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Active Set Approach for a Bilevel Portfolio Optimization Model

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Abstract
This paper presents a methodology for determining the optimal portfolio that maximizes the Sharpe ratio within a bilevel framework. The upper-level of the model maximizes the Sharpe ratio of the portfolio, while the lower-level minimizes the risk for a given expected return, which is treated as a parameter. In the methodology, a gradient-based active set approach is proposed to solve the bilevel portfolio optimization model. The proposed method generates a sequence of portfolios converging to the optimal portfolio. To validate the method, the results are tested and verified using real-world portfolio datasets collected from the Bombay Stock Exchange, India. It is observed in the numerical experiment that the Sharpe ratio obtained in a bilevel framework is better than that of the traditional method.

Keywords- Bilevel optimization problem, Portfolio selection problem, Sharpe ratio, Markowitz model.

1. Introduction

The portfolio selection problem focuses on distributing the investment budget among a set of assets to obtain minimal risk while maximizing investment returns. Investors commonly reduce the risk through portfolio diversification, which entails allocating investments across multiple assets. Key inputs to facilitate the portfolio selection model include expected return, variance, and covariance of the return of the assets. The optimization-based approach in the selection of portfolio gained significant attention after the work of Markowitz (1952). Markowitz's model is efficient, but it does not include real-world trading scenarios in its original form. Investors often adapt it with practical considerations of various trading restrictions. This involves incorporating additional constraints such as cardinality constraints, transaction costs, and bounding constraints (Lobo et al., 2007; Yen & Yen, 2014; Hooshmand et al., 2023). In addition to this, handling uncertainty in the inputs of portfolio selection problems is a prominent challenge. The uncertain parameters in a portfolio optimization model in interval form are explored in the works by Sahu et al. (2024), and Bhurjee et al. (2025). In recent years, machine learning techniques have been adopted in solving portfolio optimization models (see, Alzaman, 2024; Cui et al., 2024; Behera & Kumar, 2025).

The evaluation of portfolio performance is carried out using measures like the Sharpe ratio, Treynor ratio, Information ratio, and Sortino ratio, which are summarized in Marhfor (2016). The Sharpe ratio is relatively efficient as it accounts both the systematic and unsystematic risks. Traditionally, the optimal portfolio that maximizes the Sharpe ratio is derived by solving the capital asset pricing model within a mean–covariance framework. In this paper, we propose an alternative approach, where the Sharpe ratio is obtained through a bilevel framework.

Some portfolio optimization models can be studied in a hierarchical process, where each level is associated with a financial strategy. Many hierarchical decision-making situations are modeled as bilevel optimization problems. Bilevel optimization in portfolio selection has recently gained attention due to its hierarchical structure (see, Benita et al., 2019; Leal et al., 2020; González-Díaz et al., 2021; Kobayashi et al., 2021; Stoilov et al., 2021; Cesarone et al., 2023; Salehi et al., 2023; Bayat et al., 2024; Crisci et al., 2025). Benita et al. (2019) studied a hierarchical model in which an investor allocates a budget to determine a decentralized global investment strategy through an intermediatory acting as a follower to maximize the return. Stoilov et al. (2021) presented a bilevel model where the upper-level problem estimates the Value at Risk, while the lower-level problem addresses a multi-objective optimization with return and risk as objective functions. Leal et al. (2020) and González-Díaz et al. (2021) proposed bilevel portfolio models in which the broker acts as a leader and fixes transaction costs with the portfolio, while the investor acts as a follower and minimizes the tail risk. Salehi et al. (2023) proposed a bilevel model in which the investor (acting as leader) at the upper level allocates budgets among various subsidiaries, which are followers, thereby forming a multi-objective lower-level problem in the bilevel formulation. Bayat et al. (2024) formulated the risk budgeting problem as a bilevel model, with the upper-level allocating risk budgets and the lower level computing the associated risk budgeting portfolio. Kobayashi et al. (2021) considered a mean–risk portfolio model using conditional value-at-risk with a cardinality constraint. To handle the complexity of the cardinality constraint in the model, they reformulated it as a bilevel optimization problem and proposed a cutting-plane bilevel approach to solve it. In Cesarone et al. (2023), a financial service providers construct ESG-oriented portfolios to optimize the firm's overall ESG impact at the upper level of the bilevel model. The lower level considers multiple account holders optimizing various portfolio features such as risk, return, transaction costs, and ESG scores, while maintaining a Nash equilibrium among themselves. Crisci et al. (2025) considered a multiperiod sparse mean–variance model, incorporating uncertainty in the covariance matrix through box constraints.

The maximum Sharpe ratio of a portfolio can be computed by solving the capital asset pricing model in a mean-variance sense, which is a traditional approach. The traditional capital asset pricing model often fails to achieve the optimal Sharpe ratio due to the non-concave behavior of the Sharpe ratio. The sub-optimal solution of the corresponding optimization problem often leads inefficient portfolio. A smoothing direct search method by Chen et al. (2018) is applied to solve a bilevel model, where the Sharpe ratio is maximized at the upper-level while minimizing risk based on the Markowitz framework at the lower-level, considering expected return and weight bounds as parameters. Jing et al. (2022) incorporated cardinality constraints in the lower-level by considering a similar type of bilevel model.

The motivation of this study is to evaluate the maximum Sharpe ratio ensuring the efficient portfolio within a bilevel framework. The upper level maximizes the Sharpe ratio, while the lower level focuses on minimizing covariance risk under a parameterized expected return. The upper-level objective function is a nonconcave function, whereas the lower-level problem is a convex quadratic program. The methodology of this paper is based on the gradient-based line search technique which is different from the derivative-free method by Chen et al. (2018) and Jing et al. (2022). In the proposed methodology, the active set strategy on the lower-level problem is employed to handle the nonconvex and non-smooth nature of the bilevel formulation. Using this approach, a sequence of portfolios is generated that converge to an optimal portfolio. The methodology is tested on some portfolio problems with real datasets for validation. In addition, the simulation approach of Jorion (1992) is used to test the quality of the results obtained. From the numerical experiments, it can be concluded that the bilevel framework for maximizing the Sharpe ratio performs better than the traditional mean–variance approach.

This study is organized into several sections. Section 2 explains the bilevel portfolio optimization model and some prerequisites. The active set strategy is explained in Section 3. In Section 4, the methodology is applied to financial data taken from Bombay Stock Exchange (BSE), India.

2. Model Formulation

A bilevel optimization problem is an optimization model with a hierarchical structure in which a nested parametric problem appears in the constraint set. A general optimistic bilevel problem with equality and inequality constraints is mathematically expressed as

$$(P):\ \max_{x,y} F(x,y)$$
$$\text{s.t. } G(x,y) \leq 0, H(x,y) = 0,$$
$$y \in \underset{y}{\operatorname{argmin}}\{f(x,y) \text{ s.t. } g(x,y) \leq 0, h(x,y) = 0\},$$

where, $F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, G: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{p_1}, H: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{p_2}$, $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{q_1}$, $h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{q_2}$. The upper-level or leader's problem is,

$$\max_{x,y}\ F(x,y) \text{ s.t. } G(x,y) \leq 0, H(x,y) = 0,$$

where, y is the solution of the lower-level or follower's problem, which is,

$$\min_{y} f(x,y) \text{ s.t. } g(x,y) \leq 0, h(x,y) = 0.$$

Let the lower-level problem be uniquely solved and $y(x)$ be the optimal solution for given upper-level parameter x. Then, we say d_x is an ascent feasible direction at x if there exists $\bar{\alpha} > 0$ such that

$$F\big(x + \alpha d_x, y(x + \alpha d_x)\big) \geq F(x, y(x)),$$

and $G\big(x + \alpha d_x, y(x + \alpha d_x)\big) \leq 0, H\big(x + \alpha d_x, y(x + \alpha d_x)\big) = 0$ for each $\alpha \in (0, \bar{\alpha}]$.

A point $(x, y(x))$ is a local optimal solution of the bilevel optimization problem if there exists a neighborhood U_x about x such that $F\big(x', y(x')\big) \leq F\big(x, y(x)\big)$, where $\big(x', y(x')\big)$ satisfies $G\big(x', y(x')\big) \leq 0, H\big(x', y(x')\big) = 0$ for each $x' \in U_x$. If these conditions hold for any neighbourhood about x then $(x, y(x))$ is a global optimal solution of the bilevel optimization problems. For the necessary and sufficient conditions for the existence of a solution of bilevel optimization problems, the reader may see Chapter 5 of Dempe (2002) for the uniquely solved lower-level problem.

If the lower-level problem is convex and a suitable constraint qualification holds for the fixed upper-level variable x, the bilevel problem can be reformulated into a single-level optimization problem using its Karush-Kuhn-Tucker (KKT) optimality conditions (Allende & Still, 2013). This reformulation introduces dual variables of the lower-level problem together with complementarity constraints into the single-level formulation. Moreover, when the lower-level constraints satisfy the linear independence constraint qualification, KKT multipliers are unique, and the resulting single-level problem is equivalent to the original bilevel problem in terms of solutions (Dempe & Dutta, 2012). In this paper, we consider a bilevel optimization model in finance, known as the bilevel portfolio optimization problem, and develop an algorithm by using the active set method explained in Section 4.3 of Dempe (2002) and Khatana & Panda (2025) to estimate the optimal portfolio. The following notations are used to propose this model.

Notations

- A_i : i^{th} asset of the portfolio $P = (A_1, A_2, \cdots, A_n)$.
- T : Total number of observations for which data is collected.
- r_{ij} : Return of asset A_i at time t_j.
- σ_{ij} : Covariance between returns of the assets A_i and A_j.
- μ_i : Expected return of the asset A_i.
- μ : $(\mu_1, \mu_2, \cdots, \mu_n)^T$.
- w : $(w_1, w_2, \cdots, w_n)^T, \sum_{i=1}^{n} w_i = 1$, where w_i is the proportion of the invested capital in A_i.
- ρ : $\mu^T w$ Expected return of the portfolio or investment.
- Σ : Covariance matrix of the return of the assets.
- u : $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$.
- v : $\begin{pmatrix} v_a \\ v_b \end{pmatrix} \in \mathbb{R}^{n+n}$.
- $I(w)$: $\{i, j: -w_i + a_i = 0, w_j - b_j = 0, i, j = 1,2, \cdots, n\}$.
- $J(v)$: $\{i, j: v_{ai} > 0, v_{bj} > 0, i, j = 1,2, \cdots, n\}$.

The portfolio optimization problem determines the optimal strategy for investing capital in financial assets. The Markowitz mean-variance model aims to minimize the risk and maximize the portfolio's return. A general mean-variance model is

$$\min_{w} w^T \Sigma w \text{ s.t. } e^T w = 1, \mu^T w = \rho,$$

where, $w = (w_1, w_2, \dots, w_n)^T, w_j$ is the proportion of investment in j^{th} asset of the portfolio, $\mu = (\mu_1, \mu_2, \dots, \mu_n)^T, \mu_j$ is the expected rate of return of j^{th} asset over a time period $T, \rho \in \mathbb{R}$ is the target of return fixed by the investor, and Σ is the covariance matrix of returns. Many researchers have contributed to the variations of the Markovitz mean-variance model in different directions, which are summarized in Section 1. This paper focuses on the Sharpe ratio and structures the Markovitz model in two levels, which can be treated as a bilevel portfolio optimization model. The upper-level maximizes Sharpe ratio of the portfolio, whereas lower-level takes care of the return. The Sharpe ratio of the portfolio is defined as the ratio of the expected return of the portfolio and the square root of the risk, mathematically computed as $\frac{\mu^T w - r_f}{\sqrt{w^T \Sigma w}}$, where r_f corresponds to the rate of return for a risk-free asset. Let $\mu^T w = \rho$, where ρ is a decision variable.

The following bilevel model of a single-period investment portfolio problem is investigated here.

$$(BP): \max_{\rho, w} \frac{\rho - r_f}{\sqrt{w^T \Sigma w}}$$

$$\text{s.t. } \rho \geq \alpha,$$

$$w \in \arg\min_{w} \{w^T \Sigma w \text{ s.t. } e^T w = 1, \mu^T w = \rho, a \leq w \leq b\}.$$

In BP, $a, b, \in \mathbb{R}^n$ and the vector inequality $a \leq w \leq b$ is considered componentwise, and $\alpha > 0$ is a small real number, which ensures that the model provides a positive expected return on the investment. The upper-level of the portfolio optimization model is

$$(UP): \max_{\rho, w} \frac{\rho - r_f}{\sqrt{w^T \Sigma w}} \text{ s.t. } \rho \geq \alpha,$$

where, w is determined through a lower-level problem, which is
(LP_ρ): $\min_w \; w^T\Sigma w$ s.t. $e^T w = 1, \mu^T w = \rho, a \leq w \leq b$.

For the diversification of the optimal portfolio, the investor imposes the lower and upper bounds a and b on the weights. The lower bound on the weights ensures that a small proportion of weights can be excluded from the optimal portfolio. On the other hand, the upper bound on the weights is imposed to limit the proportion of capital allocated to a single asset to enable diversification in the portfolio.

For given expected return ρ in the LP_ρ model, an optimal risk $w(\rho)$ can be obtained by solving the LP_ρ, and the corresponding Sharpe ratio $\frac{\mu^T w(\rho) - r_f}{\sqrt{w(\rho)^T \Sigma w(\rho)}}$ can be evaluated. The problem UP determines the value of ρ that maximizes the Sharpe ratio. BP identifies the optimal portfolio weights w^* so that the Sharpe ratio is maximized at the optimal expected return ρ^*. Note that LP_ρ is a convex quadratic programming problem, which is computationally tractable. UP is a nonlinear maximization problem with a non-concave objective function. Hence this model can't be solved using traditional optimization techniques. In the next section, an iterative scheme is developed to obtain the solution of this model. A sequence of points $\{(\rho^k, w^k)\}$ is generated, starting with an initial return target ρ^0 and initial portfolio w^0, the limit point of $\{(\rho^k, w^k)\}$ is considered as $\{(\rho^*, w^*)\}$, which solves the above bilevel portfolio optimization model. This iterative scheme is in the light of an active set strategy.

3. Methodology

For a given return parameter ρ, LP_ρ is a convex quadratic programming problem. Hence, LP_ρ is uniquely solved at ρ, and the solution can be obtained by solving the KKT optimality conditions of LP_ρ. As a result, the bilevel hierarchical model is reformulated into a single-level problem with complementarity conditions, which is challenging to handle. Here, an active state strategy in the light of Section 4.4.2 of Dempe (2002) and the work in Khatana & Panda (2025) is used to address the complementarity condition. The Lagrange function for LP_ρ is given by
$L(\rho, w, u, v) = w^T\Sigma w + u_1(e^T w - 1) + u_2(\mu^T w - \rho) + v_a^T(a - w) + v_b^T(w - b)$,

where, $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$ and $(v_a, v_b) \in \mathbb{R}^n \times \mathbb{R}^n$ are the associated Lagrange multipliers with the equality constraints and inequality constraints of LP_ρ respectively. The KKT optimality conditions for the minimization problem LP_ρ are

$$2\Sigma w + u_1 e + u_2 \mu - v_a + v_b = 0 \tag{1}$$

$$e^T w = 1, \qquad \mu^T w = \rho \tag{2}$$

$$-w_i + a_i \leq 0, \quad v_{ai} \geq 0, i = 1,2,\cdots,n \tag{3}$$

$$w_j - b_j \leq 0, v_{bj} \geq 0, j = 1,2,\cdots,n \tag{4}$$

$$(-w_i + a_i)v_{ai} = 0, \quad (w_j - b_j)v_{bj} = 0, i,j = 1,2,\cdots,n \tag{5}$$

For the sake of simplicity, we further denote the objective function of UP as $F(w) = \frac{\mu^T w - r_f}{\sqrt{w^T\Sigma w}}$, and Lagrange multiplier vectors by $u = \binom{u_1}{u_2}$, and $v = \binom{v_a}{v_b}$. Using the above optimality conditions of LP_ρ, the following single-level problem can be obtained from BP.

$$\max_{\rho,w,u,v} \quad F(w).$$

s.t. $\rho \geq \alpha, (\rho, w, u, v)$ satisfies Equations (1) to (5).

The above problem is a mathematical program with complementarity constraints, which is difficult to handle due to the presence of the nonlinear and nonconvex complementarity constraints in the expression (5). Using the active set strategy on the constrains (3)-(5), these are decomposed in the linear constraints and a subproblem is constructed to determine the feasible ascent direction.

3.1 Construction of the Subproblem

Let ρ^k be the expected return and w^k be the optimal portfolio of model LP_{ρ^k} with v_a^k and v_b^k as the Lagrange multiplier vectors associated with the constraints $a - w \leq 0$ and $w - b \leq 0$ respectively. Consider the active set $I(w^k)$ and the index set $J(v^k)$ of positive multipliers of the constraints function as

$$I(w^k) = \{i, j: -w_i^k + a_i = 0, w_j^k - b_j = 0, i, j = 1,2,\cdots,n\},$$

and $J(v^k) = \{i, j: v_{ai}^k > 0, v_{bj}^k > 0, i, j = 1,2,\cdots,n\}$.

Let W be any active index set satisfying $J(v^k) \subseteq W \subseteq I(w^k)$. The complementarity constraints $(-w_i + a_i)v_{ai} = 0, (w_j - b_j)v_{bj} = 0$ of the KKT optimality conditions can be decomposed corresponding to the active set W as follows

$$-w_i + a_i = 0, v_{ai} \geq 0, i \in W \tag{6}$$

$$w_j - b_j = 0, v_{bj} \geq 0, j \in W \tag{7}$$

$$-w_i + a_i \leq 0, v_{ai} = 0, i \notin W \tag{8}$$

$$w_j - b_j \leq 0, \qquad v_{bj} = 0, j \notin W \tag{9}$$

The main objective is to impose a proper strategy to generate a sequence of portfolios $\{w^k\}$ starting with an initial portfolio w^0, which can lead to the optimal portfolio for large k. Let ρ^k be the return target and u^k, v^k are the associated KKT multiplier vectors corresponding to the portfolio w^k at k^{th} iteration. The next iterating point (ρ^{k+1}, w^{k+1}) is associated with the dual vector u^{k+1}, v^{k+1}, computed as

$$\rho^{k+1} = \rho^k + \alpha^k d_{\rho^k}, w^{k+1} = w^k + \alpha^k d_{w^k}, u^{k+1} = u^k + \alpha^k d_{u^k}, v^{k+1} = v^k + \alpha^k d_{v^k},$$

where, α^k is the step length satisfying the following relation,

$$F(w^{k+1}) \geq F(w^k) + \alpha^k \delta \nabla_w F(w^k)^T d_{w^k} \tag{10}$$

Inequality (10) ensures the increase in the objective function F at each iteration. Hence, (d_{ρ^k}, d_{w^k}) is the ascent direction at (ρ^k, w^k). Denote

$$d^k := \left(d_{\rho^k}, d_{w^k}, d_{u^k}, d_{v^k}\right).$$

The following subproblem $P_W(\rho^k, w^k)$ is solved to obtain the direction vector d^k, which is constructed using the linear approximation of the functions (1)-(2) and (6)-(9), and the objective function $F(w)$ about w^k as

$$\left(P_W\left(\rho^k, w^k\right)\right): \max_{d_\rho, d_w, d_u, d_v} \nabla_w F\left(w^k\right)^T d_w$$
$$\begin{aligned} s.t. \quad & e^T d_w = 0, \mu^T d_w = d_\rho, -\rho^k - d_\rho \leq -\alpha, \\ & 2\Sigma d_w + e d_{u_1} + \mu d_{u_2} - \sum_{i \in W} d_{v_{ai}} - \sum_{j \in W} d_{v_{bj}} = 0, \\ & d_{w_i} = 0, \qquad d_{v_{ai}} \geq 0, i \in W, \\ & d_{w_j} = 0, \qquad d_{v_{bj}} \geq 0, j \in W, \\ & -w_i^k - d_{w_i} + a_i \leq 0, \quad d_{v_{ai}} = 0, i \notin W, \\ & w_j^k + d_{w_j} - b_j \leq 0, \quad d_{v_{bj}} = 0, j \notin W. \end{aligned}$$

$P_W\left(\rho^k, w^k\right)$ is a linear programming problem, and its solution is denoted by d^k at (ρ^k, w^k) corresponding to the index set W satisfying $J\left(v^k\right) \subseteq W \subseteq I\left(w^k\right)$. Note that the active set W satisfying $J\left(v^k\right) \subseteq W \subseteq I\left(w^k\right)$ is not necessarily uniquely determined. Number of possible such W satisfying $J\left(v^k\right) \subseteq W \subseteq I\left(w^k\right)$ is $2^{(|I(w^k)|-|J(v^k)|)}$, where notation $|,|$ denote the cardinality of a set. The vector (ρ^k, w^k) is the optimal point of BP if $\nabla_w F\left(w^k\right) d_{w^k} = 0$ corresponding to each active set W satisfying $J\left(v^k\right) \subseteq W \subseteq I\left(w^k\right)$. This condition is used as a stopping criterion in Algorithm 1.

Let the sequence $\{(\rho^k, w^k)\}$ be obtained by solving the subproblem $P_W\left(\rho^k, w^k\right)$, and (ρ^*, w^*) be the limiting point of the sequence $\{(\rho^k, w^k)\}$. If (ρ^*, w^*) is not the stationary point of BP, that is, $\nabla_w F(w^*) d_{w^*} \neq 0$ for some W satisfying $J\left(v^k\right) \subseteq W \subseteq I\left(w^k\right)$, then, to address the situation, the current iterative point (ρ^k, w^k, u^k, v^k) can be updated through a two-step procedure, as explained in the Step 2 of the algorithm. This concept is followed from the reference (Zhang et al., 2004).

3.2 Outline of the Method

Let the vector $\left(\rho^k, w^k, u^k, v^k\right)$ be obtained at k^{th} iterative point. First, a new point $\left(\hat{\rho}^k, \hat{w}^k, \hat{u}^k, \hat{v}^k\right)$ with information of $\left(\rho^k, w^k, u^k, v^k\right)$ is computed for given $\delta' > 0$ as follows. Consider the approximate active set

$I_k(\delta') = \left\{i, j \in \{i, j = 1,2, \cdots, n\} : -\delta' < -w_i^k + a_i \leq 0, -\delta' < w_j^k - b_j \leq 0\right\}$,

and the approximate positive multiplier set

$J_k(\delta') = \left\{i, j \in \{i, j = 1,2, \cdots, n\} : v_{ai}^k > \delta', v_{bj}^k > \delta'\right\}$.

In the case, when $I\left(w^k\right) \neq I_k(\delta')$ and $J\left(w^k\right) \neq J_k(\delta')$, the iterative point (ρ^k, w^k, u^k, v^k) is projected on the set $S\left(I_k(\delta'), J_k(\delta')\right)$, where

$$S\left(I_k(\delta'), J_k(\delta')\right) = \left\{ (\rho, w, u, v): \begin{array}{c} e^T w = 1, \mu^T w = \rho, -\rho \leq -\alpha, \\ 2\Sigma w + u_1 e + u_2 \mu - v_{ai} + v_{bi} = 0, \\ -w_i + a_i = 0, i \in I_k(\delta'), \quad v_{ai} \geq 0, i \in J_k(\delta'), \\ w_j - b_j = 0, j \in I_k(\delta'), \quad v_{bj} \geq 0, j \in J_k(\delta'), \\ -w_i + a_i \leq 0, i \notin I_k(\delta'), \quad v_{ai} = 0, i \notin J_k(\delta'), \\ w_j - b_j \leq 0, j \notin I_k(\delta'), \quad v_{bj} = 0, j \notin J_k(\delta') \end{array} \right\}.$$

The new point using this projection is denoted by $(\hat{\rho}^k, \hat{w}^k, \hat{u}^k, \hat{v}^k)$. The next step identifies a suitable active set W from all possible choices of W satisfying $J(v^k) \subseteq W \subseteq I(w^k)$ and corresponding subproblem P_W which can provide the ascent direction $(d_{\rho^k}, d_{w^k}, d_{u^k}, d_{v^k})$. There are a finite number of choices of the active set W satisfying $J(v^k) \subseteq W \subseteq I(w^k)$ as $J(v^k)$ and $I(w^k)$ are finite sets.

A sufficient increase in the objective function F is required at each iteration of the algorithm to accelerate the iterative process, which depends on the value of $\nabla_w F(w^k)^T d_{w^k}$. According to Step 4(a), for a given $\epsilon > 0$, if $\nabla_w F(w^k)^T d_{w^k} > \epsilon$ then there is a possibility of an increase in the objective value along this direction. Hence, we proceed to compute the step size in Step 6. According to Step 4(b), if $\nabla_w F(w^k)^T d_{w^k} < \epsilon$ then we need to consider a different active set W satisfying $J(v^k) \subseteq W \subseteq I(w^k)$, and repeat Step 3 till $\nabla_w F(w^k)^T d_{w^k} > \epsilon$. According to Step 4(c), if $\nabla_w F(w^k)^T d_{w^k} < \epsilon$ for each W satisfying $J(v^k) \subseteq W \subseteq I(w^k)$ then we can update d_{w^k} corresponding to the maximum value of $\nabla_w F(w^k)^T d_{w^k}$ as follows:

$$d_{w^k} \leftarrow \arg\max \left\{ \nabla_w F(w^k)^T d_{w^k} \mid d^k \text{ solves } P_W(\rho^k, w^k), J(v^k) \subseteq W \subseteq I(w^k) \right\} \tag{11}$$

This process is outlined in Step 4 of the algorithm.

3.3 Algorithmic Steps

The following algorithm is developed based on the steps mentioned above. We denote the suitable subsets of the active set at k^{th} iteration as $\mathcal{A}^k := \{W : J(v^k) \subseteq W \subseteq I(w^k)\}$ in the following algorithm.

Algorithm 1 Iterative process for (BP)

Step 0: (Initialization) Select the initial point $\rho^0 \geq \alpha$ for given $\alpha > 0$ and compute w^0 by solving LP_{ρ^0}. The corresponding dual variables are u^0 and v^0, and the initial active set is $W^0 = I(w^0)$. Choose parameters $0 < \delta, \delta_0, \eta, \sigma, \epsilon < 1$.

Step 1: Set $\delta' = \delta_0$. Compute $I(w^k), I_k(\delta'), J(v^k),$ and $J_k(\delta')$. Obtain the set $\mathcal{A}^k$.

Step 2:(Projection step) Perform the following sub steps sequentially:

(2a). If $I_k(\delta') = I(w^k)$ and $J_k(\delta') = J(v^k)$ then update $W \leftarrow I(w^k)$. Go to Step 3.

(2b). If $S(I_k(\delta'), J_k(\delta')) \neq \emptyset$ then project (ρ^k, w^k, u^k, v^k) on the set $S(I_k(\delta'), J_k(\delta')$) and obtain $(\hat{\rho}^k, \hat{w}^k, \hat{u}^k, \hat{v}^k)$.

(2c). If $F(\hat{w}^k) > F(w^k)$ then update $(\rho^k, w^k, u^k, v^k) \leftarrow (\bar{\rho}^k, \hat{w}^k, \bar{u}^k, \bar{v}^k)$ and $W \leftarrow I(\hat{w}^k)$ and go to the Step 3.

(2d). Otherwise update $\delta' = \sigma\delta'$. Start over Step 2.

Step 3: (Direction searching step) Solve $P_W(\rho^k, w^k)$ to obtain the solution d^k.

Step 4: (Sufficient increase step) Perform the following sub steps sequentially:

(4a). If $\nabla_w F(w^k)^T d_{w^k} \geq \epsilon$ then go to Step 6.

(4b). If $\nabla_w F(w^k)^T d_{w^k} < \epsilon$ and $\mathcal{A}^k \neq \emptyset$ then select $W \in \mathcal{A}^k$, set $\mathcal{A}^k = \mathcal{A}^k \setminus W$, and go to Step 3.

(4c). If $\mathcal{A}^k = \emptyset$ then set $\epsilon = \sigma\epsilon$ and determine d_{w^k} using Equation (11). Go to Step 5.

Step 5: (Stopping criteria) If $\nabla_w F(w^k)^T d_{w^k} = 0$ then terminate with (ρ^k, w^k).

Step 6: (Step length computation) Compute $\alpha_k = \eta^j, j \in \{0,1,2, \dots\}$ such that j is the smallest number and α_k satisfies Armijo condition (11).

Step 7: Update $\left(\rho^{k+1}, w^{k+1}, u^{k+1}, v^{k+1}\right) = \left(\rho^k + \alpha_k d_{\rho^k}, w^k + \alpha_k d_{w^k}, u^k + \alpha_k d_{u^k}, v^k + \alpha_k d_{v^k}\right)$, and set $k = k + 1$, and go back to Step 1.

4. Results and Discussions

Suppose an investor is interested in trading n risky assets, $A_i, i = 1,2, \dots, n$. Let $\mathbf{r}_i$ be the return of asset A_i, which is a random variable following normal distribution $N(\mu, \sigma), \sigma_{ik}$ be the covariance of returns of the assets A_i and A_k, and r_{ij} denote the return of asset A_i at time t_j, where $j = 1,2, \dots, T$. The expected return of the asset A_i is $\mu_i := \frac{1}{T}\sum_{j=1}^{T} r_{ij}$ and $\sigma_{ik} := \frac{1}{T-1}\sum_{j=1}^{T} \left(r_{ij} - \mu_i\right)\left(r_{kj} - \mu_k\right)$.

A simulation approach is used to test the results obtained through the model. In the process, historical data including daily and hourly data from different periods in the Bombay Stock Exchange India, are collected. To ensure diversity in the data, stocks are selected from various categories such as banking, information technologies, automobiles, and others, which are listed under the category Nifty 50, Nifty Small cap 50, and Nifty Midcap 50 indexes. **Table 1** displays the historical data between March 2018 and November 2024. The dataset is divided in two subsets, which are training and testing data sets. The training data (in-sample data) contain the first part of the dataset, which is used to estimate the optimal portfolio using the simulation technique. The testing data (out-of-sample data) comprises the second part of the dataset, used to test the quality of the results obtained from the in-sample period. Here, Data 1, 2, and 3 are collected on the daily basis, while Data 4, 5, and 6 are collected on the hourly basis.

Table 1. Details of the data sets.

Data set	n	T	In sample data	Out sample data
Data 1	10	498	16 September 2021 to 25 April 2023	29 April 2023 to 15 September 2023
Data 2	25	639	3 January 2022 to 24 July 2023	25 July 2023 to 31 July 2024
Data 3	50	639	3 January 2022 to 25 Jan 2024	29 January 2024 to 31 July 2024
Data 4	70	11409	12 March 2018 to 26 October 2021	26 October 2021 to 29 November 2024
Data 5	100	5704	26 October 2022 to 4 April 2023	4 April 2023 to 29 November 2024
Data 6	139	11409	12 March 2018 to 23 December 2022	23 December 2022 to 29 November 2024

The high and low peak in the historical data along with the uncertainty in the data, is addressed using the smoothing technique. To reduce the effect of the noise in the data, the data are first processed to smooth fluctuations and then used in the experiments for the estimation of the optimal parameters. A suitable process cleans hourly data before the applicability. Missing entries in the rate matrix of the dataset are filled using the rolling mean of data from the previous trading days, with a seven-point window. If some entries remain missing, the corresponding periods across all stocks are removed for the sake of consistency. To smooth the rate matrix and reduce noise, an exponential moving average technique with a 35-point span is applied, which is equivalent to one week of trading. This provides a more stable representation of the rate of change for each stock and improves the reliability of the analysis. We perform a well-known simulation approach (see (Jorion, 1992)) in portfolio optimization to evaluate the performance of the model. For $\xi \in (0,1)$, we use the first $T\xi$ observations to compute the mean and covariance as follows:
$\overline{\mu_i} = \frac{1}{\xi T}\sum_{j=1}^{\xi T} r_{ij}, \overline{\sigma_{ik}} = \frac{1}{\xi T-1}\sum_{j=1}^{\xi T} \left(r_{ij} - \mu_i\right)\left(r_{kj} - \mu_k\right)$.

Based on this information, we simulate the data for the next $(1 - \xi)T$ observations using the inbuilt code MVNRND from the machine learning toolbox of MATLAB. Let μ^{in} and C^{in} denote the mean and covariance matrix of the simulated data. The bilevel model BP is then solved using these estimated values of mean and covariance using Algorithm 1.

- Algorithm 1 is implemented by developing MATLAB code, and all the numerical experiments are conducted on a Windows 10 PC using MATLAB (2023b).
- The following accuracies are accepted in the code: In Step 1 of Algorithm 1, the index sets are determined with a tolerance of 10^{-6}, and the algorithm is terminated at Step 5 with an optimal tolerance of 10^{-8}. The various parameters values in the algorithm are as follows.
  $a = -e, b = e,\ \delta = 10^{-3}, \delta_0 = 10^{-4}, \eta = 0.5, \sigma = 10^{-3},\ \epsilon = 10^{-1}, \alpha = 10^{-3}, r_f = 0.$
- LINPROG function of MATLAB is used to solve the subproblem $P_W(\rho^k, w^k)$ in Step 4 for calculation of d_{w^k}.

Let ρ^* be the optimal expected return and w^* be the corresponding optimal portfolio allocation. In that case in-sample Sharpe ratio is calculated using the formula $\frac{\rho^* - r_f}{\sqrt{w^{*T} C^{in} w^*}}$. Next, let μ^{out} and C^{out} represent the mean and covariance matrix of the out-of-sample data. Using this data, out-sample Sharpe Ratio is calculated as $\frac{\rho^* - r_f}{\sqrt{w^{*T} C^{out} w^*}}$. The result obtained from the out-sample data represents one observation of the optimal portfolio. The out-sample data results are then compared with the results obtained with a naive strategy. The naive strategy, also known as the equal-weight strategy, is obtained by assigning equal weights to all assets and is widely accepted as a benchmark in portfolio selection problems to test the quality of the results. **Figure 1** and **Figure 2** represent in-sample Sharpe ratio and out-sample Sharpe ratio plotted with respect to the expected return ρ. The optimal expected return ρ^* is plotted to indicate the optimal Sharpe ratio, represented by dashed line. In general, to evaluate the Sharpe ratio, investor determines the risk-free return corresponding to the investment in the government securities, bonds, or treasury bills etc. However, for simplicity, we consider the risk-free return $r_f = 0$.

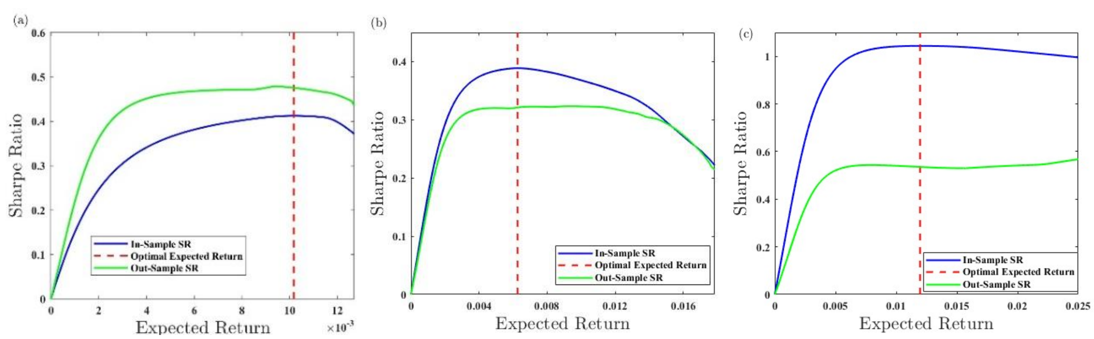

Figure 1. Sharpe ratio on daily data.

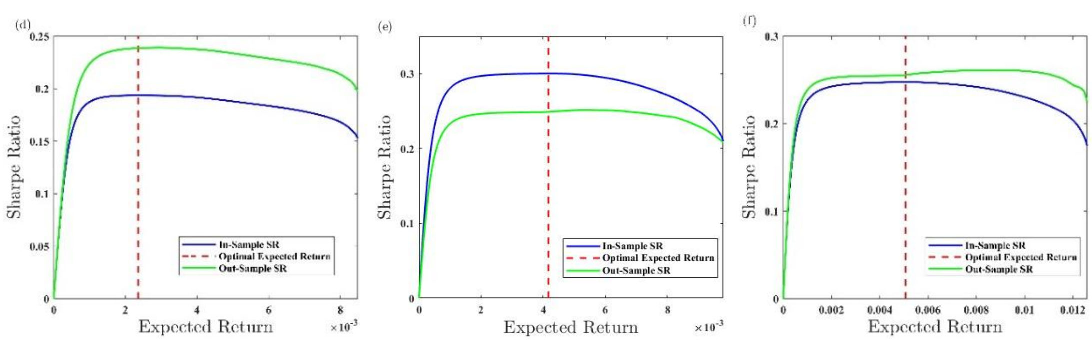

Figure 2. Sharpe ratio on hourly data.

The results obtained on the data sets of **Table 1** are summarized in **Table 2**. In this table, the optimal expected return is computed by solving the model with the algorithm's code using the in-sample data estimates (μ^{in}, C^{in}). The corresponding in-sample Sharpe ratio is reported using the simulated data. Using the out-of-sample estimates ($\mu^{\text{out}}, C^{\text{out}}$), optimal Sharpe ratio is obtained by solving the bilevel model, which we report as the actual portfolio Sharpe ratio.

Table 2. Numerical results for bilevel model.

Data set	**Data 1**	**Data 2**	**Data 3**	**Data 4**	**Data 5**	**Data 6**
Optimal Expected Return	1.02×10^{-2}	6.23×10^{-3}	1.19×10^{-2}	2.36×10^{-3}	4.17×10^{-3}	5.06×10^{-3}
Actual Sharpe ratio	5.31×10^{-1}	3.87×10^{-1}	1.10	1.55×10^{-1}	2.47×10^{-1}	2.65×10^{-1}
In Sample Sharpe ratio	4.13×10^{-1}	3.89×10^{-1}	1.00	1.94×10^{-1}	3.00×10^{-1}	2.48×10^{-1}
Out Sample Sharpe ratio	4.76×10^{-1}	3.21×10^{-1}	5.35×10^{-1}	2.39×10^{-1}	2.49×10^{-1}	2.56×10^{-1}
Average Sharpe ratio	4.93×10^{-1}	8.95×10^{-2}	1.46×10^{-1}	2.33×10^{-3}	6.61×10^{-2}	5.78×10^{-2}

Figure 3 is plotted to visualize the results of **Table 2**. The following observations are made from this figure.

- Better results are reported with hourly data compared to daily data, as the gap among in-sample Sharpe ratio, out-of-sample Sharpe ratio, and actual Sharpe ratio are relatively smaller. This is because hourly data provides a better estimation of the covariance matrix.
- For the nonstationary data, the gap between the in-sample and out-of-sample results is significant, as the performance of the model deteriorates due to sudden changes and regime shifts in the market data.
- The gap between the in-sample Sharpe ratio and the out-sample Sharpe ratio increased with the size of portfolio for the daily data.
- In-sample Sharpe ratio provides a good approximation of the actual Sharpe ratio.
- By comparing with the naive strategy, we see a larger Sharpe ratio is obtained than the naive strategy for each data set.

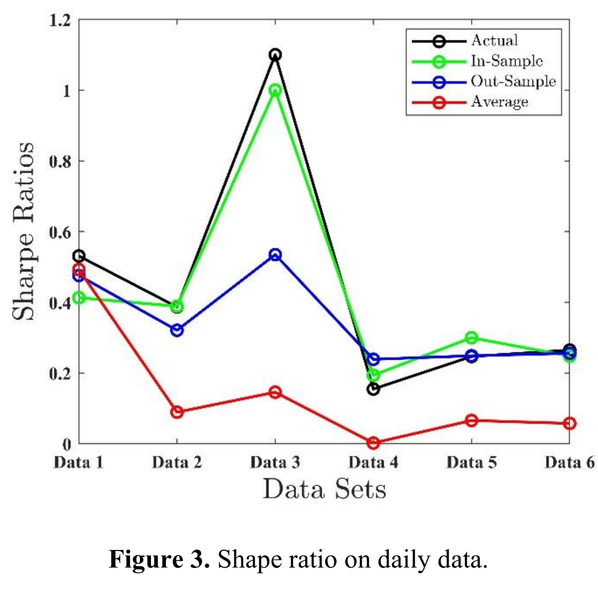

Figure 3. Shape ratio on daily data.

***Comparison with traditional approach*:** The proposed maximum Sharpe ratio model is formulated in two levels, and the methodology developed for the model is a gradient-based approach in which the active set plays an important role at every iteration. It is well-known that the maximum Sharpe ratio of a portfolio can be obtained by solving the Capital Asset Pricing Model (CAPM). We test the Sharpe ratio of our model against the Markowitz model and Capital Asset Pricing Model (CAPM) using out-of-sample data of the data sets of **Table 1**. The results obtained by various methods are listed in **Table 3**. The following notations are used in the table. BP: Sharpe ratio obtained in the bilevel framework using **Algorithm 1**.

CAPM: Sharpe ratio obtained by solving the CAPM model. The capital asset pricing model is a nonconcave maximization problem, which is solved using the particle swarm optimization method. It is a derivative-free approach.

Markowitz: Sharpe ratio obtained after solving Markowitz mean-variance model corresponding to the optimal portfolio. The Markowitz model is a convex quadratic problem, which we solve using quadprog in MATLAB.

Table 3 displays these results, and empirically conclude that the two-level model BP outperforms these standard models in estimating the Sharpe ratio.

Table 3. Comparison of Sharpe ratio on different models.

Data set	Data 1	Data 2	Data 3	Data 4	Data 5	Data 6
BP	5.31×10^{-1}	3.87×10^{-1}	1.10	1.55×10^{-1}	2.47×10^{-1}	2.65×10^{-1}
Markowitz	3.37×10^{-1}	5.06×10^{-2}	2.18×10^{-1}	2.61×10^{-2}	3.84×10^{-2}	4.06×10^{-2}
CAPM	4.09×10^{-1}	1.99×10^{-1}	4.23×10^{-1}	6.66×10^{-2}	9.08×10^{-2}	9.81×10^{-2}

5. Conclusion

In this paper, an iterative method is developed using an active set strategy to obtain the efficient portfolio of a bilevel portfolio optimization model for maximizing the Sharpe ratio. The method finds the optimal expected return in the bilevel model, corresponding to an efficient portfolio. The scheme is implemented in a financial data set. The results are compared with the capital asset pricing model. The existing methods (Chen et al., 2018; Jing et al., 2022) that deal with these models are direct search approaches, which compare the objective values and require a large number of function evaluations and lower-level solution computations. The proposed approach is based on the gradient information of the objective function and adopts an active set strategy at the lower-level problem.

The key limitation of the proposed method is the brute-force technique for the search of suitable active set in Step 4 of the algorithm, which may become computationally expensive for large size portfolios. However, these computations are not required at every iteration, and in practice, the search is not costly for small-sized portfolios.

The current model considers only covariance-based risk, and does not account for various market nonstationary conditions. The model could be extended to include value-at-risk (VaR) or conditional value-at-risk (CVaR) as risk measures, which may be considered as future contributions of the present work.

Conflict of Interest

The authors confirm that there are no conflicts of interest associated with this work.

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AI Disclosure
The author(s) confirm that generative AI tools were not used in the preparation of this article.

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