International Journal of Mathematical, Engineering and Management Sciences
Vol. 10, No. 1, 189-206, 2025
https://doi.org/10.33889/IJMEMS.2025.10.1.011

Ram Arti
Publishers

Numerical Solution of Conformable Fractional Periodic Boundary Value Problems by Shifted Jacobi Method

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(Received on May 10, 2024; Revised on July 26, 2024 & September 5, 2024 & October 1, 2024; Accepted on November 7, 2024)

Abstract
This paper presents the so-called shifted Jacobi method, an efficient numerical technique to solve second-order periodic boundary value problems with finitely many singularities involving nonlinear systems of two points. The method relies on the Jacobi polynomials used as natural basis functions in the conformable sense of fractional derivative. A study is carried out to compare the outcomes of the shifted Jacobi approach with those of other methods that are currently in use. In the same vein, a theoretical result for establishing a bound of the error generated from the proposed approximate solution is proved accordingly. The efficiency and effectiveness of the shifted Jacobi technique with conformable fractional derivative are discussed numerically.

Keywords- Conformable fractional derivative, Nonlinear boundary value problems, Nonlinear fractional differential equations, Jacobi orthogonal polynomials.

1. Introduction

In order to provide tools for modeling anomalous diffusion and memory effects in a variety of physical systems, fractional calculus expands on classical calculus by permitting differentiation and integration to non-integer orders. Riemann-Liouville and Caputo derivatives are basic operations in fractional calculus that generalize integer-order differentiation and integration. In domains where conventional models are insufficient, such viscoelasticity, electrical engineering, and control theory, it finds use. Systems with complicated dynamics and inherited properties can be better described by fractional calculus. There is constant research into new definitions and applications for the rich and ever-evolving mathematical framework (Kilbas et al., 2006; Podlubny, 1999).

Fractional differential system with periodic boundary value problems (BVPs) has a strong influence in many different areas of mathematics and physics, see (Abdelhalim et al., 2023; Al-Nana et al., 2023; Anakira et al., 2022; Bouchenak et al., 2024; Hammad et al., 2023). For this, we find that it has received the attention of many mathematicians who still use it to solve and extract multiple mathematical problems as accurately as possible, see for instance (Kilbas et al., 2006; Lakshmikantham et al., 2009; Miller and Ross, 1993; Podlubny, 1999; Samko et al., 1993).

According to Kilbas et al. (2006), Singular Fractional differential system with periodic boundary conditions.

$$u_1^{\alpha}(t)+\frac{a_1}{b_1}u_1'(t)+\frac{c_1}{d_1}u_1(t)=f_1(t,u_1(t),u_2(t)) \tag{1}$$

$$u_2^{\alpha}(t)+\frac{a_1}{b_1}u_2'(t)+\frac{c_1}{d_1}u_2(t)=f_2\big(t,u_1(t),u_2(t)\big).$$

subject to the boundary conditions

$$u_1(0)=u_1(1), \qquad u_1'(0)=u_1'(1) \tag{2}$$

$$u_2(0)=u_2(1), \qquad u_2'(0)=u_2'(1).$$

When solving differential equations, one benefit of employing Jacobi polynomials is that the solution may be found in terms of the Jacobi parameters, α and β, as shown in Bojdi et al. (2013), Doha et al. (2012), and Doha et al. (2015). Differential equations of diverse kinds are numerically solved using some particular examples of the Jacobi parameters α and β.The study of existence theory for real-world issues, which can be represented by fractional differential equations with multi-point boundary conditions, is gaining popularity and is a field of study that is expanding quickly. Numerous scholars have attempted to quickly achieve the best accuracy by utilizing a variety of techniques in the numerical analysis literature to solve singular fractional differential systems with periodic boundary conditions. One of the various numerical methods is the shifted Jacobi approach.

The main objective of this study is to solve systems of singular fractional differential equations with periodic boundary value problems numerically using the Jacobi method. To determine the precision and robustness of the proposed solution, it would be helpful to contrast these outcomes with those from other contemporary techniques. Many researchers have been able to obtain more precise and realistic solutions to nonlinear boundary value issues for fractional differential equations because of the development of nonlinear analytic tools. Therefore, this study may contribute to finding the solutions of the general 2nd-order nonlinear fractional boundary value problems.

The arrangement of this paper is as follows: In the subsequent sections, we remember a few definitions involving conformable fractional derivative as well as the definition of shifted Jacobi polynomial. In

Section 3, the necessary foundations in relation to the operational matrices that are used for constructing the shifted Jacobi polynomials are presented. The rectification of the proposed approach is verified by its application to several numerical examples in Section 4. The work's conclusion and future work are presented in the last two sections, respectively.

2. Preliminaries

This section presents some key definitions, characteristics, and some fundamentals related to Jacobi polynomials and the conformable fractional calculus.

2.1 Conformable Fractional Derivative

This part is devoted to remember the most useful basics and fundamentals in connection to the conformable fractional calculus.

Definition 2.1 (Khalil et al., 2014; Kitzhofer et al., 2007) The conformable fractional derivative of order σ of a function $\psi: [0,\infty) \to \mathbb{R}$ might be given by

$$T_\sigma(\psi)(\tau) = \psi^{(\sigma)}(\tau) = \lim_{\varepsilon\to 0} \frac{\psi(\tau+\varepsilon\tau^{1-\sigma})-\psi(\tau)}{\varepsilon} \tag{3}$$

Theorem 2.1 (Khalil et al., 2014; Kitzhofer et al., 2007) In light of Definition 2.1, we have:

(i) The linearity property is satisfied by the conformable fractional derivative, that is,

$$T_\sigma(a\psi + b\phi) = aT_\sigma(\psi) + bT_\sigma(\phi), \quad \forall a, b \in \mathbb{R} \tag{4}$$

(ii) In the event that f is n times differentiable, we obtain

$$T_\sigma\psi(\tau) = \tau^{n+1-\sigma}\psi^{[\sigma]}(\tau), \ \forall\sigma \in (n, n+1] \tag{5}$$

(iii) Given a constant c, we obtain $T_\sigma(c) = 0$.

(iv) for $\tau > 0$ and $\sigma \in (0,1)$. Given that $\lim_{\tau\to 0^+} \psi^{(\sigma)}(\tau)$ exists for which ψ is σ-differentiable in some $(0, a)$, $a > 0$, then one may denote $\psi^{(\sigma)}(0) = \lim_{\tau\to 0^+} \psi^{(\sigma)}(\tau)$.

(v) We have

$$T_\sigma(\tau^s) = \begin{cases} \frac{\Gamma(s+1)}{\Gamma(s-n)}\tau^{s-\sigma} & , \forall s \in \mathbb{N} \ and \ s > \sigma \\ 0 & , \ \forall s \in \mathbb{N} \ and \ s < \ \sigma \end{cases} \tag{6}$$

where, $\sigma \in (n, n+1]$ and Γ indicates Gamma function.

2.2 Shifted Jacobi Polynomials

The purpose of this section is to help the reader recall the most important foundations and principles related to the main concepts of the so-called shifted Jacobi polynomials.

Definition 2.2 (Bhrawy et al., 2014) The following formula defines the shifted Jacobi polynomials with parameters $(\wp, q)$ defined on the interval $[0, \mathcal{L}]$:

$$\mathcal{J}_{\mathcal{L},\iota}^{(\wp,q)}(\tau) = \sum_{\kappa=0}^{\iota}(-1)^{\iota+\kappa}\frac{\Gamma(\iota+q+1)\Gamma(\iota+\kappa+\wp+q+1)}{\Gamma(\kappa+q+1)\Gamma(\iota+\wp+q+1)\Gamma(\iota-\kappa+1)\Gamma(\kappa+1)\mathcal{L}^\kappa}\tau^\kappa \tag{7}$$

for $\iota \in \mathbb{N}^{\circ}$, $p, q > -1$, and $p + q \neq -1$ where,

$$\mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(0) = (-1)^{\iota} \frac{\Gamma(\iota+q+1)}{\Gamma(q+1)\iota!} \tag{8}$$

and

$$\mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\mathcal{L}) = \frac{\Gamma(\iota+p+1)}{\Gamma(p+1)\iota!} \tag{9}$$

Also, for any positive integer r, we have

$$\frac{d^r}{d\tau^r} \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) = \frac{\Gamma(r+\iota+p+q+1)}{\Gamma(\iota+p+q+1)} \mathcal{J}_{\mathcal{L},\iota-r}^{(p+r,q+r)}(\tau) \tag{10}$$

The wight function $\mathcal{W}(\tau) = (\mathcal{L} - \tau)^{p} \tau^{q}$ is orthogonal to the shifted Jacobi polynomials. The orthogonality relation is given by

$$\int_0^{\mathcal{L}} \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) \mathcal{J}_{\mathcal{L},j}^{(p,q)}(\tau) \mathcal{W}(\tau) d\tau = \begin{cases} k_{\iota,\mathcal{L}} & , \quad \iota = j \\ 0 & , \quad \iota \neq j \end{cases}$$

where,

$$k_{\iota,\mathcal{L}} = \frac{\Gamma(\iota + p + 1)\Gamma(\iota + q + 1)}{(2\iota + p + q + 1)\Gamma(\iota + 1)\Gamma(\iota + p + q + 1)} \mathcal{L}^{p+q+1}.$$

If $\psi(\tau) \in L^2_{\mathcal{W}(\tau)}[0, \mathcal{L}]$, then the shifted Jacobi orthogonal can be used to represent it as follows:

$$\psi(\tau) = \sum_{\iota=0}^{\infty} c_\iota \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau),$$

where, $c_\iota = \frac{1}{k_{\iota,\mathcal{L}}} \int_0^{\mathcal{L}} \psi(\tau) \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) \mathcal{W}(\tau) d\tau$, $\iota = 0,1,2, \dots$. By approximating $\psi(\tau)$ to the first $(\mathrm{N} + 1)$-terms, we get:

$$\psi(\tau) \cong \psi_N(\tau) = \sum_{\iota=0}^{N} c_\iota \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) = \mathrm{C}^T \Psi(\tau),$$

such that $\mathrm{C}^T = [c_0, c_1, \dots, c_\mathrm{N}]$ and $\Psi(\tau) = \left[\mathcal{J}_{\mathcal{L},0}^{(p,q)}(\tau), \mathcal{J}_{\mathcal{L},1}^{(p,q)}(\tau), \dots, \mathcal{J}_{\mathcal{L},\mathrm{N}}^{(p,q)}(\tau)\right]^T$.

3. Operational Matrices

The two subsequent results are used to evaluate the conformable fractional derivative using shifted Jacobi polynomials. These results are proven based on certain properties of this derivative.

Theorem 3.1 Suppose that the approximated function in terms of shifted Jacobi polynomials $\psi_N(\tau)$ is given by

$$\psi_\mathrm{N}(\tau) = \sum_{\iota=0}^{\mathrm{N}} c_\iota \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) \tag{11}$$

Then, the conformable fractional derivative of $\psi_\mathrm{N}(\tau)$ can be written as

$$T_\sigma \psi_\mathrm{N}(t) = \sum_{\iota=n+1}^{\mathrm{N}} \sum_{k=0}^{\iota-n-1} c_\iota \mathcal{M}_{\iota,k}^{\sigma} t^{k-\sigma}, \sigma \in (n, n+1] \tag{12}$$

for which

$$\mathcal{M}_{\iota,k}^{\sigma} = (-1)^{\iota+k} \frac{\Gamma(\iota+q+1)\Gamma(\iota+\kappa+p+q+1)}{\Gamma(\kappa+q+1)\Gamma(\iota+p+q+1)\Gamma(\iota-\kappa+1)\Gamma(\kappa-n+1)\mathcal{L}^{\kappa}}.$$

Proof**:** Utilizing the approximated function $\psi_{\mathrm{N}}(\tau)$ defined in Equation (11) and the linearity of the conformable fractional differentiator provided in Equation (4), we get

$$T_\sigma \psi_{\mathrm{N}}(t) = \sum_{\iota=0}^{N} c_\iota T_\sigma \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) \tag{13}$$

Moreover, from Equations (4), (5) and (6) in Theorem 2.1, we get

$$T_\sigma \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) = 0, \;\; i = 0,1, \ldots\ldots \sigma-1, \; n \in \mathbb{N}, \quad n < \sigma \le n+1 \tag{14}$$

Therefore, from Equation (7), we have

$$T_\sigma \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) = \sum_{k=0}^{i-n-1} (-1)^{i+k} \frac{\Gamma(\iota+q+1)\Gamma(\iota+\kappa+p+q+1)}{\Gamma(\kappa+q+1)\Gamma(\iota+p+q+1)(\iota-\kappa)!(\kappa-n)!\mathcal{L}^{\kappa}} t^{k-\sigma} \tag{15}$$

for $k > n$. By combinations Equations (13), (14) and (15), we obtain

$$T_\sigma\, \psi_{\mathrm{N}}(t) = \sum_{i=n+1}^{N} \sum_{k=0}^{i-n-1} c_n (-1)^{i+k} \frac{\Gamma(\iota+q+1)\Gamma(\iota+\kappa+p+q+1)}{\Gamma(\kappa+q+1)\Gamma(\iota+p+q+1)(\iota-\kappa)!(\kappa-n)!\mathcal{L}^{\kappa}} t^{k-\sigma}$$

for $n \le \sigma \le n+1$.

Theorem 3.2 The bound for the error between $T_\sigma \psi_{\mathrm{N}}(\tau)$ and $T_\sigma \psi(\tau)$ is given by

$$|\, T_\sigma \psi(\tau) - T_\sigma \psi_{\mathrm{N}}(\tau)| \le \sum_{i=N+1}^{\infty} \sum_{j=0}^{N} \sum_{k=0}^{i-n-1} |c_\iota| \left|\mathcal{B}_{i,,\iota,k}\right| \left|\mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau)\right|,$$

for which

$$\mathcal{B}_{i,j,k} = \sum_{m=0}^{\iota} (-1)^{i+\iota+k+1} \frac{\Gamma(\iota+q+1)\Gamma(\iota+\kappa+p+q+1)\hbar_{\iota,\mathcal{L}}}{\Gamma(\kappa+q+1)\Gamma(\iota+p+q+1)(\iota-\kappa)!\,(\kappa-n)!\,\mathcal{L}^{\kappa}}.$$

Proof**:** By approximating $t^{k-\sigma}$ with $(N+1)$-terms of Jacobi polynomial, one can have

$$t^{k-\sigma} \cong \sum_{\iota=0}^{\mathrm{N}} b_{k,\iota} \mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau) \tag{16}$$

Now, with respect $\mathcal{W}(\tau) = 1$, we have

$$b_{k,\iota} = \sum_{m=0}^{\iota} (-1)^{i-k} \frac{\Gamma(m+q+1)\Gamma(m+\kappa+p+q+1)}{\Gamma(\kappa+q+1)\Gamma(m+p+q+1)\Gamma(m-\kappa+1)\Gamma(\kappa-n+1)\Gamma(k-\sigma+m+1)\mathcal{L}^{\kappa}} \hbar_{\iota,\mathcal{L}} \tag{17}$$

where,

$$\hbar_{\iota,\mathcal{L}} = \frac{\Gamma(\iota+p+1)\Gamma(\iota+q+1)}{(2\iota+p+q+1)\Gamma(\iota+1)\Gamma(\iota+p+q+1)} \mathcal{L}^{p+q+1}.$$

By combining Equations (11), (16) and (17), we obtain

$$|\, T_\sigma \psi(\tau) - T_\sigma \psi_{\mathrm{N}}(\tau)| \le \sum_{\iota=0}^{N} |c_\iota| \left|\mathcal{B}_{i,,\iota,k}\right| \left|\mathcal{J}_{\mathcal{L},\iota}^{(p,q)}(\tau)\right|,$$

for $n-1 \leq \sigma \leq n$.

Generally speaking, the adaptation procedure makes sure that the strong characteristics of Jacobi polynomials are kept while expanding their use to conformable fractional derivative problems. This makes it possible to use proven analytical and numerical methods to a larger class of fractional differential equations. Jacobi polynomials offer strong numerical features for spectral approaches, superior orthogonality with respect to a weight function on a defined interval, and variable boundary behavior through changeable parameters. These benefits make them especially useful in situations where complicated differential equations and peculiar endpoint behaviors are needed.

4. Numerical Results

Three numerical examples are presented in this section to validate the key conclusions of our suggested numerical technique.

Example 4.1 Consider the following differential equation of second-order:
$y''(x) + (4x^2 - 4x + 1)y(x) = 2\sin(x - x^2),$

with boundary conditions
$y(0) = y(1), \;\; y'(0) = y'(1).$

The exact solution of the above problem is given by $u(x) = \cos(x - x^2)$. In particular, one might find this exact solution by first considering the following associated homogeneous equation:
$y''(x) + (4x^2 - 4x + 1)y(x) = 0.$

This equation represents a second-order linear differential equation with variable coefficients. The term $(4x^2 - 4x + 1)$ can be analyzed for its behavior, particularly its discriminant is calculated as
$D = (-4)^2 - 4.4.1 = 0$.

Since the discriminant is zero, the characteristic polynomial has a double root. This indicates that the solutions to the homogeneous equation will involve terms like
$y_h(x) = C_1 y_1(x) + C_2 y_2(x),$

where, $y_1(x)$ and $y_2(x)$ are linearly independent solutions. Secondly, we consider the non-homogeneous term so that the right-hand side of the equation, $2\sin(x - x^2)$ suggests looking for a particular solution of a similar form. However, instead of trying to find a solution directly from this term, we can hypothesize that the solution has a certain form based on its structure. Thirdly, given that $2\sin(x - x^2)$ is a sine function, we might think of solutions that include sine and cosine functions. However, the specific form $y(x) = 2\cos(x - x^2)$ suggests a particular insight. Fourthly, to verify that $y(x) = \cos(x - x^2)$ is indeed a solution, we can directly compute its second derivative and substitute it back into the original equation. In particular, we carry out the following steps:

- We calculate $y'(x)$ and $y''(x)$ to obtain
  $y'(x) = -2\sin(x - x^2)(1 - 2x)$
  and
  $y''(x) = 2\sin(x - x^2) - \cos(x - x^2)(1 - 2x)^2.$

- We substitute into the differential equation by plugging $y'(x)$ and $y''(x)$ back into the differential equation, we will check if:

$$y''(x) + (4x^2 - 4x + 1)\cos(x - x^2) = 2\sin(x - x^2).$$

After simplification, you would find that both sides match, confirming that $y(x) = \cos(x - x^2)$ satisfies the equation.

Finally, we need to check if this solution satisfies the boundary conditions at $x = 0$ and $x = 1$, i.e., $y(0) = \cos(0) = 1,\ y(1) = \cos(1 - 1^2) = 1$.

Thus, $y(0) = y(1)$ holds. For the remainder derivatives, we can also verify $y'(0) = y'(1)$ by computing $y'(0)$ and $y'(1)$ similarly. This leads to the conclusion that the exact solution is indeed $y(x) = \cos(x - x^2)$.

On the other hand, if one wants to solve the problem reported in Example 4.1 by using our proposed numerical approach, we use Theorem 3.1 with $N = 10$, $\alpha = \beta = L = 1, \mu = 2$. Consequently, we obtain the approximate solution that is depicted in **Figure 1**, whereas the errors can be shown in **Table 1**, **Figure 2**, and **Figure 3**.

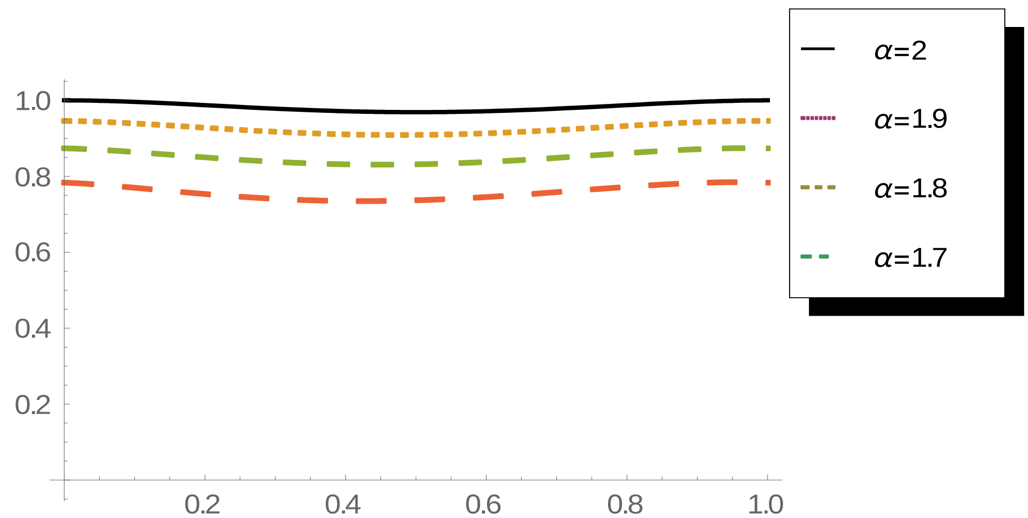

Figure 1. The exact and approximate solutions of Example 4.1 according to different values of α for $N = 10$, $\alpha = \beta = L = 1, \mu = 2$.

Table 1. The absolute and the relative errors between the approximate and exact solutions of Example 4.1 for $N = 10$, $\alpha = \beta = L = 1, \mu = 2$.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.0	1.000000	1.000000	$1.309146744 \times 10^{-7}$	$1.309146744 \times 10^{-7}$
0.1	0.995953	0.995953	$1.325669166 \times 10^{-7}$	$1.331056306 \times 10^{-7}$
0.2	0.987227	0.987227	$1.33538675 \times 10^{-7}$	$1.352663943 \times 10^{-7}$
0.3	0.978031	0.978031	$1.337035589 \times 10^{-7}$	$1.367068841 \times 10^{-7}$
0.4	0.971338	0.971338	$1.336176949 \times 10^{-7}$	$1.375604562 \times 10^{-7}$
0.5	0.968912	0.968913	$1.340519761 \times 10^{-7}$	$1.383530369 \times 10^{-7}$
0.6	0.971338	0.971338	$1.336176949 \times 10^{-7}$	$1.375604562 \times 10^{-7}$
0.7	0.978031	0.978031	$1.33703559 \times 10^{-7}$	$1.367068842 \times 10^{-7}$
0.8	0.987227	0.987227	$1.33538675 \times 10^{-7}$	$1.352663943 \times 10^{-7}$
0.9	0.995953	0.995953	$1.325669166 \times 10^{-7}$	$1.331056306 \times 10^{-7}$

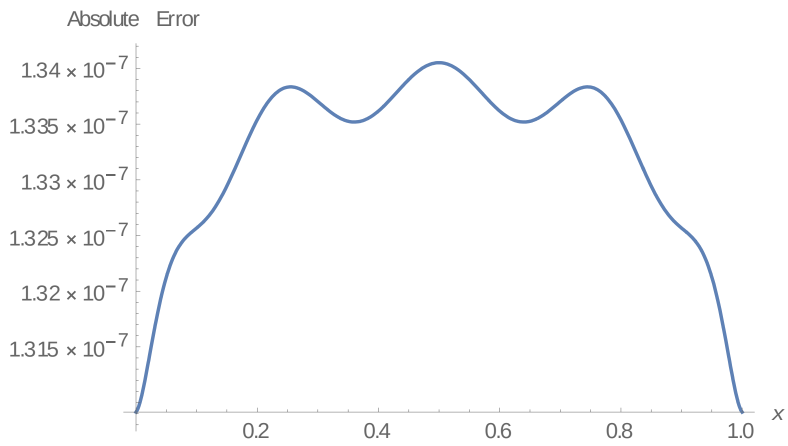

Figure 2. The absolute error of Example 4.1 for $N = 10, \alpha = \beta = L = 1, \mu = 2$.

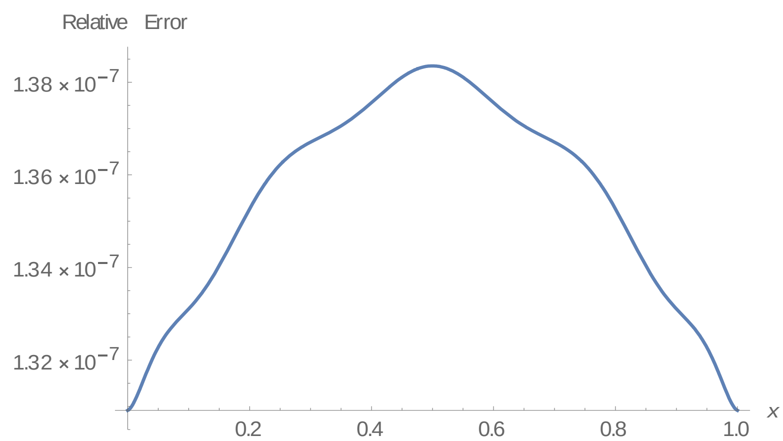

Figure 3. The relative error of Example 4.1 for $N = 10, \alpha = \beta = L = 1, \mu = 2$.

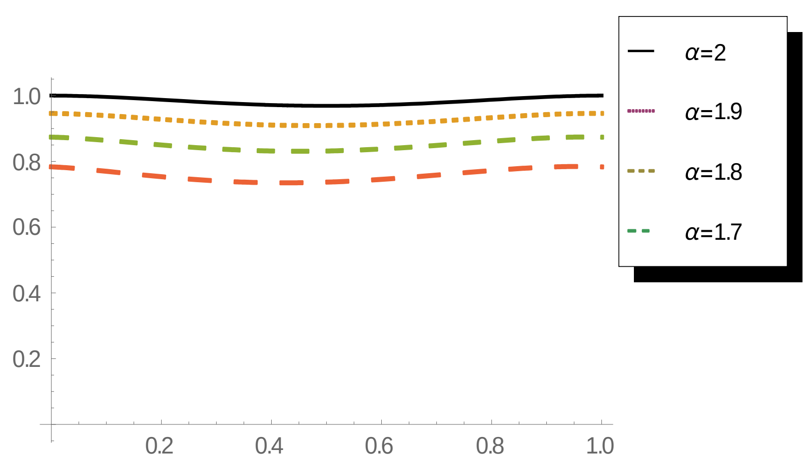

Figure 4. The exact solution and the approximate solution of Example 4.1 for $N = 10, \alpha = \beta = 0.5, L = 1, \mu = 2$.

Table 2. The absolute and the relative errors between the approximate and exact solutions of Example 4.1 for $N = 10, \alpha = \beta = 0.5, L = 1, \mu = 2$.

x	Exact solution	Approximate solution $y(x)$	Absolute error	Relative error
0.0	1.000000	1.000000	$4.417082455 \times 10^{-8}$	$4.417082455 \times 10^{-8}$
0.1	0.995953	0.995953	$4.462151815 \times 10^{-8}$	$4.480284723 \times 10^{-8}$
0.2	0.987227	0.987227	$4.530544373 \times 10^{-8}$	$4.589160419 \times 10^{-8}$
0.3	0.978031	0.978031	$4.513401031 \times 10^{-8}$	$4.614783605 \times 10^{-8}$
0.4	0.971338	0.971338	$4.503665718 \times 10^{-8}$	$4.636558886 \times 10^{-8}$
0.5	0.968912	0.968912	$4.561085509 \times 10^{-8}$	$4.707428047 \times 10^{-8}$
0.6	0.971338	0.971338	$4.503665718 \times 10^{-8}$	$4.636558886 \times 10^{-8}$
0.7	0.978031	0.978031	$4.513401031 \times 10^{-8}$	$4.614783605 \times 10^{-8}$
0.8	0.987227	0.987227	$4.530544373 \times 10^{-8}$	$4.589160419 \times 10^{-8}$
0.9	0.995953	0.995953	$4.462151793 \times 10^{-8}$	$4.480284701 \times 10^{-8}$
1.0	1.000000	1.000000	$4.417082433 \times 10^{-8}$	$4.417082433 \times 10^{-8}$

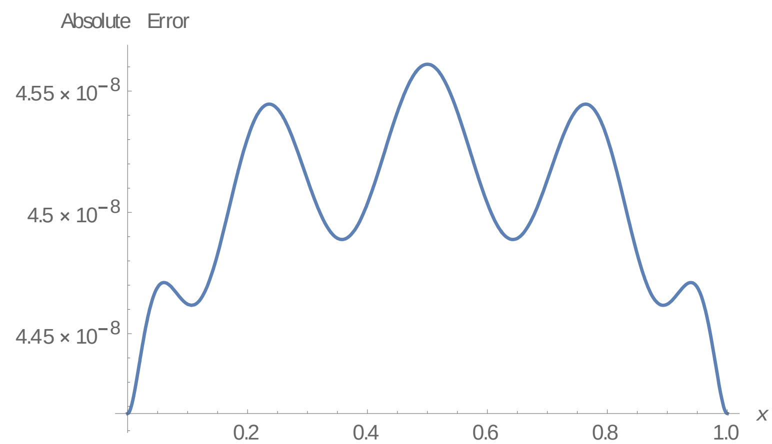

Figure 5. The absolute error of Example 4.1 for $N = 10, \alpha = \beta = 0.5, L = 1, \mu = 2$.

Now, by taking $N = 10, \alpha = \beta = 0.5, L = 1, \mu = 2$, we can obtain other numerical results of Example 4.1. These results include an approximate solution of the considered example that is shown in **Figure 4** and the generated errors that can be shown in **Table 2**, **Figure 5**, and **Figure 6**.

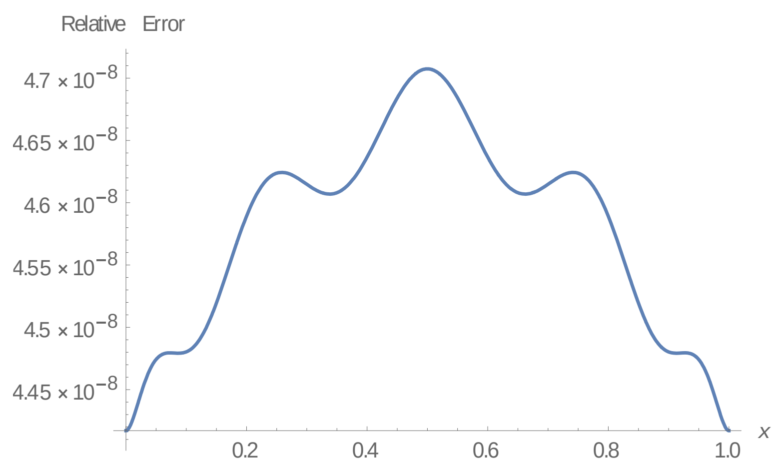

Figure 6. The relative error of Example 4.1 for $N = 10, \alpha = \beta = 0.5, L = 1, \mu = 2$.

Example 4.2 Consider the following 2nd-order system of differential equations:

$$u''_1(x)+\frac{2}{\ln x}u'_1(x)+\sinh[u_1(x)-x^2(2x-3)]-(u_2(x))^2=f_1,$$

$$u''_2(x)+\frac{x^3}{1-x}u'_2(x)+(u_2(x)+x^2)^3=f_2,$$

subject to the boundary conditions
$u_1(0)=u_1(1),\ u_2(0)=u_2(1),\quad u'_1(0)=u'_1(1),\quad u'_2(0)=u'_2(1),$

where,

$$f_1=-2-12(-1+x)x-\frac{4(-1+x)x(-1+2x)}{\log x}-\log(1+(-1+x)^2x^2)^2-\sinh(x^2-(x^2(-2+x^2))$$

and

$$f_2=\frac{2(1-2x)x^4}{1+(-1+x)^2x^2}+\frac{2+12(-1+x)x}{1+(-1+x)^2x^2}-\frac{4(x-3x^2+2x^3)}{(1+(-1+x)^2x^2)^2}+(x^2+\log(1+(-1+x)^2x^2)^3.$$

The exact solution of the above system is $(u_1(x),u_2(x))=(x^4+2x^3-x^2,\ln(x^4-2x^3+x^2+1))$. With the use of Theorem 3.1 with $N=10,\alpha=\beta=1$, we obtain an approximate solution of the above system that can be shown in **Figure 7** and **Figure 8** at various values of α, whereas the corresponding errors can be shown in **Table 3**, **Table 4**, and **Figure 9**.

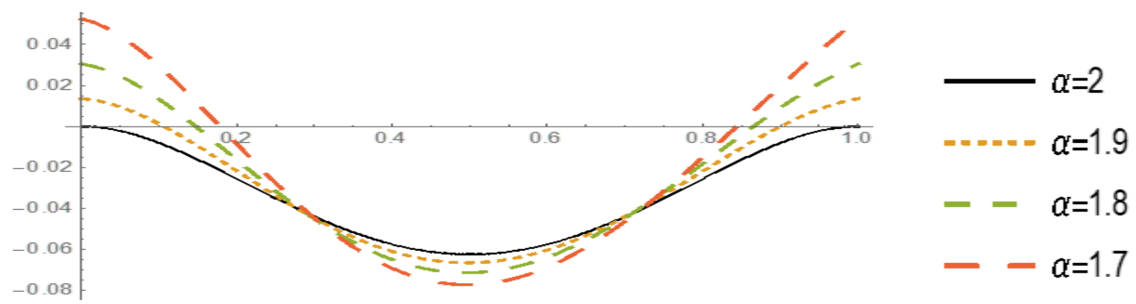

Figure 7. The behavior approximate solutions of Example 4.2 for $N=10,\alpha=\beta=1$ at various of α.

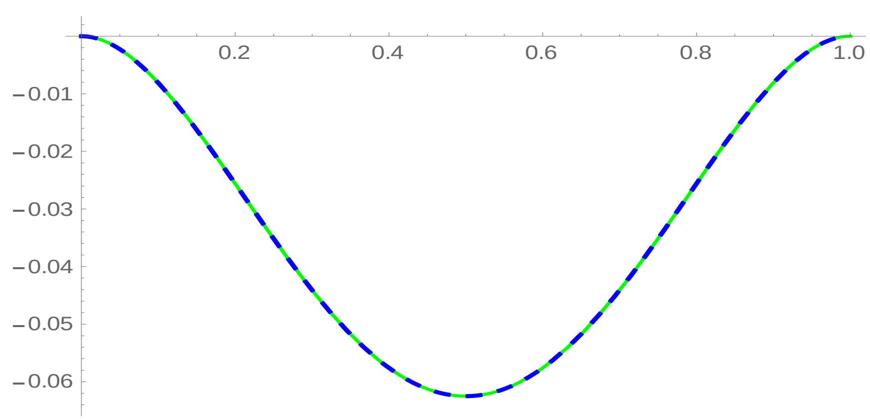

Figure 8. The approximate and exact solutions of Example 4.2 for $N=10,\alpha=\beta=1$.

Table 3. The absolute and the relative errors between the approximate solution $u_1(x)$ and exact solution of Example 4.2 for $N = 10, \alpha = \beta = 1$.

x	Exact solution	Approximate solution $u_1(x)$	Absolute error	Relative error
0.0	0.0000	0.0000	$3.992270089 \times 10^{-12}$	0.0000
0.1	-0.0081	-0.0081	$1.44295513 \times 10^{-12}$	-0.0081
0.2	-0.0256	-0.0256	$6.373790384 \times 10^{-13}$	-0.0256
0.3	-0.0441	-0.0441	$1.582393938 \times 10^{-12}$	-0.0441
0.4	-0.0567	-0.0567	$3.305619667 \times 10^{-13}$	-0.0567
0.5	-0.0625	-0.0625	$2.18662588 \times 10^{-12}$	-0.0625
0.6	-0.0567	-0.0567	$1.862121568 \times 10^{-13}$	-0.0567
0.7	-0.0441	-0.0441	$9.365147546 \times 10^{-13}$	-0.0441
0.8	-0.0256	-0.0256	$5.512083845 \times 10^{-13}$	-0.0256
0.9	-0.0081	-0.0081	$2.79627016 \times 10^{-13}$	-0.0081
1.0	0.0000	0.0000	$3.992266167 \times 10^{-12}$	0.0000

Table 4. The absolute and the relative errors between the approximate solution $u_2(x)$ and exact solution of Example 4.2 for $N = 10, \alpha = \beta = 1$.

x	Exact solution	Approximate solution $u_2(x)$	Absolute error	Relative error
0.0	0.00000000	0.00000000	$3.745751578 \times 10^{-9}$	0.00000000
0.1	0.00806737	0.00806737	$4.101568325 \times 10^{-9}$	0.00806737
0.2	0.0252778	0.0252778	$4.15143726 \times 10^{-9}$	0.02527780
0.3	0.0431553	0.0431553	$1.302711758 \times 10^{-9}$	0.0431553
0.4	0.0560022	0.0560022	$3.452931413 \times 10^{-10}$	0.0560022
0.5	0.0606246	0.0606246	$5.585241186 \times 10^{-9}$	0.0606246
0.6	0.0560022	0.0560022	$5.803486094 \times 10^{-9}$	0.0560022
0.7	0.0431553	0.0431553	$1.429653249 \times 10^{-8}$	0.0431553
0.8	0.0252778	0.0252778	1.3535506×10^{-8}	0.0252778
0.9	0.00806737	0.00806737	$1.942234495 \times 10^{-8}$	0.00806737
1.0	0.00000000	0.00000000	$3.74575158 \times 10^{-9}$	0.00000000

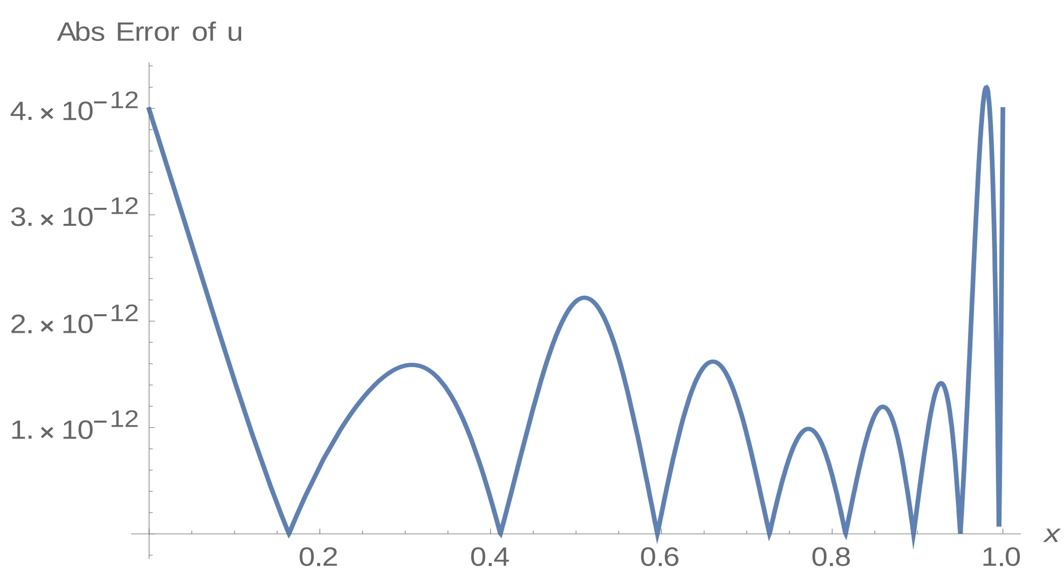

Figure 9. Absolute error of Example 4.2 for N =10, $\alpha = \beta = 1$.

In the same regard, with the help of using Theorem 3.1 and with $N = 10, \alpha = \beta = \frac{1}{2}$, we obtain the approximate solutions of Example 4.2 at several of α, which can be shown **Figure 10** and **Figure 11**, whereas the corresponding errors can be shown in **Table 5**, **Table 6**, and **Figure12**.

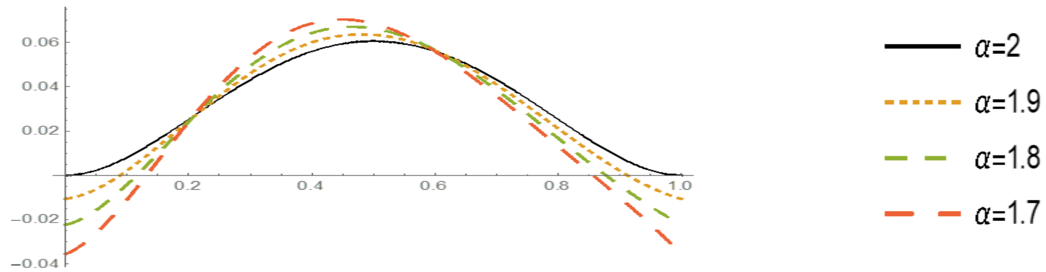

Figure 10. The behavior approximate solutions of Example 4.2 at various of α for $N = 10, \alpha = \beta = \frac{1}{2}$.

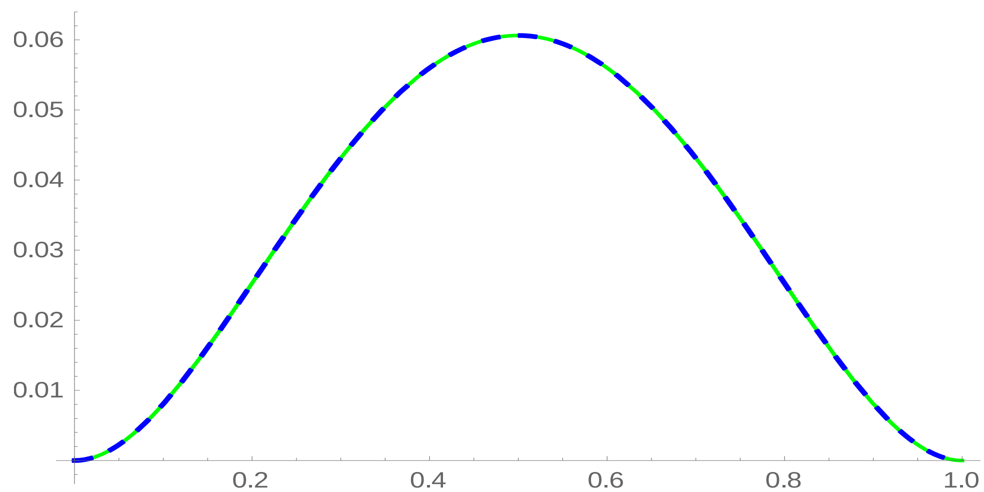

Figure 11. The approximate solution and the exact solution of Example 4.2 for $N = 10, \alpha = \beta = \frac{1}{2}$.

Table 5. The absolute and the relative errors between the approximate solution $u_1(x)$ and exact solution of Example 4.2 for $N = 10, \alpha = \beta = \frac{1}{2}$.

x	Exact solution	Approximate solution $u_1(x)$	Absolute error	Relative error
0.0	0.0000	0.0000	$3.840311455 \times 10^{-10}$	0.0000
0.1	-0.0081	-0.0081	$2.543730816 \times 10^{-10}$	-0.0081
0.2	-0.0256	-0.0256	$9.152329589 \times 10^{-11}$	-0.0256
0.3	-0.0441	-0.0441	$8.122100215 \times 10^{-12}$	-0.0441
0.4	-0.0567	-0.0567	$1.394506247 \times 10^{-10}$	-0.0567
0.5	-0.0625	-0.0625	$3.69283118 \times 10^{-10}$	-0.0625
0.6	-0.0567	-0.0567	$2.719820411 \times 10^{-10}$	-0.0567
0.7	-0.0441	-0.0441	$2.513053723 \times 10^{-10}$	-0.0441
0.8	-0.0256	-0.0256	$2.400460941 \times 10^{-10}$	-0.0256
0.9	-0.0081	-0.0081	$2.228329864 \times 10^{-10}$	-0.0081
1.0	0.0000	0.0000	$4.75352532 \times 10^{-10}$	0.0000

Table 6. The absolute and the relative errors between the approximate solution $u_2(x)$ and exact solution of Example 4.2 for $N = 10, \alpha = \beta = \frac{1}{2}$.

x	Exact solution	Approximate solution $u_2(x)$	Absolute error	Relative error
0.0	0.00000000	0.00000000	$3.40500247 \times 10^{-8}$	0.00000000
0.1	0.00806737	0.00806737	$1.38792113 \times 10^{-7}$	0.00806737
0.2	0.0252778	0.0252778	$2.263154496 \times 10^{-7}$	0.0252778
0.3	0.0431553	0.0431553	$1.78483935 \times 10^{-7}$	0.0431553
0.4	0.0560022	0.0560022	$4.138881937 \times 10^{-7}$	0.0560022
0.5	0.0606246	0.0606246	$3.235697437 \times 10^{-7}$	0.0606246
0.6	0.0560022	0.0560022	$1.403079832 \times 10^{-7}$	0.0560022
0.7	0.0431553	0.0431553	$3.556736469 \times 10^{-7}$	0.0431553
0.8	0.0252778	0.0252778	$3.014283997 \times 10^{-7}$	0.0252778
0.9	0.00806737	0.00806737	$2.169483741 \times 10^{-7}$	0.00806737
1.0	0.00000000	0.00000000	$4.015075761 \times 10^{-7}$	0.00000000

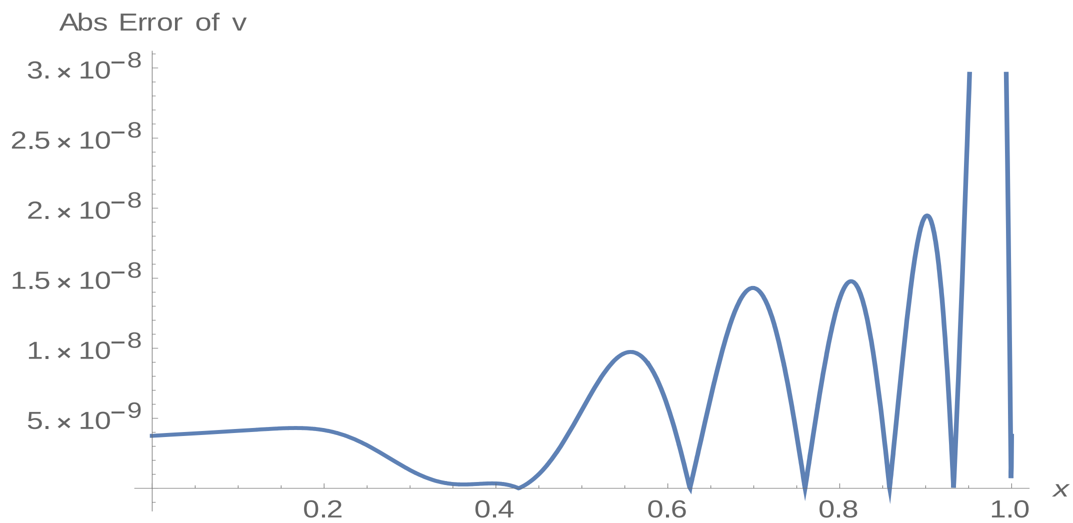

Figure 12. Absolute error of Example 4.2 for N =10, $\alpha = \beta = \frac{1}{2}$.

Example 4.3 Consider the following fractional differential equation:

$$D^{\alpha} u_1(x) - \frac{1}{x(x-1)} u_2^2(x) = 2 - 11\,x + 7\,x^2 + 8\,x^3 - 4\,x^4,$$

$$D^{\alpha} u_2(x) + \frac{1}{(x-1)} u_1'(x) = -6 + 10\,x + 4x^2.$$

subject to the boundary conditions
$u_1(0) = u_1(1),\ u_2(0) = u_2(1), \quad u'_1(0) = u'_1(1), \quad u'_2(0) = u'_2(1).$

where, $(u_1(x), u_2(x)) = (x^4 - 2x^3 + x^2, 2x^3 - 3x^2 + x)$ is the exact solution at $\alpha = 2$. By using Theorem 3.1 with $N = 45, \alpha = \beta = \frac{1}{2},\ L = 1,\ \mu = 2$, we plot in **Figure 13** the approximate solution at various of α. In the same regard, the generated errors are shown in **Table 7**, **Table 8**, and **Figure 14**.

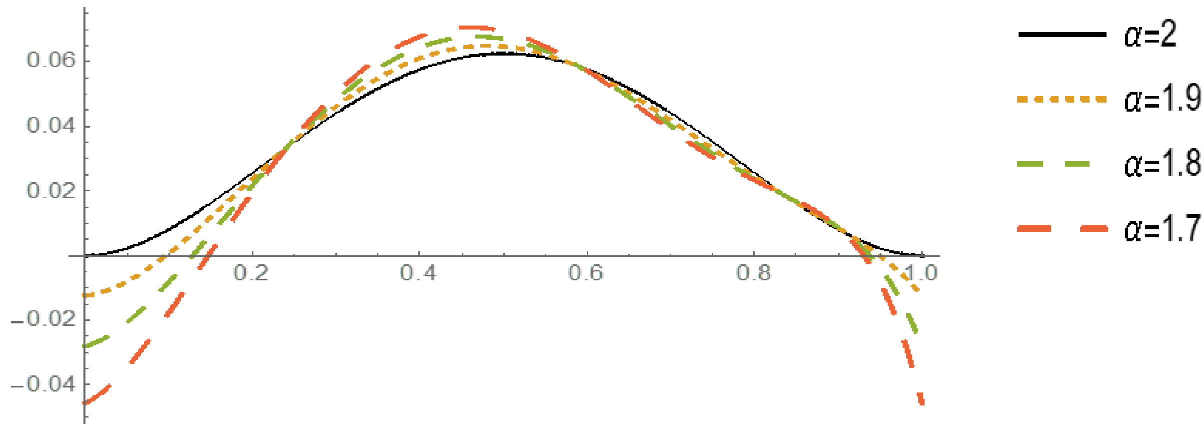

Figure 13. The behavior approximate solution of Example 4.3 at various values of α for $N = 45, \alpha = \beta = \frac{1}{2}, L = 1$, $\mu = 2$.

Table 7. The absolute and the relative errors between the approximate solution $u_1(x)$ and exact solution of Example 4.3 for $N = 45, \alpha = \beta = \frac{1}{2}, L = 1,\ \mu = 2$.

x	Exact solution	Approximate solution $u_1(x)$	Absolute error	Exact solution
0.0	0 .000	0 .000	$1.199481702 \times 10^{-11}$	0 .000
0.1	0.0081	0.0081	$1.19948166 \times 10^{-11}$	0.0081
0.2	0.0256	0.0256	$1.199482874 \times 10^{-11}$	0.0256
0.3	0.0441	0.0441	$1.19948218 \times 10^{-11}$	0.0441
0.4	0.0567	0.0567	$1.199414179 \times 10^{-11}$	0.0567
0.5	0.0625	0.0625	$1.199212951 \times 10^{-11}$	0.0625
0.6	0.0567	0.0567	$1.191101384 \times 10^{-11}$	0.0567
0.7	0.0441	0.0441	$9.47653761 \times 10^{-12}$	0.0441
0.8	0.0256	0.0256	$1.11284662 \times 10^{-11}$	0.0256
0.9	0.0081	0.0081	$8.370777682 \times 10^{-11}$	0.0081
1.0	0 .000	0 .000	$5.052967948 \times 10^{-11}$	0 .000

Table 8. The absolute and the relative errors between the approximate solution $u_2(x)$ and exact solution of Example 4.2 for $N = 45, \alpha = \beta = \frac{1}{2}, L = 1,\ \mu = 2$.

x	Exact solution	Approximate solution $u_2(x)$	Absolute error	Exact solution
0.0	0.000	0.000	$1.175575442 \times 10^{-25}$	0.000
0.1	0.072	0.072	0.00000000	0.072
0.2	0.096	0.096	$1.387778781 \times 10^{-17}$	0.096
0.3	0.084	0.084	0.00000000	0.084
0.4	0.048	0.048	$5.551115123 \times 10^{-17}$	0.048
0.5	0.000	0.000	$8.256520146 \times 10^{-31}$	0.000
0.6	-0.048	-0.048	0.00000000	-0.048
0.7	-0.084	-0.084	$5.551115123 \times 10^{-17}$	-0.084
0.8	-0.096	-0.096	$9.714451465 \times 10^{-17}$	-0.096
0.9	-0.072	-0.072	$5.551115123 \times 10^{-17}$	-0.072
1.0	0.000	0.000	$1.175063538 \times 10^{-25}$	0.000

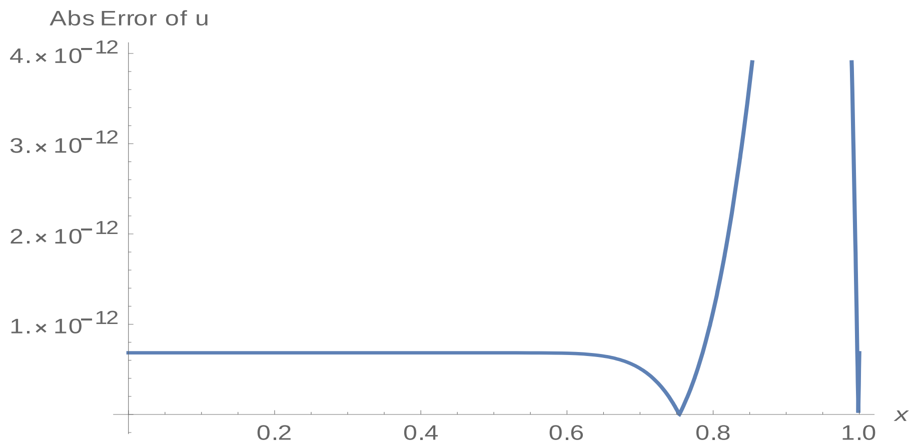

Figure 14. Absolute error of Example 4.3 for $N = 45, \alpha = \beta = \frac{1}{2}$.

In a similar manner, with the use of Theorem 3.1 and with $N = 45, \alpha = \beta = 1, L = 1, \mu = 2$, we obtain the approximate solution of Example 4.3 at various values of α that can be shown in **Figure 15**, whereas the corresponding errors can be shown in **Table 9**, **Table 10**, and **Figure 16**.

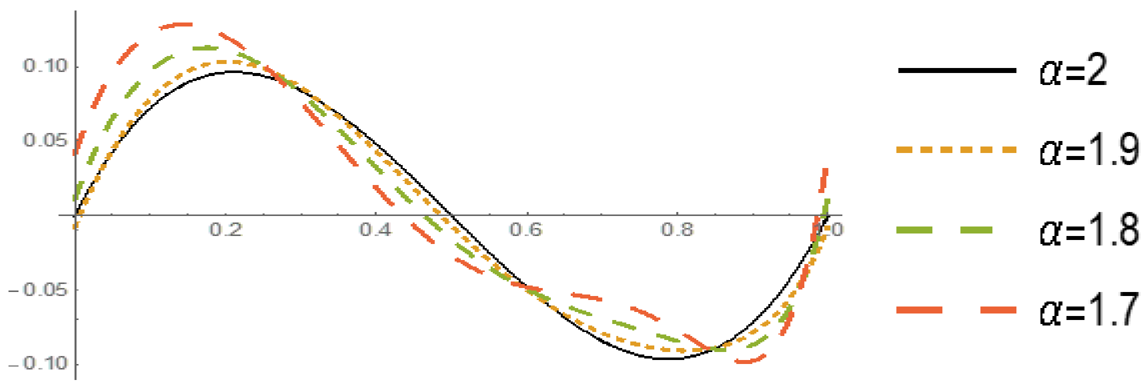

Figure 15. The behavior approximate solutions of Example 4.3 at various values of α for $N = 45, \alpha = \beta = 1, L = 1, \mu = 2$.

Table 9. The absolute and the relative errors between the approximate solution $u_1(x)$ and exact solution of Example 4.3 for $N = 45, \alpha = \beta = 1, L = 1, \mu = 2$.

x	Exact solution	Approximate solution $u_1(x)$	Absolute error
0.0	0 .000	0 .000	$6.828778928 \times 10^{-13}$
0.1	0.0081	0.0081	$6.828704269 \times 10^{-13}$
0.2	0.0256	0.0256	$6.828669574 \times 10^{-13}$
0.3	0.0441	0.0441	$6.828773658 \times 10^{-13}$
0.4	0.0567	0.0567	$6.828287935 \times 10^{-13}$
0.5	0.0625	0.0625	$6.827455268 \times 10^{-13}$
0.6	0.0567	0.0567	$6.778952399 \times 10^{-13}$
0.7	0.0441	0.0441	$5.096409406 \times 10^{-13}$
0.8	0.0256	0.0256	$1.139185968 \times 10^{-12}$
0.9	0.0081	0.0081	$7.589829806 \times 10^{-12}$
1.0	0 .000	0 .000	$6.828729616 \times 10^{-13}$

Table 10. The absolute and the relative errors between the approximate solution $u_2(x)$ and exact solution of Example 4.3 for $N = 45, \alpha = \beta = 1,\ L = 1,\ \mu = 2$.

x	Exact solution	Approximate solution $u_2(x)$	Absolute error
0.0	0 .000	0 .000	$6.538352438 \times 10^{-26}$
0.1	0.072	0.072	$5.551115123 \times 10^{-17}$
0.2	0.096	0.096	0.0000000000000
0.3	0.084	0.084	0.0000000000000
0.4	0.048	0.048	$5.551115123 \times 10^{-17}$
0.5	0 .000	0 .000	$9.176748161 \times 10^{-28}$
0.6	-0.048	-0.048	$6.938893904 \times 10^{-18}$
0.7	-0.084	-0.084	$5.551115123 \times 10^{-17}$
0.8	-0.096	-0.096	$9.714451465 \times 10^{-17}$
0.9	-0.072	-0.072	0.0000000000000
1.0	0 .000	0 .000	$6.538352438 \times 10^{-26}$

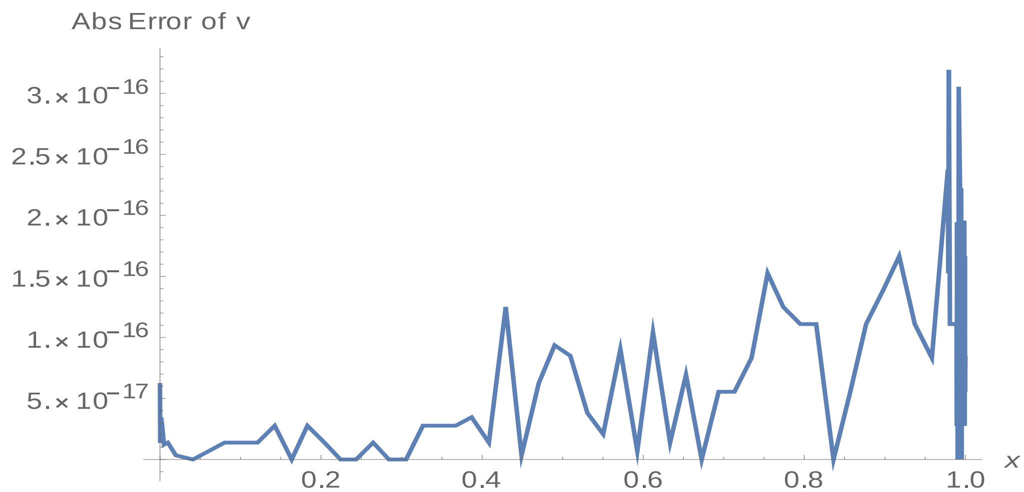

Figure 16. Absolute error of Example 4.3 for $N = 45, \alpha = \beta = 1$.

Based on the previous numerical results, one might observe that the shifted Jacobi approach with conformable fractional derivative can provide high accurate approximate solutions to the conventional periodic boundary value problems. Furthermore, it is worth mentioning that such an approach can generate several numerical solutions in accordance with various fractional-order values. However, it is important to acknowledge that the suggested approach is strong in handling boundary conditions and providing efficient spectral approximations for differential equations. However, it can be computationally intensive and complex to implement compared to simpler polynomial bases as well as it may converge slowly, particularly for large systems or matrices with poorly scaled elements.

5. Discussion and Conclusion

The shifted Jacobi approach has been presented in this study based on the Jacobi polynomials and the conformable fractional derivative. A certain theoretical result has been proved correspondingly to determine a bound of the error arising from the suggested approximate solution. Several numerical discussions have been provided to the efficiency and effectiveness of the shifted Jacobi approach with conformable fractional derivative. One may note from the prior numerical results that the conventional periodic boundary value problems can have highly accurate approximation solutions solved by using the shifted Jacobi technique with conformable fractional derivative. It's also important to note that this method can produce multiple numerical solutions according to different fractional-order values. Finally, it should be mentioned, regarding the shifted Jacobi method's computational intensity and complexity, that they can be addressed by optimizing algorithms for polynomial evaluations, leveraging fast transform techniques, and utilizing parallel computing.

6. Future Work

Research on conformable periodic fractional boundary value problems using the shifted Jacobi method may go forward by investigating the applicability of the method to a larger spectrum of complicated, real-world systems and creating more effective algorithms for higher-dimensional problems. In addition, the computational intensity and complexity shifted Jacobi method for conformable fractional periodic boundary value problems should be the subject of future research, especially for systems that are more complicated and higher dimensional.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgments

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The authors would like to thank the editor and anonymous reviewers for their comments that help improve the quality of this work.

References

Abdelhalim, N., Ouannas, A., Rezzoug, I., & Batiha, I.M. (2023). A study of a high-order time-fractional partial differential equation with purely integral boundary conditions. *Fractional Differential Calculus, 13*(2), 199-210. https://doi.org/10.7153/fdc-2023-13-13.

Al-Nana, A.A., Batiha, I.M., & Momani, S. (2023). A numerical approach for dealing with fractional boundary value problems. *Mathematics, 11*(19), 4082. https://doi.org/10.3390/math11194082.

Anakira, N.R., Ababneh, O.Y., Heilat, A.S., & Batiha, I.M. (2022). A new accurate approximate solution of singular two-point boundary value problems. *General Letters in Mathematics, 12*(1), 31-39. https://doi.org/10.31559/glm2022.12.1.4.

Bhrawy, A.H., Tharwat, M.M., & Alghamdi, M.A. (2014). A new operational matrix of fractional integration for shifted Jacobi polynomials. http://math.usm.my/bulletin/pdf/acceptedpapers/2012-05-057-r1.pdf.

Bojdi, Z.K., Ahmadi-Asl, S., & Aminataei, A. (2013). The general shifted Jacobi matrix method for solving the general high order linear differential- difference equations with variable coefficients. *Journal of Mathematical Research and Applications, 1*(1), 10-23. http://www.academicpub.org/downloadpaper.aspx?paperid=14506.

Bouchenak, A., Batiha, I.M., Aljazzazi, M., Jebril, I.H., Al-Horani, M., & Khalil, R. (2024). Atomic exact solution for some fractional partial differential equations in Banach spaces. *Partial Differential Equations in Applied Mathematics, 9*, 100626. https://doi.org/10.1016/j.padiff.2024.100626.

Doha, E.H., Bhrawy, A.H., & Ezz-Eldien, S.S. (2012). A new Jacobi operational matrix: an application for solving fractional differential equations. *Applied Mathematical Modelling, 36*(10), 4931-4943. https://doi.org/10.1016/j.apm.2011.12.031.

Doha, E.H., Bhrawy, A.H., Baleanu, D., Ezz-Eldien, S.S., & Hafez, R.M. (2015). An efficient numerical scheme based on the shifted orthonormal Jacobi polynomials for solving fractional optimal control problems. *Advances in Difference Equations, 2015*(15), 1-17. https://doi.org/10.1186/s13662-014-0344-z.

Hammad, M.A., Alrowaily, A.W., Shah, R., Ismaeel, S.M.E., & El-Tantawy, S.A. (2023). Analytical analysis of fractional nonlinear Jaulent-Miodek system with energy-dependent Schrödinger potential. *Frontiers in Physics, 11*, 1-10. https://doi.org/10.3389/fphy.2023.1148306.

Khalil, R., Horani, M.A., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics, 264*, 65-70. https://doi.org/10.1016/j.cam.2014.01.002.

Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations*. Elsevier, Netherland.

Kitzhofer, G., Koch, O., Lima, P., & Weinmüller, E. (2007). Efficient numerical solution of the density profile equation in hydrodynamics. *Journal of Scientific Computing*, *32*(3), 411-424. https://doi.org/10.1007/s10915-007-9141-0.

Lakshmikantham, V., Leela, S., & Vasundhara Devi, J. (2009). *Theory of fractional dynamic systems*. Cambridge Academic Publishers: Cambridge, UK. http://ci.nii.ac.jp/ncid/ba90497599.

Miller, K.S., & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. John Wiley & Sons, New York/Singapore. http://ci.nii.ac.jp/ncid/ba20696637.

Podlubny, I. (1999). *Fractional differential equations*. Academic Press, San Diego. https://openlibrary.org/books/ol7328592m/fractional_differential_equations_(mathematics_in_science_and_engineering).

Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional integrals and derivatives: theory and applications*. Gordon and Breach Science Publishers. Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional integrals and derivatives: theory and applications*. Gordon and Breach Science Publishers, Yverdon.