, and adoption (Hassinger, 1959). There is no doubt that the division was quite convenient at that time, and defined the adoption process very well. Many researchers highlighted the impact of word of mouth in the adoption process (Martilla, 1971; Cheung et al., 2008). In the early stage of product sales, Sales are primarily due to interpersonal relationships such as contacts of the manufacturer, contacts of sellers associated with the company, and so on and the sales occur later as a result of the effect of word of mouth. Ettlie (1980) defined the adoption stage as a six-stage process incorporating awareness, interest, evaluation, trial, adoption and implantation. In addition to the above-discussed product adoption process, rogers defined the most well-known stages incorporating knowledge, persuasion, decision, implementation and confirmation (Rogers et al., 2014). These five stages can be studied under three-stage in which knowledge and persuasion can be clubbed and studied under the awareness stage, decision and implementation can be studied under the motivation stage

and conformation can be studied in the eventual adoption stage (Anand et al., 2014). Below mentioned are some key points regarding the three clubbed stages.

Awareness: Awareness is a stage when individuals get to know about the presence of a product in the market and try to collect information about the product (Hsiao et al., 2014). Companies have different types of attractive and effective advertisements to spread awareness about their product. Potential customers can get acquainted about the existence of the product in many ways; some potential Customers get aware of through the company's promotional strategy, some by word of mouth from previous buyers/adopters (Merikle, 1984).

Motivation: When an individual became aware about the presence of a product in the market, he becomes interested to know about the product due to the general tendency of human beings and he starts collecting more and more information about the product from society or mass media (Singhal et al., 2019b). According to the information received, an individual forms his opinion about the product, if the individual is satisfied with the information received about the product, then he gets motivated to buy that product. Such individual is called motivated (Anand et al., 2016; Sachdeva, 2017).

Eventual Adoption: If an individual takes complete information about the product and he finds the product fulfilling his needs, then the customer accepts the product. In this way, an individual passes through all steps of adoption and reaches the step of adoption and that individual became an adopter of the product (Agarwal et al., 2017). The Figure 1 shown below can be understood for the same.

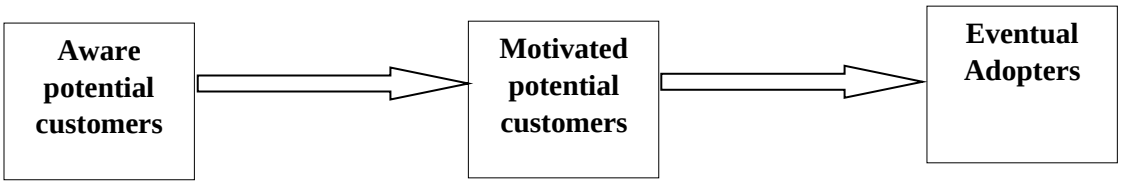

Figure 1. Customer adoption process.

Despite the fact that there has been a great deal of work performed in this field. Table 1 shows the quality and diversity of the work carried out in this field and specifically in the work presented in this article. The whole idea of designing the proposal has been a two-field perspective. Adoption process has been studied by many researchers but very few have focussed on bringing this analogy from epidemic modelling framework. Secondly, of having suitable solution methodology for the designed problem as many time the proposal does not lead to any significant outcome. This methodology helps to deal with such problems.

Table 1. Comparison of the proposed model and existing models.

Earlier set of work	Ability to estimate potential adopters	Ability to estimate Aware potential customer	Ability to estimate Motivated potential customer	Applicable for different rates	Epidemiology based
Kapur et al. (2004)	Yes	No	No	No	No
Anand et al. (2014)	Yes	Yes	Yes	No	No
Singhal et al. (2019)	Yes	Yes	Yes	Yes	No
Proposed model	Yes	Yes	Yes	Yes	Yes

This paper is divided into several parts to make it easier to grasp the modelling framework and results of this article. In the first part, the relevance of the proposed research to current and real-life scenarios has been proposed. In the second section, the authors have provided notations and some basic boundary lines (assumptions). The mathematical modelling framework has been proposed in the third section. Data analysis and model validation has been provided in the fourth section. The Managerial implication, future scope, and conclusions are discussed in the 5th 6th and 7th section respectively followed by acknowledgements at last. A list of references is also provided under reference section.

2. Notations and Assumptions

2.1 Notations

m = Potential market size.

$N_{AW}(t)$ = Aware potential customers by time t.

$N_{MO}(t)$ = Motivated potential customer by time t.

$N_{AD}(t)$ = Eventual adoption by time t.

b_1 = Rate of awareness.

b_2 = Rate of motivation.

b_3 =Rate of adoption.

β_1 = Learning parameter for the awareness.

β_2 = Learning parameter for motivation.

β_3 = Learning parameter for eventual adoption.

2.2 Assumptions

In this paper, the structure has been prepared on the basis of some basic assumptions, which are also compatible with the market. Some parsimonious assumptions on which this work stand are as follows:

a. Market size is fixed during the whole life cycle of product and rejection of the product is negligible.
b. All potential customers who are aware of the existence of the product will be motivated to buy the product in future, and these motivated potential customers will buy the product in future and become the eventual adopters of the product.
c. Only one adoption per adopter is taking place. i.e. there is no repeat buying happening.
d. Spread of awareness, motivation toward product and eventual adoption follows logistic growth rate.

3. Model Development

As mentioned in the introduction of this paper, the Epidemiological modelling framework has been utilized. To model the product adoption process, the authors need to first explain the epidemiological modelling framework, and then the similarities between the epidemiological modelling framework and the product adoption process are discussed in order to move ahead.

3.1 Epidemiological Modelling Framework

An epidemic can be defined as an outbreak/disease occurring over a short period of time (Green et al., 2002). It is important to understand that a disease can spread literally from person to person, either through direct contact or by indirect mediums, such as air, water, creatures, objects, or other mediums (Shore et al., 2019). There have been a number of mathematical models presented in the field of epidemics by many researchers (Satsuma et al., 2004; Kabir et al., 2019). In the epidemic modelling framework, there are many

mathematical models based on SIR- Model (S-Susceptible, I-Infected, R-Recover) available (Shulgin et al., 1998; Zaman et al., 2008). SIR- Model is the basic mathematical model in epidemiology, in this model, the population is divided into three parts. In the initial stage, everyone is susceptible to the epidemic, from these susceptible people become infected by direct or indirect contact with the disease, and in the final stage, these infected populations become to recover. These stages are studied under the SIR modelling framework, where S indicate susceptible, I indicate infected, and R indicates the Recover population at any particular time.

3.2 Markov Chain Process

A Markov chain process is a stochastic model that depicts a series of potential occurrences where the probability of occurring each event in future entirely depends on the state obtained in the present event (Chib and Greenberg, 1996). Markov process is a well-known method that is frequently used in mathematical modelling in epidemiology to decipher the behaviour of an individual (Hamra et al., 2013). As previously noted, the process of product adoption by potential customers and the spread of diseases in epidemics are comparable in several ways, therefore the adoption may be simply understood using the Markov process (Bauckhage et al., 2015).

The behaviour of potential customers of the product can be easily understood with help of the following transition diagram as in Figure 2. If at any particular time, i proportion became motivated (in case of epidemiology, infected) then $1-i$ is the proportion of remaining aware potential customers (suspected). If out of motivated potential customers, j proportion adopts the product then $1-j$ is the proportion of the remaining number of motivated potential customers. The adoption process can be expressed in the form of a transition matrix as shown below.

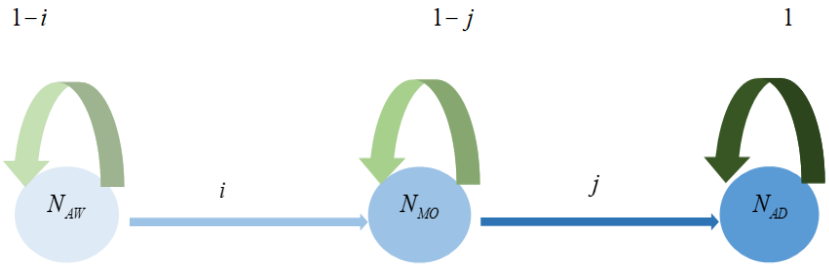

Figure 2. Markov Chain model for the adoption process.

$$\begin{bmatrix} N_{AW}^{t} \\ N_{MO}^{t} \\ N_{AD}^{t} \end{bmatrix} = \begin{bmatrix} 1-i & 0 & 0 \\ i & 1-j & 0 \\ 0 & \mathrm{j} & 1 \end{bmatrix} \begin{bmatrix} N_{AW}^{t-1} \\ N_{MO}^{t-1} \\ N_{AD}^{t-1} \end{bmatrix} \quad (1)$$

With the initial conditions

$$\begin{bmatrix} N_{AW}^{0} \\ N_{MO}^{0} \\ N_{AD}^{0} \end{bmatrix} = \begin{bmatrix} \in \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

In above equation N_{AW}^{t} and N_{AW}^{t-1} denote the number of aware potential customer at time points t and $t-1$, N_{MO}^{t} and N_{MO}^{t-1} denote the number of motivated potential customers but at time points t and $t-1$, N_{AD}^{t} and N_{AD}^{t-1} denote the potential adopters at time point t and $t-1$. As per simple initial conditions, which indicate $\in$ number of potential aware customers, zero motivated potential customers, and zero adopters of the product in the initial stage. Where $\in$ is a very small number. Since the size of the market is very large so, $\in$ can be taken at 0 at the initial stage. With the help of the aforementioned discrete model, authors can effectively estimate that potential buyers who are aware of the product's presence in the market but are not motivated, are motivated but have not yet adopt the product, and the product's sales at any time point. This is a discrete model based on time, but because time is a continuous variable, authors can also portray the adopters' behaviour in a continuous form.

3.3 Similarities Between Epidemiology Framework and the Proposed Adoption Process

During an epidemic, a person can become infected with the disease by direct or indirect contact with a diseased person. In the same way, an aware potential customer of the product in the market can become motivated to buy the product through any promotion and advertisement of the product or from information obtained by the previous user of the product.

In an epidemic, a person infected with a disease recovers from the disease after treatment. As in the SIR model, it is assumed that a person who has recovered from the disease may not be susceptible to the disease again, Similarly, the person motivated for the product in the market buys the product and after buying the product will be out of the adoption process i.e., a potential customer buys the product only once. It is these set of similar functionalities that has motivated us to consider the analogy and take the route of epidemic modelling framework and understand how the innovation adoption process would like when considered along with these parameters.

3.4 The Continuous Transition Form: Proposed Modeling Framework

Before purchasing a product, an individual becomes aware of its existence; here aware individuals are referred to as aware potential customers; they then become motivated to purchase the product; these motivated aware potential customers are referred to as motivated potential customers, and these motivated potential customers adopt the product and become eventual adopters of the product. In this way, it can say that the adoption process is a multi-stage process, which has also already discussed in the introduction part. Let us now understand the same using mathematical modelling framework.

Stage 1: To get the attention of potential customers, during launching a new product, the company aims to make them aware of the product's existence. Different strategies are used by companies to achieve this goal. The awareness rate of the product's presence in the market is proportional to the number of potential customers who remain aware of the product's existence.

$$\frac{dN_{AW}(t)}{dt} = \frac{b_1}{1+\beta e^{-bt}}\left(m - N_{AW}(t)\right) \tag{3}$$

Stage 2: There is a theory in epidemiology that susceptible people become infected when they come into contact with an infected person or a virus, and in the same manner the aware potential customers become motivated potential customer by the efforts made by the company or by the effect of word of mouth spread by customers. So, in this way infected persons can take as motivated potential customer in marketing. Motivated potential customer can be given as following equation:

$$\frac{dN_{MO}(t)}{dt} = \frac{b_2}{1+\beta_2 e^{-b_2 t}}\left(N_{AW}(t) - N_{MO}(t)\right) \tag{4}$$

Companies have to adopt various marketing strategies to increase the sales of the product. The main objective of these various strategies is to make the non-aware potential customers aware of the product, motivate the aware customers about product performance and convert these motivated potential customers into adopters. If a company tries to induce the product through these aware and non-motivated customers, then the company will not be able to do enough justification even with their good promotional campaigns. This would indirectly or directly impact the sales pattern of the product. It is therefore, imperative for companies to estimate and keep into account the aware, non-aware, motivated and non-motivated potential customers of the product.

Since the number of aware and motivated customers have already been taken care of in equation number 3 and 4 respectively, an important observation can be made by making use of their leftovers; that is; the Number of unaware and Non-motivated potential customers. But they cannot be studied in isolation but will be required to be taken under one canopy. Equation 5 given below takes care of this combination of these two attributes.

$$N'(t) = m - N_{MO}(t)) \tag{5}$$

Furthermore, as an analogy is being studied here, one can see that as depicted in an SIR-model, the graph of susceptible (aware) population is decreasing one in nature and a similar type of observation can be made here as well.

Stage 3: As is evident from literature, one of the basic assumptions of the SIR model is that after a person has been treated for a disease, he or she becomes recovered, and that he or she will not become susceptible again. In the same way, it has been argued that as when someone who is motivated toward using a product adopts it, and as he or she has already encountered the awareness and motivation phases, so he or she will not again face these phases. So number of adopters can be given by following differential equation:

$$\frac{dN_{AD}(t)}{dt} = \frac{b_3}{1+\beta_3 e^{-b_3 t}}\left(N_{MO}(t) - N_{AD}(t)\right) \tag{6}$$

This equation will help us out in evaluation the total number of eventual sales for that product.
But, as studied and stated earlier, the number of aware and non- motivated customers; those who have not adopted yet are also a keen participants in the system. This number can be understood using the mathematical structure given using equation (7):

$$N'_{MO}(t) = N_{MO}(t) - N_{AD}(t) \tag{7}$$

To move further, equations (3), (4) and (6) are the equations to study the diffusion dynamics and the total number of products sold in the market.

But an interesting observation can be made after looking at these equations. The closed form solution is not available for these set of simultaneous equations. To cater to this issue and having some concrete thing as an outcome, one can make use of the services offered by approximation methods. That is, numerical solutions of these ordinary differential equations can be found using approximation methods. Literature is evident of having many methodologies to solve them, like; Picard's method, Euler method, Runge-Kutta method, to name a few. Picard method and Euler methods are based on the Taylor series and the Taylor series doesn't have a closed form and needs higher-order derivatives to find Approximate values (Süli, 2010; Atkinson et al., 2011). There are some other methods like Huen's method, Runge-Kutta method of 2^{nd} order, Runge-Kutta method of 4^{th} order and so on, also exists, which are appropriate versions of the Taylor series-based method. In order to solve the problem proposed in this manuscript, Runge-Kutta mthod of 4^{th} order has been utilised. As per the literature, it is one of the most reliable techniques for solving differential equations, since its error in estimating the approximate value of the original equation was negligible (Tan et al., 2012).

4. Data Analysis and Model Validation

In order to validate the proposed set of mathematical modeling framework; authors have used data from four different durable products. The first data is of room AC and is named as Data Set 1 (DS 1), the second set of data is of Unitary AC which is termed as DS 2, Third data is of Nokia Mobile Phone (DS 3) and the fourth one is of Microwave oven (DS 4). All these data sets have been fetched from Anand et al. (2014). A stated earlier, the closed form could not be obtained for the proposed set of models, so their parameter estimation could not be carried out using traditional approaches. Thereby, Runge Kutta method of 4^{th} order has been utilized to begin with the problem evaluation. The estimates given in Table 2 are obtained using the same and have in line interpretation associated with them.

Table 2. The approximated parameter values for proposed model.

Parameter	m	b_1	b_2	b_3	β_1	β_2	β_3
DS 1	101.08	0.842	0.765	0.260	10	5	9.375
DS 2	1300	0.928	0.728	0.065	0.8	0.0014	0.015
DS 3	570.65	0.896	0.266	0.0582	0.5	0.0582	0.002
DS 4	101.6	0.823	0.423	0.301	12	0.9	64.79

According to Table 2, the rate of transformation from one stage to the next stage is decreasing in order. As from table value of b_1 is very high which indicate almost all potential customers become aware about existence of product in a very short time period. The value of b_2 is less as compared to b_1, which implies that potential aware customer gathers information about product before getting motivated towards it. The value of b_3 is very less as compared to b_2 and b_1. Reason behind this can be thought of given the fact that aware and motivated potential observe entire market along with the offerings by competitors before finally adopting the product.

Moreover, to demonstrate the usefulness and reality of the presented model, different statistical errors with respect to the original values of the data set and the values that were detected by the model are calculated. These values are presented in the Table 3 given next.

Table 3. Values of different statistical errors in proposed model.

statistical errors	R-square	MSE	MAE	RMSE
DS1	0.999	3.9682	1.774	1.99
DS2	0.993	588.09	21.33	24.25
DS3	0.980	286.37	36.05	16.92
DS4	0.999	1.08	0.916	1.04

The values of R-square, MSE, MAE and RMSE can be said to be in acceptable mode. Moreover, the fairly good values of R-square is 0.999, 0.993, 0.980, and 0.999 indicate that the proposed modeling framework is well fitted on the all data sets. Values of MSE, MAE and RMSE also shows the closeness of proposal on real life data sets.

A further clarity can be seen using the goodness of fit curves obtained by plotting the actual sales versus the predicted sales. Figures 3, 4, 5 and 6 presents the goodness of fit graphs for the four data sets under consideration, respectively.

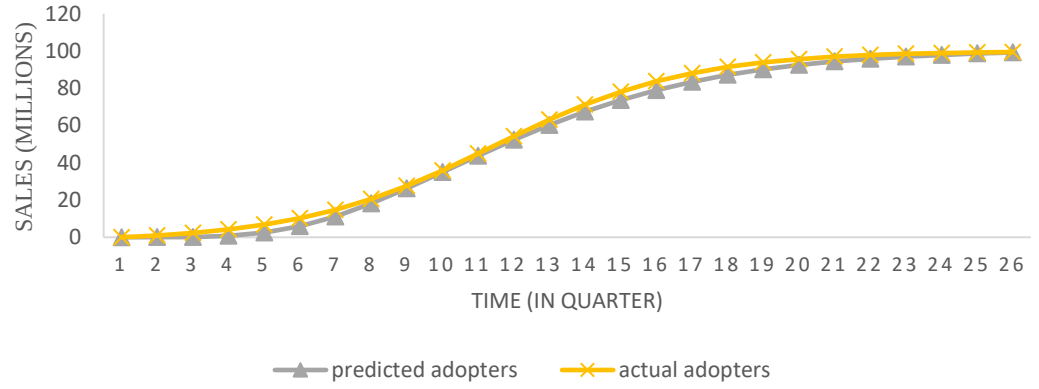

Figure 3. Goodness of fit for DS1.

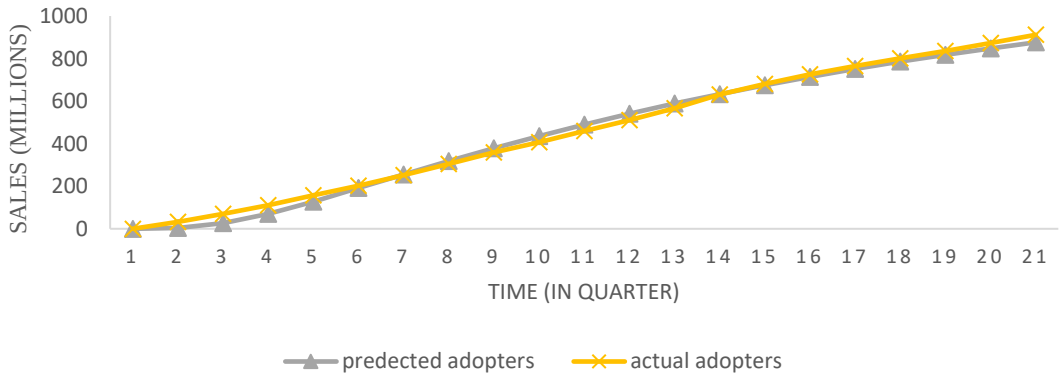

Figure 4. Goodness of fit for DS2.

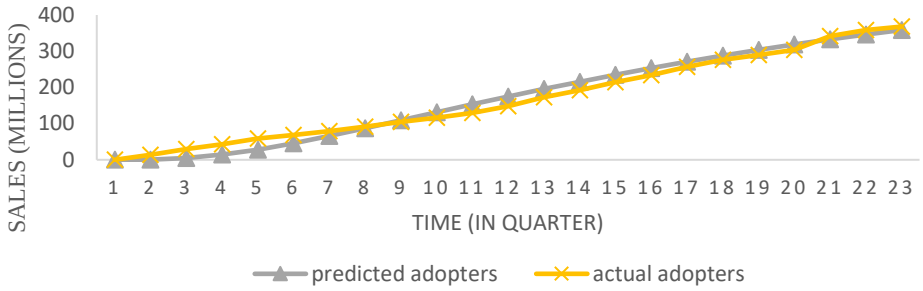

Figure 5. Goodness of fit for DS3.

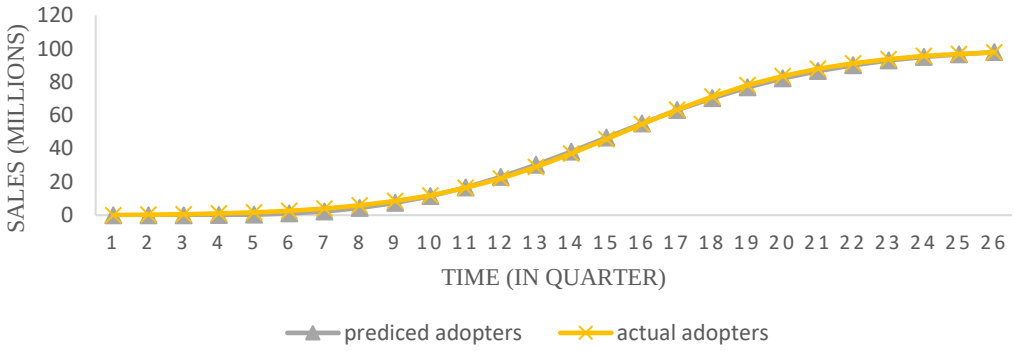

Figure 6. Goodness of fit for DS4.

A company is always required to carry out marketing strategies related to product promotion and dissemination, for this company must need to know, the number of potential customers who are aware and unaware of the product, motivated but non-adopter potential customers and total adopters. Equations (5), (6), and (7) can be used to have some fair interpretation. The resulting graphs are displayed for all of the data sets using Figures 7, 8, 9 and 10 respectively.

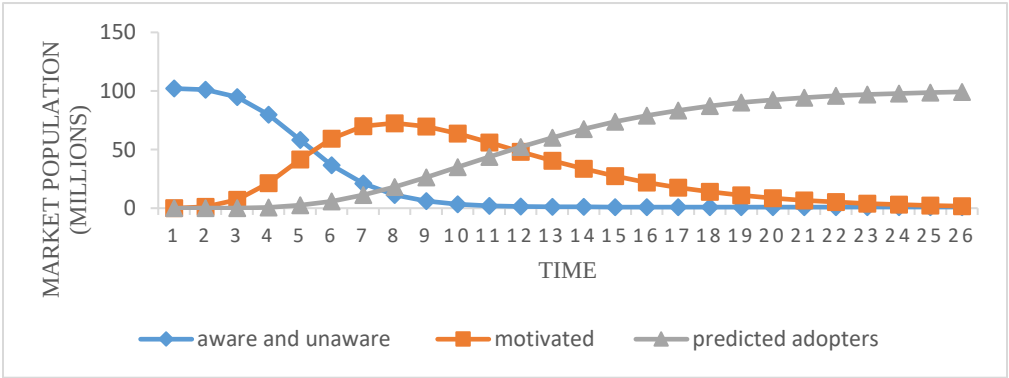

Figure 7. Customers under different role for DS1.

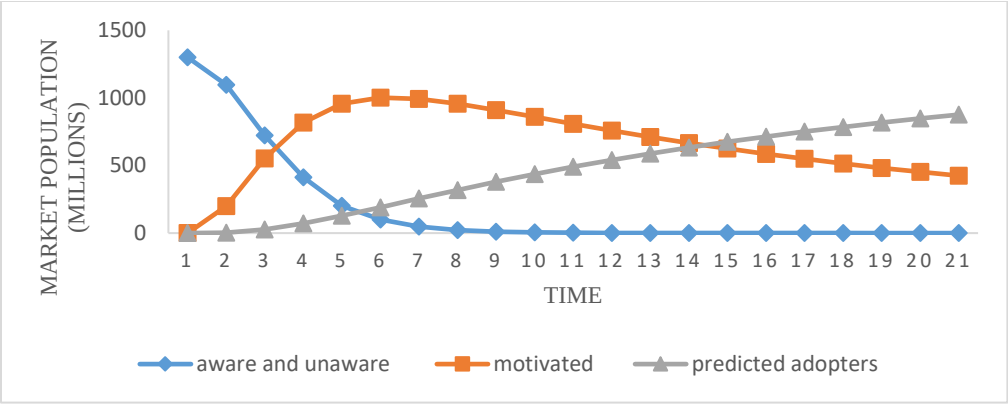

Figure 8. Customers under different role for DS2.

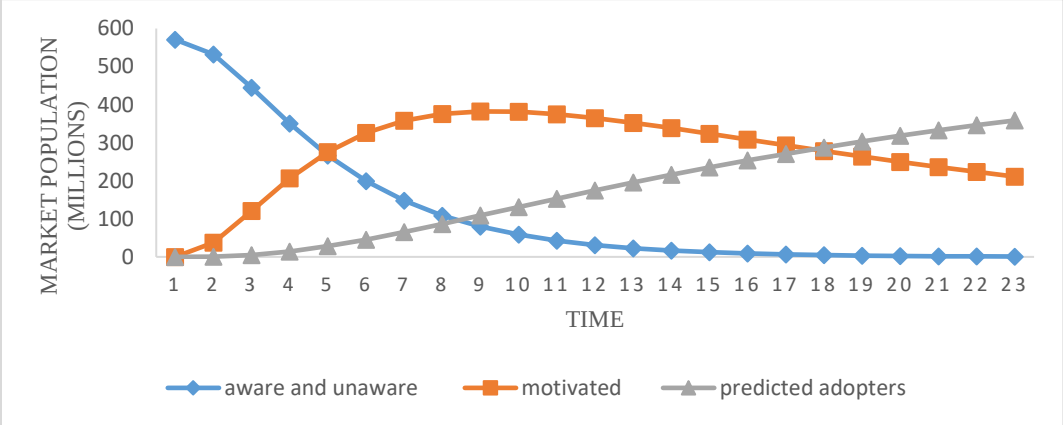

Figure 9. Customers under different role for DS3.

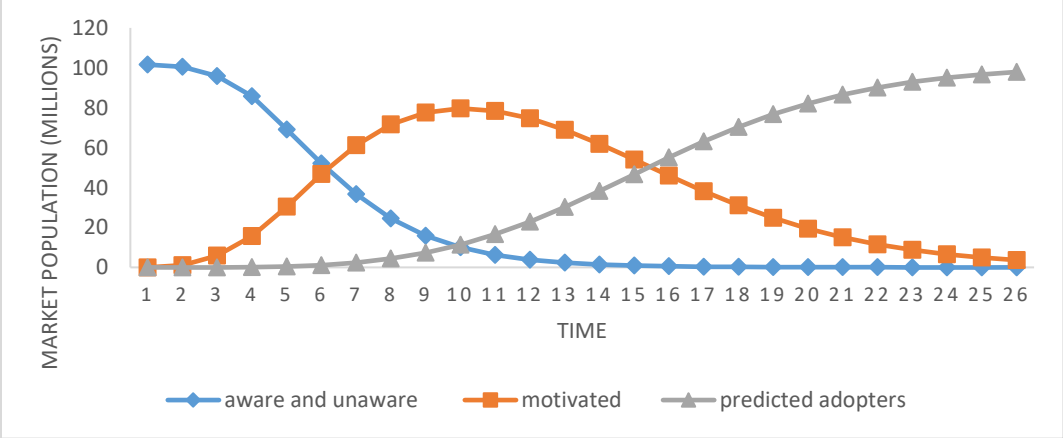

Figure 10. Customers under different role for DS4.

A company can conveniently measure the fully aware, motivated (but non-adopters), and adopters of the product at each and every time of the horizon. Like; for the time point 10^{th} for DS1, there is around 2.140597 million populations, that represents people who are yet to gain awareness., there are around 56.17649 million people who are motivated but have not adopted by that time, and 47.76291 million customers have adopted the product. Similar type of observations can be made from the remaining three data sets as well. These estimates would help the managers to choose future tactics for marketing and promotional campaigns to boost product sales.

5. Managerial Implications

With the help of the concept discussed in this article, given certain set of assumptions, a company will not only be able to approximate the sales of the product, but also have fair knowledge about the number of motivated potential customers along with the remaining number of aware potential customers for the product. These all are important attributes in order to have full-fledged information about the diffusion process and discuss its dynamics. Firms can accordingly plan to design their marketing plan around the non-aware and motivated potential adopters and increase the sales. Furthermore, they can strategically play with coming of new generations as long as there is a rise in the number of motivated potential customers. If the company notices that its rate of motivated clients is low, it must work and enhance its promotion and advertising plan.

6. Future Scope

The model proposed in this paper can be extended in future course of action. An important assumption that the work proposed here carries is that the product's potential market is constant in size. In future, this market size can be studied under dynamic environment. Moreover, some randomness can also be studied in the market size in the later stages.

7. Conclusion

In this study, the authors suggested a comprehensive and innovative way to simulate knowledge and opinions to model the diffusion process in marketing. The use of the basic epidemic modeling framework (SIR model) to comprehend product sales and potential customers' behavior has added to the contribution further. In literature, there exist several problems in the pipeline which have been not used because of not having the closed form solution. The problem of not having the closed form solution to the set of equations at hand has been dealt with the usage of Runge-Kutta 4^{th} order method and many more and the approximated values have been studied in order to reach a in depth analytical point. Four separate data sets have been used to evaluate the proposed model, and the outcomes are studied based on the approximated values thus obtained.

Conflict of Interest

The authors certify that there are no conflicts of interest in this article's content.

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