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A Characterization of Generalized Boolean Functions Employed in CDMA Communications

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Abstract
In design of secure cryptosystems and CDMA communications, the negabent functions play a significant role. The generalized Boolean functions have been extensively studied by Schmidt and established several important results in this setup. In this paper, several characteristics of the generalized nega-Hadamard transform (GNT) of generalized Boolean functions like inverse of GNT, generalized nega-cross correlation, generalized nega-Parseval's identity, relationship between GNT and generalized nega-cross correlation have analyzed. We studied the GNT for the derivative of this setup of functions and established the connection of generalized Walsh-Hadamard transform and GNT of derivatives of these functions. Also, the GNT of composition of vectorial Boolean function and generalized Boolean function is presented. Further, the generalized nega-convolution theorem for generalized Boolean function is obtained.

Keywords- Generalized nega-Hadamard transform (GNT), Generalized negabent function, Generalized nega-cross correlation, Generalized boolean function.

1. Introduction

Boolean functions have received dan immense consideration in the area of cryptography. The Boolean functions which have maximum distance from the set of all affine functions are termed as Boolean bent functions (equivalently, if it has flat spectrum of Walsh-Hadamard transform spectrum). These functions were instigated by Rothaus (1976) and are crucial components for various applications in symmetric cryptography and in construction of error correcting codes. The concept of bent functions is further generalized by Riera and Parker (2006). They analyzed Boolean functions having flat spectrum against one or more transforms obtained from n-fold tensor product of the matrix$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H =$

$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $N=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}$. The Walsh-Hadamard transform (WT) is the tensor product of several Hadamard matrices H, equivalently, the tensor product of several nega-Hadamard matrices N is known as the nega-Hadamard transform (NT). As in the WT, a Boolean function that have flat spectrum under NT is called negabent function. In the last few years, many researchers have proposed the several generalizations for Boolean functions and studied the effect of WT on them. Kumar et al. (1985) presented a q-ary functions as a generalization of Boolean functions defined from $\mathbb{Z}_q^n$ to $\mathbb{Z}_q$ with a positive integer $q \geq 2$. They provided the constructions of$q$-ary bent functions for even n and $q \neq 2 \pmod 4$. Also, Schmidt (2009) presented the generalization of Boolean functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$ and established several important results in this setup. We call these functions as generalized Boolean functions. Later, Solé and Tokareva (2009) has considered these generlizations and obtained a relationship between the bent functions, q-ary (for $q = 4$) bent and the generalized bent functions. Further, Stănică et al. (2013) has obtained the characteristics of generalized bent functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$, and characterized the generalized bent functions on $\mathbb{Z}_2^n$ with values in $\mathbb{Z}_8$ and $\mathbb{Z}_4$. Also, Chaturvedi and Gangopadhyay (2013) considered the generalization of Schmidt and analyzed some characteristics of generalized nega-Hadamard transform (GNT) in his generalized setup. Kaur and Sharma (2018) characterized and constructed the generalized negabent functions with values in $\mathbb{Z}_8$ and $\mathbb{Z}_{16}$. Paul (2018) has raised a problem in his thesis for the general construction of generalized (q-ary) negabent functions from $\mathbb{Z}_q^n$ to $\mathbb{Z}_q$. Recently, Sharma and Tiwari (2022) solved this problem partially and obtained the various properties of these functions in respect of NT. They also obtained the relationship of generalized nega-autocorrelation and GNT. Çeşmelioğlu and Meidl (2023) considered the generalized q-ary functions from $\mathbb{Z}_q^n$ into the cyclic group $\mathbb{Z}_{q^k}$ and studied the equivalence of these generalized Boolean functions. Su and Guo (2023) presented a further study in accordance with Rothaus's bent function for the construction approach for bent and self-dual bent functions. Carlet and Villa (2023) studied the general class of those Boolean functions having property that for all non zero $\boldsymbol{a} \in \mathbb{Z}_2^n$, the derivative of $f, D_{\boldsymbol{a}}f(\boldsymbol{s}) = f(\boldsymbol{s}) + f(\boldsymbol{s}+\boldsymbol{a})$ admits that atleast one derivative $D_{\boldsymbol{b}}D_{\boldsymbol{a}}f(\boldsymbol{s}) = f(\boldsymbol{s}) + f(\boldsymbol{s}+\boldsymbol{a}) + f(\boldsymbol{s}+\boldsymbol{b}) + f(\boldsymbol{s}+\boldsymbol{a}+\boldsymbol{b})$ is equal to constant 1. These functions are called cubic-like bent. They also provided the characterization and construction and showed the existence of cubic-like bent Boolean functions of any algebraic degree between 2 and $\frac{n}{2}$. These generalizations of Boolean functions given by the various researchers are helpful in the construction of some efficient cryptographic algorithms for smooth digital communication. For Boolean functions in cryptography and error correcting codes, we may refer to MacWilliams (1977), Carlet (2010), Singh et al. (2013), Zhuo and Chong (2015), Gangopadhyay et al. (2019), Singh and Paul (2019), Mandal and Gangopadhyay (2021), Mandal et al. (2022), Singh et al. (2022), Tiwari and Sharma (2023). In the present article, by considering the Schmidt's generalization, we discuss some more results of generalized Boolean functions in respect of GNT. Especially, we tried to find out the solution of some basic questions like whether nega-Parseval identity holds in the current setup, whether nega-convolution theorem is true in the present setup, whether the inverse of GNT exists in the current setup, if so, what will be its form. Fortunately, we could provide answer to all these questions in subsequent sections of the manuscript Apart from this; we have provided GNT for the composition of a vectorial and a generalized Boolean function in the current setup. Also, we obtained the GNT for the derivatives of generalized Boolean functions for the current setup.

Let n and $q \geq 2$ be the positiveinteger. Let $\mathbb{Z}_q$ be the ring of integers modulo q. Suppose $\mathbb{Z}_2$ and $\mathbb{Z}_2^n$ are the prime field of order 2 and vector space of dimension n respectively. A Boolean function on n-variables is a function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2$ and the set of all these functions is denoted by $\mathcal{B}_n$. Let $\oplus$ denotes the addition over $\mathbb{Z}_2^n$. If $\boldsymbol{r} = (r_1, r_2, \dots, r_n)$ and $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$ are two elements in $\mathbb{Z}_2^n$, then the inner (scalar) product $<\boldsymbol{r}, \boldsymbol{s}>$ and the intersection $\boldsymbol{r} * \boldsymbol{s}$ is given by,

$$< \boldsymbol{r}, \boldsymbol{s} >= r_1 s_1 \oplus r_2 s_2 \oplus \cdots \oplus r_n s_n \text{ and } \boldsymbol{r} * \boldsymbol{s} = (r_1 s_1, r_2 s_2, \dots, r_n s_n).$$

Let $z = \alpha + \beta i$ be a complex number. Then, the absolute value of z is defined as $|z| = \sqrt{\alpha^2 + \beta^2}$ and the conjugate of z is given as $\overline{z} = \alpha - \beta i$, where $\alpha, \beta \in \mathbb{R}$. The Algebraic normal form (ANF) of any $f \in \mathcal{B}_n$ is written as,

$$f(r_1, r_2, \dots, r_n) = \bigoplus_{\boldsymbol{a}=(a_1, a_2, \dots, a_n) \in \mathbb{Z}_2^n} \mu_{\boldsymbol{a}} \left(\prod_{i=1}^{n} r_i^{a_i} \right),$$

where, $\mu_{\boldsymbol{a}} \in \mathbb{Z}_2$. We define the algebraic degree of $f \in \mathcal{B}_n$ as $\deg(f) \coloneqq \max_{\boldsymbol{a} \in \mathbb{Z}_2^n} \{Wt(\boldsymbol{a}): \mu_{\boldsymbol{a}} \neq 0\}$. Boolean functions whose algebraic degree less than or equal to one is called an affine Boolean function. Suppose $\mathcal{A}_n$ denotes the set of all affine Boolean functions on n-variables. The Hamming weight of an element $\boldsymbol{c} = (c_1, c_2, \dots, c_n) \in \mathbb{Z}_2^n$ is $Wt(\boldsymbol{c}) \coloneqq \sum_{i=1}^{n} c_i$. The Hamming distance between the functions $f, h \in \mathcal{B}_n$ is given by $d(f, h) = |\{\boldsymbol{c}: f(\boldsymbol{c}) \neq h(\boldsymbol{c}),\ \boldsymbol{c} \in \mathbb{Z}_2^n\}|$. The least Hamming distance of $f \in \mathcal{B}_n$ and all affine functions is known as nonlinearity of a function f. The WT of $f \in \mathcal{B}_n$ at any point $\boldsymbol{u} \in \mathbb{Z}_2^n$ is given by,

$$\mathcal{T}_f(\boldsymbol{u}) = \sum_{\boldsymbol{c} \in \mathbb{Z}_2^n} (-1)^{f(\boldsymbol{c}) \oplus <\boldsymbol{u}, \boldsymbol{c}>}.$$

A function $f \in \mathcal{B}_n$ is called a bent if and only if $\left|\mathcal{T}_f(\boldsymbol{u})\right| = 1$ for every $\boldsymbol{u} \in \mathbb{Z}_2^n$. Any Boolean function $f \in \mathcal{B}_n$ is termed as balanced Boolean function if its output in truth table has equal number of ones and zeros. A function $f \in \mathcal{B}_n$ is said to be semi-bent if and only if $\mathcal{T}_f(\boldsymbol{u}) \in \{0, 2^n, -2^n\}$. Any function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$, witha positive integer $q \geq 2$ is said to be a generalized Boolean function. Let $\mathcal{B}_n^q$ denotes the set of all these generalized functions. Suppose the qth-primitive root of unity is $\zeta = e^{2\pi i/q}$. The generalized Walsh-Hadamard transform (GWT) of $f \in \mathcal{B}_n^q$ at any vector $\boldsymbol{u} \in \mathbb{Z}_2^n$ is given as,

$$\mathcal{T}_f^q(\boldsymbol{u}) = 2^{-\frac{n}{2}} \sum_{c \in \mathbb{Z}_2^n} \zeta^{f(\boldsymbol{c})} (-1)^{<\boldsymbol{u}, \boldsymbol{c}>}.$$

A function $f \in \mathcal{B}_n^q$ is known as generalized bent if and only if $|\mathcal{T}_f(\boldsymbol{u})| = 1$ for every $\boldsymbol{u} \in \mathbb{Z}_2^n$. The NT of $f \in \mathcal{B}_n$ at any $\boldsymbol{u} \in \mathbb{Z}_2^n$ is the complex valued function defined as,

$$\mathcal{N}_f(\boldsymbol{u}) = 2^{-\frac{n}{2}} \sum_{\boldsymbol{c} \in \mathbb{Z}_2^n} (-1)^{f(\boldsymbol{c}) \oplus <\boldsymbol{u}, \boldsymbol{c}>} i^{Wt(\boldsymbol{c})}.$$

A function $f \in \mathcal{B}_n$ is known as a negabent if and only if $\left|\mathcal{N}_f(\boldsymbol{u})\right| = 1$ for every $\boldsymbol{u} \in \mathbb{Z}_2^n$. The GNT of $f \in \mathcal{B}_n^q$ at $\boldsymbol{u} \in \mathbb{Z}_2^n$ is given by,

$$\mathcal{N}_f^q(\boldsymbol{u}) = 2^{-\frac{n}{2}} \sum_{\boldsymbol{c} \in \mathbb{Z}_2^n} \zeta^{f(\boldsymbol{c})} (-1)^{<\boldsymbol{u}, \boldsymbol{c}>} i^{Wt(\boldsymbol{c})}.$$

Any function $f \in \mathcal{B}_n^q$ is known as a generalized negabent if and only if $\left|\mathcal{N}_f^q(\boldsymbol{u})\right| = 1$ for every $\boldsymbol{u} \in \mathbb{Z}_2^n$. The nega-cross correlation of $g, h \in \mathcal{B}_n$ at $\boldsymbol{u} \in \mathbb{Z}_2^n$ is defined as

$$R_{g,h}(\boldsymbol{u}) = \sum_{\boldsymbol{s} \in \mathbb{Z}_2^n} (-1)^{g(\boldsymbol{s}) \oplus h(\boldsymbol{s} \oplus \boldsymbol{u})} (-1)^{<\boldsymbol{u}, \boldsymbol{s}>}.$$

If $g = h$, then the nega auto correlation of $h \in \mathcal{B}_n$ at $\boldsymbol{u} \in \mathbb{Z}_2^n$ is defined as,

$$R_h(\boldsymbol{u}) = \sum_{\boldsymbol{s}\in\mathbb{Z}_2^n} (-1)^{h(\boldsymbol{s})\oplus h(\boldsymbol{s}\oplus\boldsymbol{u})}(-1)^{<\boldsymbol{u},\boldsymbol{s}>}.$$

The generalized nega-cross correlation of $g, h \in \mathcal{B}_n^q$ at $\boldsymbol{u} \in \mathbb{Z}_2^n$ is given by,

$$C_{g,h}(\boldsymbol{u}) = \sum_{\boldsymbol{s}\in\mathbb{Z}_2^n}(-1)^{g(\boldsymbol{s})-h(\boldsymbol{s}\oplus\boldsymbol{u})}(-1)^{<\boldsymbol{u},\boldsymbol{s}>}.$$

and for $g = h, C_{g,h}(\boldsymbol{u}) = C_h(\boldsymbol{u})$ is called generalized auto-cross correlation of $h \in \mathcal{B}_n^q$ at $\boldsymbol{u} \in \mathbb{Z}_2^n$.

We shall use the following well-known identity given by MacWilliams and Sloane (1977),

$$Wt(\boldsymbol{r} \oplus \boldsymbol{s}) = Wt(\boldsymbol{r}) \oplus Wt(\boldsymbol{s}) - 2Wt(\boldsymbol{r} * \boldsymbol{s}) \tag{1}$$

We recall the following preliminary results.

Lemma 1. [Cusik and Stănică (2009)] Suppose $\boldsymbol{d} \in \mathbb{Z}_2^n$. Then, we have

$$\sum_{\boldsymbol{r}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{d},\boldsymbol{r}>} = \begin{cases} 2^n, & \text{if} \boldsymbol{d} = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2. [Schmidt et al. (2008)] For any $\boldsymbol{c} \in \mathbb{Z}_2^n$, we have

$$\sum_{\boldsymbol{r}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{c}\oplus\sigma(\boldsymbol{s}),\boldsymbol{r}>} i^{Wt(\boldsymbol{r})} = 2^{-\frac{n}{2}}\omega^n i^{-Wt(\sigma(\boldsymbol{s})\oplus\boldsymbol{c})}.$$

Lemma 3. [Stănică et al. (2012)] Let $f \in \mathcal{B}_n^q$.The inverse of the GWT is given as

$$\zeta^{f(\boldsymbol{s})} = 2^{-\frac{n}{2}} \sum_{\boldsymbol{c}\in\mathbb{Z}_2^n} \mathcal{T}_f^q(\boldsymbol{c})(-1)^{<\boldsymbol{c},\boldsymbol{s}>}.$$

The article is organized as follows: The results and discussions in detail for the article are presented in section 2. This section is further organised in three subsections in which subsection 2.1 presents some properties of GNT of generalized Boolean functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$ like the inverse of GNT, generalized nega-cross correlation, generalized nega-Parseval's identity, relationship between GNT and generalized nega-cross correlation. Subsection 2.2 presents some results of the GNT of the derivative of the generalized Boolean functions. In subsection 2.3, the composition theorem of vectorial functions and generalized Boolean functions is obtained. Further, the generalized nega convolution theorem for generalized Boolean functions is obtained. Section 3 presents conclusion and future scope.

2. Results and Discussions

This section contains all the original results derived in this article. In subsection 2.1, we obtain some properties of GNT like the inverse of GNT, generalized nega-cross correlation, generalized nega-Parseval's identity, relationship between GNT and generalized nega-cross correlation of generalized Boolean functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$. Subsection 2.2 presents some results of the GNT of the derivative of the generalized Boolean functions. In subsection 2.3, the composition theorem of a vectorial and generalized Boolean functions is obtained. Further, the generalized nega convolution theorem for generalized Boolean functions is obtained.

2.1 Properties of GNT

In this subsection, we investigate some properties of the GNT in connection with generalized nega-cross correlation of generalized Boolean functions. These properties are helpful to solve various further results for GNT of these functions.

Theorem 1. Suppose $f \in \mathcal{B}_n^q$. Then, the inverse of GNT is given by,

$$\zeta^{f(\boldsymbol{r})} = 2^{-\frac{n}{2}} \sum_{\boldsymbol{\lambda} \in \mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{\lambda})\, (-1)^{<\boldsymbol{\lambda}, \boldsymbol{r}>}\, i^{-Wt(\boldsymbol{r})}.$$

Proof. Taking

$$2^{-\frac{n}{2}} \sum_{\boldsymbol{\lambda} \in \mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{\lambda})(-1)^{<\boldsymbol{\lambda}, \boldsymbol{r}>}\, i^{-Wt(\boldsymbol{r})} = 2^{-\frac{n}{2}} \sum_{\boldsymbol{\lambda} \in \mathbb{Z}_2^n} 2^{-\frac{n}{2}} \sum_{\boldsymbol{\gamma} \in \mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\gamma})}\, (-1)^{<\boldsymbol{\lambda}, \boldsymbol{\gamma}>} i^{Wt(\boldsymbol{\gamma})}\, (-1)^{<\boldsymbol{\lambda}, \boldsymbol{r}>} i^{-Wt(\boldsymbol{r})}$$

$$= 2^{-n} \sum_{\boldsymbol{\gamma} \in \mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\gamma})} \sum_{\boldsymbol{\lambda} \in \mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda}, \boldsymbol{\gamma} \oplus \boldsymbol{r}>} i^{Wt(\boldsymbol{\gamma})}\, i^{-Wt(\boldsymbol{r})}.$$

Using Lemma 1, we get

$$\sum_{\boldsymbol{\lambda} \in \mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda}, \boldsymbol{\gamma} \oplus \boldsymbol{r}>} = \begin{cases} 2^n, & \text{if} \boldsymbol{\gamma} \oplus \boldsymbol{r} = 0 \\ 0, & \text{otherwise.} \end{cases}$$

If $\boldsymbol{\gamma} \oplus \boldsymbol{r} = 0$, then $\boldsymbol{\gamma} = \boldsymbol{r}$ in $\mathbb{Z}_2^n$. So, we have,

$$2^{-\frac{n}{2}} \sum_{\boldsymbol{\lambda} \in \mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{\lambda})(-1)^{<\boldsymbol{\lambda}, \boldsymbol{r}>}\, i^{-Wt(\boldsymbol{r})} = 2^{-n} \zeta^{f(\boldsymbol{r})} 2^n i^{Wt(\boldsymbol{r})} i^{-Wt(\boldsymbol{r})}$$

$$= \zeta^{f(\boldsymbol{r})}.$$

Theorem 2. If $f, g \in \mathcal{B}_n^q$, then the generalized nega-cross correlation is

$$C_{f,g}(\boldsymbol{t}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_2^n} (-1)^{f(\boldsymbol{\alpha}) - g(\boldsymbol{\alpha} \oplus \boldsymbol{t})} (-1)^{<\boldsymbol{\alpha}, \boldsymbol{t}>} = i^{Wt(\boldsymbol{t})} \sum_{\boldsymbol{c} \in \mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{c}) \overline{\mathcal{N}_g^q(\boldsymbol{c})} (-1)^{<\boldsymbol{c}, \boldsymbol{t}>}.$$

Proof. Taking

$$i^{Wt(\boldsymbol{t})} \sum_{\boldsymbol{c} \in \mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{c}) \overline{\mathcal{N}_g^q(\boldsymbol{c})} (-1)^{<\boldsymbol{c}, \boldsymbol{t}>} = i^{Wt(\boldsymbol{t})} \sum_{\boldsymbol{c} \in \mathbb{Z}_2^n} (-1)^{<\boldsymbol{c}, \boldsymbol{t}>}\, 2^{-\frac{n}{2}} \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha})}\, (-1)^{<\boldsymbol{c}, \boldsymbol{\alpha}>} i^{Wt(\boldsymbol{\alpha})}$$

$$\times\ 2^{-\frac{n}{2}} \sum_{\boldsymbol{y} \in \mathbb{Z}_2^n} \zeta^{-g(\boldsymbol{y})}\, (-1)^{<\boldsymbol{c}, \boldsymbol{y}>} i^{-Wt(\boldsymbol{y})}$$

$$= 2^{-n} \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_2^n} \sum_{\boldsymbol{y} \in \mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha}) - g(\boldsymbol{y})}\, i^{Wt(\boldsymbol{\alpha}) - Wt(\boldsymbol{y}) + Wt(\boldsymbol{t})} \sum_{\boldsymbol{c} \in \mathbb{Z}_2^n} (-1)^{<\boldsymbol{c}, \boldsymbol{\alpha} \oplus \boldsymbol{y} \oplus \boldsymbol{t}>}.$$

By using Lemma 1, we obtain,

$$\sum_{\boldsymbol{c}\in\mathbb{Z}_2^n}(-1)^{<\boldsymbol{c},\boldsymbol{\alpha}\oplus\boldsymbol{y}\oplus\boldsymbol{t}>}=\begin{cases}2^n, & \text{if}\boldsymbol{\alpha}\oplus\boldsymbol{y}\oplus\boldsymbol{t}=0\\ 0, & \text{otherwise.}\end{cases}$$

If $\boldsymbol{\alpha}\oplus\boldsymbol{y}\oplus\boldsymbol{t}=0$, then $\boldsymbol{y}=\boldsymbol{\alpha}\oplus\boldsymbol{t}\text{in}\mathbb{Z}_2^n$. Thus, we have,

$$i^{Wt(\boldsymbol{t})}\sum_{\boldsymbol{c}\in\mathbb{Z}_2^n}\mathcal{N}_f^q(\boldsymbol{c})\overline{\mathcal{N}_g^q(\boldsymbol{c})}(-1)^{<\boldsymbol{c},\boldsymbol{t}>}=2^{-n}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}\zeta^{f(\boldsymbol{\alpha})-g(\boldsymbol{\alpha}\oplus\boldsymbol{t})}\left(i^{Wt(\boldsymbol{\alpha})-Wt(\boldsymbol{\alpha}\oplus\boldsymbol{t})+Wt(\boldsymbol{t})}\right)2^n$$

$$=\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}\zeta^{f(\boldsymbol{\alpha})-g(\boldsymbol{\alpha}\oplus\boldsymbol{t})}\,i^{Wt(\boldsymbol{\alpha})-\left(Wt(\boldsymbol{\alpha})\oplus Wt(\boldsymbol{t})-2Wt(\boldsymbol{\alpha}*\boldsymbol{t})\right)+Wt(\boldsymbol{t})}$$

$$=\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}\zeta^{f(\boldsymbol{\alpha})-g(\boldsymbol{\alpha}\oplus\boldsymbol{t})}\,i^{2Wt(\boldsymbol{\alpha}*\boldsymbol{t})}$$

$$=\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}\zeta^{f(\boldsymbol{\alpha})-g(\boldsymbol{\alpha}\oplus\boldsymbol{t})}\,(-1)^{<\boldsymbol{\alpha},\boldsymbol{t}>}$$

$$=C_{f,g}(\boldsymbol{t}).$$

If we consider $f=g$ in the previous theorem, then we get

$$C_{f,f}(\boldsymbol{t})=C_f(\boldsymbol{t})=\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}\zeta^{f(\boldsymbol{\alpha})-f(\boldsymbol{\alpha}\oplus\boldsymbol{t})}\,(-1)^{<\boldsymbol{\alpha},\boldsymbol{t}>}$$

$$=i^{Wt(\boldsymbol{t})}\sum_{\boldsymbol{c}\in\mathbb{Z}_2^n}\mathcal{N}_f^q(\boldsymbol{c})\overline{\mathcal{N}_g^q(\boldsymbol{c})}(-1)^{<\boldsymbol{c},\boldsymbol{t}>}$$

$$=i^{Wt(\boldsymbol{t})}\sum\nolimits_{\boldsymbol{c}\in\mathbb{Z}_2^n}\left|\mathcal{N}_f^q(\boldsymbol{c})\right|^2(-1)^{<\boldsymbol{c},\boldsymbol{t}>}\tag{2}$$

This is the generalized nega-autocorrelation of generalized Boolean functions. Since both generalized Walsh-Hadamard and GNT are preserving energy. Thus, Parseval's theorem for both generalized transformations holds. We may refer to Stănică et al. (2012) for generalized Parseval's identity. Substituting $t = 0$ in (2), we obtain the following generalized nega-Parseval's identity.

Corollary 1. The generalized nega-Parseval's identity is

$$\sum_{\boldsymbol{c}\in\mathbb{Z}_2^n}\left|\mathcal{N}_f^q(\boldsymbol{c})\right|^2=2^n.$$

Lemma 4. A function $f\in\mathcal{B}_n^q$ is generalized negabent if and only if $C_f(\boldsymbol{t})=0$ for every $\boldsymbol{t}\in\mathbb{Z}_2^n\backslash\{0\}$.

Proof. First, suppose that a function $f\in\mathcal{B}_n^q$ is generalized negabent, then $\left|\mathcal{N}_f^q(\boldsymbol{c})\right|=1$ for every $\boldsymbol{c}\in\mathbb{Z}_2^n$. This implies that the Equation (2) is $C_f(\boldsymbol{t})=i^{Wt(\boldsymbol{t})}\sum_{\boldsymbol{c}\in\mathbb{Z}_2^n}(-1)^{<\boldsymbol{c},\boldsymbol{t}>}$. By using Lemma 1, for all $\boldsymbol{t}\neq 0$,we obtain $\sum_{\boldsymbol{c}\in\mathbb{Z}_2^n}(-1)^{<\boldsymbol{c},\boldsymbol{t}>}=0$ which implies $C_f(\boldsymbol{t})=0$. The converse part follows from the Equation (2).

Now, in the following result, we provide a relation of GNT and generalized cross-correlation in the current setup.

Lemma 5. Suppose $f \in \mathcal{B}_n^q$ and $\boldsymbol{\lambda} \in \mathbb{Z}_2^n$. Then, we have,

$$\left|\mathcal{N}_f^q(\boldsymbol{\lambda})\right|^2 = 2^{-n} \sum_{\boldsymbol{t}\in\mathbb{Z}_2^n} C_f(\boldsymbol{t})\,(-1)^{<\boldsymbol{\lambda},\boldsymbol{t}>} i^{-Wt(\boldsymbol{t})}.$$

Proof. We know that

$$\left|\mathcal{N}_f^q(\boldsymbol{\lambda})\right|^2 = \mathcal{N}_f^q(\boldsymbol{\lambda})\overline{\mathcal{N}_f^q(\boldsymbol{\lambda})}$$

$$= 2^{-\frac{n}{2}} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha})}\,(-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} i^{Wt(\boldsymbol{\alpha})} \left(2^{-\frac{n}{2}} \sum_{\boldsymbol{d}\in\mathbb{Z}_2^n} \zeta^{-f(\boldsymbol{d})}\,(-1)^{<\boldsymbol{\lambda},\boldsymbol{d}>} i^{-Wt(\boldsymbol{d})}\right)$$

$$= 2^{-n} \sum_{\boldsymbol{\alpha},\boldsymbol{d}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha})-f(\boldsymbol{d})}\, i^{Wt(\boldsymbol{\alpha})-Wt(\boldsymbol{d})} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>+<\boldsymbol{\lambda},\boldsymbol{d}>}$$

$$= 2^{-n} \sum_{\boldsymbol{\alpha},\boldsymbol{t}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha})-f(\boldsymbol{\alpha}\oplus\boldsymbol{t})}\, i^{Wt(\boldsymbol{\alpha})-Wt(\boldsymbol{\alpha}\oplus\boldsymbol{t})} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>+<\boldsymbol{\lambda},\boldsymbol{\alpha}\oplus\boldsymbol{t}>}$$

$$= 2^{-n} \sum_{\boldsymbol{\alpha},\boldsymbol{t}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha})-f(\boldsymbol{\alpha}\oplus\boldsymbol{t})}\, i^{Wt(\boldsymbol{t})+2Wt(\boldsymbol{\alpha}*\boldsymbol{t})} (-1)^{2<\boldsymbol{\lambda},\boldsymbol{\alpha}>} (-1)^{<\boldsymbol{\lambda},\boldsymbol{t}>} \qquad \text{(using (1))}$$

$$= 2^{-n} \sum_{\boldsymbol{\alpha},\boldsymbol{t}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha})-f(\boldsymbol{\alpha}\oplus\boldsymbol{t})} (-1)^{<\boldsymbol{\alpha},\boldsymbol{t}>}\, i^{-Wt(\boldsymbol{t})} (-1)^{<\boldsymbol{\lambda},\boldsymbol{t}>}$$

$$= 2^{-n} \sum_{\boldsymbol{t}\in\mathbb{Z}_2^n} C_f(\boldsymbol{t})\,(-1)^{<\boldsymbol{\lambda},\boldsymbol{t}>}\, i^{-Wt(\boldsymbol{t})}.$$

Theorem 3. Let $f, g \in \mathcal{B}_n^q$ and $\boldsymbol{\lambda}, \boldsymbol{\gamma} \in \mathbb{Z}_2^n$. Then,

$$2^n \sum_{\boldsymbol{\lambda}\in\mathbb{Z}_2^n} \left|\mathcal{N}_f^q(\boldsymbol{\lambda})\right|^2 \left|\mathcal{N}_g^q(\boldsymbol{\gamma}\oplus\boldsymbol{\lambda})\right|^2 = 2^{-n} \sum_{\boldsymbol{a}\in\mathbb{Z}_2^n} C_f(\boldsymbol{a})\,C_g(\boldsymbol{a})\, i^{-2Wt(\boldsymbol{a})} (-1)^{<\gamma,\boldsymbol{a}>}.$$

Proof. Using previous lemma, we have,

$$2^n \sum_{\boldsymbol{\lambda}\in\mathbb{Z}_2^n} \left|\mathcal{N}_f^q(\boldsymbol{\lambda})\right|^2 \left|\mathcal{N}_g^q(\boldsymbol{\gamma}\oplus\boldsymbol{\lambda})\right|^2$$

$$= 2^{-2n} \sum_{\boldsymbol{\lambda}\in\mathbb{Z}_2^n} \sum_{\boldsymbol{a}\in\mathbb{Z}_2^n} C_f(\boldsymbol{a})\, i^{-Wt(\boldsymbol{a})} (-1)^{<\boldsymbol{\lambda},\boldsymbol{a}>} \sum_{\boldsymbol{b}\in\mathbb{Z}_2^n} C_g(\boldsymbol{b})\, i^{-Wt(\boldsymbol{b})} (-1)^{<\boldsymbol{\gamma}\oplus\boldsymbol{\lambda},\boldsymbol{b}>}$$

$$= 2^{-2n} \sum_{\boldsymbol{a},\boldsymbol{b},\lambda \in \mathbb{Z}_2^n} C_f(\boldsymbol{a}) C_g(\boldsymbol{b})\ i^{-Wt(\boldsymbol{a})-Wt(\boldsymbol{b})} (-1)^{<\boldsymbol{\lambda},\boldsymbol{a}>+<\boldsymbol{\gamma}\oplus\boldsymbol{\lambda},\boldsymbol{b}>}$$

$$= 2^{-2n} \sum_{\boldsymbol{a},\boldsymbol{b} \in \mathbb{Z}_2^n} C_f(\boldsymbol{a}) C_g(\boldsymbol{b})\ i^{-Wt(\boldsymbol{a})-Wt(\boldsymbol{b})} (-1)^{<\boldsymbol{\gamma},\boldsymbol{b}>} \sum_{\lambda \in \mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda},\boldsymbol{a}\oplus\boldsymbol{b}>}$$

$$= 2^{-n} \sum_{\boldsymbol{a} \in \mathbb{Z}_2^n} C_f(\boldsymbol{a}) C_g(\boldsymbol{a})\ i^{-2Wt(\boldsymbol{a})} (-1)^{<\boldsymbol{\gamma},\boldsymbol{a}>} \quad \text{(by using Lemma 1)}$$

In particular, if $f = g$, then, we have

$$2^n \sum_{\boldsymbol{\lambda} \in \mathbb{Z}_2^n} \left|\mathcal{N}_f^q(\boldsymbol{\lambda})\right|^2 \left|\mathcal{N}_f^q(\boldsymbol{\gamma}-\boldsymbol{\lambda})\right|^2 = \sum_{\boldsymbol{a} \in \mathbb{Z}_2^n} (C_f(\boldsymbol{a}))^2\ i^{-2Wt(\boldsymbol{a})} (-1)^{<\boldsymbol{\gamma},\boldsymbol{a}>}.$$

If $2^{m-1} < q \leq 2^m$,we associate an individual sequence of Boolean functions $b_j \in \mathcal{B}_n$(where $j = 0, 1, \dots, m-1$) for any $f \in \mathcal{B}_n^q$ such that,

$$f(\boldsymbol{\alpha}) = b_0(\boldsymbol{\alpha}) + 2\, b_1(\boldsymbol{\alpha}) + \cdots + 2^{m-1} b_{m-1}(\boldsymbol{\alpha}) \ \text{ for all } \boldsymbol{\alpha} \in \mathbb{Z}_2^n \tag{3}$$

Suppose the qth-primitive root of unity is $\zeta = e^{\frac{2\pi i}{q}}$ and $f: \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_q$ as in (3). The GNT of $f$can be described in terms of NT of b_j, where b_j denotes the Boolean component.

Theorem 4. The GNT of$f: \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_q$, $2^{m-1} < q \leq 2^m$, where $f(\boldsymbol{\alpha}) = \sum_{j=0}^{m-1} b_j(\boldsymbol{\alpha}) 2^j$, $b_j \in \mathcal{B}_n$is given as

$$\mathcal{N}_f^q(\boldsymbol{\lambda}) = 2^{-m} \sum_{P \subseteq \{0,\dots,m-1\}} \zeta^{\sum_{j\in P} 2^j} \sum_{J \subseteq P, K \subseteq \bar{P}} (-1)^{|J|}\, \mathcal{N}_{\sum_{l\in J\cup K} b_l(\boldsymbol{\alpha})}(\boldsymbol{\lambda}) i^{Wt(\boldsymbol{\alpha})}.$$

Proof. Let $\zeta_j = \zeta^{2^j}$and we observe that for $k \in \mathbb{Z}_2$, we have

$$z^k = \frac{1+(-1)^k}{2} + \frac{1-(-1)^k}{2} z,$$

and we know the identities $\zeta_j^{b_j(\boldsymbol{\alpha})} = \frac{1}{2}\left(B_j + B_j' \zeta_j\right)$, where, $B_j = 1 + (-1)^{b_j(\boldsymbol{\alpha})}$, $B_j' = 1 - (-1)^{b_j(\boldsymbol{\alpha})}$ and for some subset P of $\{0, 1, 2, 3, \dots, m-1\}$, the complement $\bar{P} \coloneqq \{0, 1, 2, 3, \dots, m-1\} \backslash P$. The GNT of f is

$$2^{\frac{n}{2}} \mathcal{N}_f^q(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_2^n} \zeta^{f(\boldsymbol{\alpha})} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} i^{Wt(\boldsymbol{\alpha})}$$

$$= \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_2^n} \zeta^{\sum_{j=0}^{m-1} b_j(\boldsymbol{\alpha}) 2^j} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} i^{Wt(\boldsymbol{\alpha})}$$

$$= \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} i^{Wt(\boldsymbol{\alpha})} \prod_{j=0}^{m-1} (\zeta^{2^j})^{b_j(\boldsymbol{\alpha})}$$

$$= \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} \iota^{Wt(\boldsymbol{\alpha})} \prod_{j=0}^{m-1} \frac{1}{2}\left(1+(-1)^{b_j(\boldsymbol{\alpha})}+\left(1-(-1)^{b_j(\boldsymbol{\alpha})}\right)\zeta_j\right)$$

$$= 2^{-m} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} \iota^{Wt(\boldsymbol{\alpha})} \sum_{P\subseteq\{0,\dots,m-1\}} \prod_{j\in P, i\in\bar{P}} \zeta_j\, B_j' B_i$$

$$= 2^{-m} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} \iota^{Wt(\boldsymbol{\alpha})} \sum_{P\subseteq\{0,\dots,m-1\}} \zeta^{\sum_{j\in P} 2^j} \prod_{j\in P, i\in\bar{P}} B_j' B_i$$

$$= 2^{-m} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} \iota^{Wt(\boldsymbol{\alpha})} \sum_{P\subseteq\{0,\dots,m-1\}} \zeta^{\sum_{j\in I} 2^j} \sum_{J\subseteq P, K\subseteq\bar{P}} (-1)^{|J|} (-1)^{\sum_{i\in J} b_i(\boldsymbol{\alpha}) \oplus \sum_{k\in K} b_k(\boldsymbol{\alpha})}$$

$$= 2^{-m} \sum_{P\subseteq\{0,\dots,m-1\}} \zeta^{\sum_{j\in P} 2^j} \sum_{J\subseteq P, K\subseteq\bar{P}} (-1)^{|J|} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{\lambda},\boldsymbol{\alpha}>} (-1)^{\sum_{l\in J\cup K} b_l(\boldsymbol{\alpha})} \iota^{Wt(\boldsymbol{\alpha})}.$$

Hence, we get the result.

2.2 GNT for Derivative of $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{\mathcal{B}_n^q}$

In this subsection, we establish the connection of GNT and generalized Walsh-Hadamard transform for derivative of $f, g \in \mathcal{B}_n^q$. The derivative for $f, g \in \mathcal{B}_n^q$ at $\boldsymbol{a} \in \mathbb{Z}_2^n$ is defined as $D_{f,g}(\boldsymbol{a}) = f(\boldsymbol{r}) - g(\boldsymbol{r}+\boldsymbol{a})$. If $f = g$, then derivative of f at $\boldsymbol{a} \in \mathbb{Z}_2^n$ is $D_f(\boldsymbol{a}) = f(\boldsymbol{r}) - f(\boldsymbol{r}+\boldsymbol{a})$.

Lemma 6. Let $f, g\ and\ h \in \mathcal{B}_n^q$ such that $h(\boldsymbol{y}) = f(\boldsymbol{y}) - g(\boldsymbol{y})$ for $\boldsymbol{y} \in \mathbb{Z}_2^n$.Then

$$\mathcal{N}_h^q(\boldsymbol{c}) = 2^{-\frac{n}{2}} \sum_{\boldsymbol{d}\in\mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{c}\oplus\boldsymbol{d})\overline{\mathcal{T}_g^q(\boldsymbol{d})}.$$

Proof. Let $\boldsymbol{c} \in \mathbb{Z}_2^n$. Then by definition, we get

$$\sum_{\boldsymbol{d}\in\mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{c}\oplus\boldsymbol{d})\overline{\mathcal{T}_g^q(\boldsymbol{d})} = 2^{-\frac{n}{2}} \sum_{\boldsymbol{d}\in\mathbb{Z}_2^n} \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{y})} (-1)^{<\boldsymbol{c}\oplus\boldsymbol{d},\boldsymbol{y}>} \iota^{Wt(\boldsymbol{y})} \left(2^{-\frac{n}{2}} \sum_{\boldsymbol{s}\in\mathbb{Z}_2^n} \zeta^{-g(\boldsymbol{s})} (-1)^{-<\boldsymbol{d},\boldsymbol{s}>}\right)$$

$$= 2^{-n} \sum_{\boldsymbol{y},\boldsymbol{s}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{y})-g(\boldsymbol{s})} (-1)^{<\boldsymbol{c},\boldsymbol{y}>} \iota^{Wt(\boldsymbol{y})} \sum_{\boldsymbol{d}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{d},\boldsymbol{y}-\boldsymbol{s}>}$$

$$= \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} \zeta^{f(\boldsymbol{y})-g(\boldsymbol{y})} (-1)^{<\boldsymbol{c},\boldsymbol{y}>} \iota^{Wt(\boldsymbol{y})} \qquad \text{(by using Lemma 1)}$$

$$= \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} \zeta^{h(\boldsymbol{y})} (-1)^{<\boldsymbol{c},\boldsymbol{y}>} \iota^{Wt(\boldsymbol{y})}$$

$$= 2^{\frac{n}{2}} \mathcal{N}_h^q(\boldsymbol{c}).$$

Theorem 5. If $f, g \in \mathcal{B}_n^q$ and $\boldsymbol{c} \in \mathbb{Z}_2^n$. Then,

$$\mathcal{N}_{D_{f,g}(\boldsymbol{t})}^q(\boldsymbol{c}) = 2^{-\frac{n}{2}} \sum_{\boldsymbol{e}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{d},\boldsymbol{t}>} \mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{e}) \overline{\mathcal{T}_g^q(\boldsymbol{e})}$$

and

$$\mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{e}) \overline{\mathcal{T}_g^q(\boldsymbol{e})} = 2^{-\frac{n}{2}} \sum_{\boldsymbol{t}\in\mathbb{Z}_2^n} (-1)^{-<\boldsymbol{e},\boldsymbol{t}>} \mathcal{N}_{D_{f,g}(\boldsymbol{t})}^q(\boldsymbol{c}).$$

Proof. For simplicity, we denoted $g_{\boldsymbol{t}} = g(\boldsymbol{t} \oplus \boldsymbol{y})$. Then we have

$$\begin{aligned}
\mathcal{T}_{g_{\boldsymbol{t}}}^q(\boldsymbol{e}) &= \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} \zeta^{g_{\boldsymbol{t}}(\boldsymbol{y})} (-1)^{<\boldsymbol{e},\boldsymbol{y}>} \\
&= \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} \zeta^{g(\boldsymbol{t}\oplus\boldsymbol{y})} (-1)^{<\boldsymbol{e},\boldsymbol{y}>} \\
&= \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} \zeta^{g(\boldsymbol{t}+\boldsymbol{y})} (-1)^{<\boldsymbol{e},\boldsymbol{y}\oplus\boldsymbol{t}-\boldsymbol{t}>} \\
&= (-1)^{-<\boldsymbol{e},\boldsymbol{t}>} \sum_{\boldsymbol{u}\in\mathbb{Z}_2^n} \zeta^{g(\boldsymbol{t}\oplus\boldsymbol{y})} (-1)^{<\boldsymbol{e},\boldsymbol{y}\oplus\boldsymbol{t}>} \\
&= (-1)^{-<\boldsymbol{e},\boldsymbol{t}>} \mathcal{T}_g^q(\boldsymbol{e}).
\end{aligned}$$

Replacing g by $g_{\boldsymbol{t}}$ and h by $D_{f,g}(\boldsymbol{t})$ in Lemma 6, we get,

$$\mathcal{N}_{D_{f,g}(\boldsymbol{t})}^q(\boldsymbol{c}) = 2^{-\frac{n}{2}} \sum_{\boldsymbol{e}\in\mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{e}) \overline{\mathcal{T}_{g_{\boldsymbol{t}}}^q(\boldsymbol{e})}.$$

Thus, we have

$$\mathcal{N}_{D_{f,g}(\boldsymbol{t})}^q(\boldsymbol{c}) = 2^{-\frac{n}{2}} \sum_{\boldsymbol{d}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{e},\boldsymbol{t}>} \mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{e}) \overline{\mathcal{T}_g^q(\boldsymbol{e})}.$$

Therefore,

$$\begin{aligned}
\sum_{\boldsymbol{t}\in\mathbb{Z}_2^n} (-1)^{-<\boldsymbol{e},\boldsymbol{t}>} \mathcal{N}_{D_{f,g}(\boldsymbol{t})}^q(\boldsymbol{c}) &= \sum_{\boldsymbol{t}\in\mathbb{Z}_2^n} (-1)^{-<\boldsymbol{e},\boldsymbol{t}>} 2^{-\frac{n}{2}} \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{y},\boldsymbol{t}>} \mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{y}) \overline{\mathcal{T}_g^q(\boldsymbol{y})} \\
&= 2^{-\frac{n}{2}} \sum_{\boldsymbol{y}\in\mathbb{Z}_2^n} \mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{y}) \overline{\mathcal{T}_g^q(\boldsymbol{y})} \sum_{\boldsymbol{t}\in\mathbb{Z}_2^n} (-1)^{<\boldsymbol{y}-\boldsymbol{e},\boldsymbol{t}>} \\
&= 2^{\frac{n}{2}} \mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{e}) \overline{\mathcal{T}_g^q(\boldsymbol{e})}. \qquad \text{(by using Lemma 1)}
\end{aligned}$$

Theorem 6. Suppose $f, g \in \mathcal{B}_n^q$ and V is a subspace of $\mathbb{Z}_2^n$ of dimension k. Then for every $\boldsymbol{c} \in \mathbb{Z}_2^n$, we have,

$$\sum_{\boldsymbol{d}\in V} \mathcal{N}_f^q(\boldsymbol{c} \oplus \boldsymbol{d}) \overline{\mathcal{T}_g^q(\boldsymbol{d})} = 2^{\frac{2k-n}{2}} \sum_{\boldsymbol{\alpha}\in V^\perp} \mathcal{N}_{D_{f,g}(\boldsymbol{\alpha})}^q(\boldsymbol{c}),$$

where $V^\perp$ is the dual of V.

Proof. From Theorem 5, we have

$$\sum_{\boldsymbol{d}\in V}\mathcal{N}_f^q(\boldsymbol{c}\oplus\boldsymbol{d})\overline{\mathcal{T}_g^q(\boldsymbol{d})}=\sum_{\boldsymbol{d}\in V}2^{-\frac{n}{2}}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}(-1)^{-<\boldsymbol{d},\boldsymbol{\alpha}>}\mathcal{N}_{D_{f,g}(\boldsymbol{\alpha})}^q(\boldsymbol{c})$$
$$=2^{-\frac{n}{2}}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}\mathcal{N}_{D_{f,g}(\boldsymbol{\alpha})}^q(\boldsymbol{c})\sum_{\boldsymbol{d}\in V}(-1)^{-<\boldsymbol{d},\boldsymbol{\alpha}>},$$

where, $\sum_{\boldsymbol{d}\in V}(-1)^{-<\boldsymbol{d},\boldsymbol{\alpha}>}\neq 0$ if and only if $\boldsymbol{\alpha}\in V^{\perp}$.

Therefore, we get

$$\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_2^n}\mathcal{N}_{D_{f,g}(\boldsymbol{\alpha})}^q(\boldsymbol{c})\sum_{\boldsymbol{d}\in V}(-1)^{-<\boldsymbol{d},\boldsymbol{\alpha}>}=2^k\sum_{\boldsymbol{\alpha}\in V^{\perp}}\mathcal{N}_{D_{f,g}(\boldsymbol{\alpha})}^q(\boldsymbol{c}).$$

Hence,

$$\sum_{\boldsymbol{d}\in V}\mathcal{N}_f^q(\boldsymbol{c}\oplus\boldsymbol{d})\overline{\mathcal{T}_g^q(\boldsymbol{d})}=2^{\frac{2k-n}{2}}\sum_{\boldsymbol{\alpha}\in V^{\perp}}\mathcal{N}_{D_{f,g}(\boldsymbol{\alpha})}^q(\boldsymbol{c}).$$

2.3 Some Results on Generalized Composition and Convolution Theorems

A function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^m$ is known as a vectorial Boolean function, where n and m are positive integers. Suppose $H:\mathbb{Z}_2^n\to\mathbb{Z}_2^m$ is a vectorial function and $K:\mathbb{Z}_2^m\to\mathbb{Z}_q$ is a generalized Boolean function on m-variables defined as $K\circ H(\boldsymbol{y})=K(H(\boldsymbol{y}))$ for $\boldsymbol{y}\in\mathbb{Z}_2^n$. Here, In the following result, the GNT for composition of H and K is presented.

Theorem 7. Suppose $H:\mathbb{Z}_2^n\to\mathbb{Z}_2^m$ and $K:\mathbb{Z}_2^m\to\mathbb{Z}_q$. Then, the GNT for composition of H and K at any $\boldsymbol{c}\in\mathbb{Z}_2^n$, $\boldsymbol{d}\in\mathbb{Z}_2^m$ is given as,

$$\mathcal{N}_{K\circ H}^q(\boldsymbol{c})=2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}_K^q(\boldsymbol{d})\mathcal{N}_{L_{\boldsymbol{d}}\circ H}^q(\boldsymbol{c}).$$

where, the function $L_{\boldsymbol{d}}(\boldsymbol{y})=\frac{q}{2}<\boldsymbol{d},\boldsymbol{y}>$ is a linear and $L_{\boldsymbol{d}}\circ H(\boldsymbol{y})=\frac{q}{2}<\boldsymbol{d},H(\boldsymbol{y})>$.

Proof. From the inverse WT, it follows that,

$$\zeta^{K(y)}=2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}_K^q(\boldsymbol{d})(-1)^{<\boldsymbol{d},\boldsymbol{y}>}.$$

Let $\boldsymbol{s}=H(\boldsymbol{y})$. Then

$$\zeta^{K\circ H(\boldsymbol{y})}=\zeta^{K(H(\boldsymbol{y}))}$$
$$=\zeta^{K(\boldsymbol{s})}$$
$$=2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}_K^q(\boldsymbol{d})(-1)^{<\boldsymbol{d},\boldsymbol{s}>}$$
$$=2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}_K^q(\boldsymbol{d})(-1)^{<\boldsymbol{d},H(\boldsymbol{y})>}$$
$$=2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}_K^q(\boldsymbol{d})(-1)^{\frac{2}{q}L_{\boldsymbol{d}}\circ H(\boldsymbol{y})}$$

Therefore,

$$\begin{aligned}\mathcal{N}^q_{K\circ H}(\boldsymbol{c}) &= 2^{-\frac{n}{2}}\sum_{\boldsymbol{y}\in\mathbb{Z}_2^n}\zeta^{(K\circ H)(\boldsymbol{y})}(-1)^{<\boldsymbol{y},\boldsymbol{c}>}\iota^{Wt(\boldsymbol{y})}\\ &= 2^{-\frac{n}{2}}\sum_{\boldsymbol{y}\in\mathbb{Z}_2^n}2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}^q_K(\boldsymbol{d})(-1)^{\frac{2}{q}(L_{\boldsymbol{d}}\circ H)(\boldsymbol{y})+<\boldsymbol{y},\boldsymbol{c}>}\iota^{Wt(\boldsymbol{y})}\\ &= 2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}^q_K(\boldsymbol{d})2^{\frac{-n}{2}}\sum_{\boldsymbol{y}\in\mathbb{Z}_2^n}\zeta^{(L_{\boldsymbol{d}}\circ H)(\boldsymbol{y})}(-1)^{<\boldsymbol{y},\boldsymbol{c}>}\iota^{Wt(\boldsymbol{y})}\\ &= 2^{-\frac{m}{2}}\sum_{\boldsymbol{d}\in\mathbb{Z}_2^m}\mathcal{T}^q_K(\boldsymbol{d})\mathcal{N}^q_{L_{\boldsymbol{d}}\circ H}(\boldsymbol{c}).\end{aligned}$$

The nega-convolution theorem for Boolean functions has been given by Jiang et al. (2022). Here, we discuss the generalized nega-convolution theorem for the generalized Boolean function $h \in \mathcal{B}^q_n$.

Theorem 8. Let $f_1(\boldsymbol{r}), \dots, f_p(\boldsymbol{r}) \in \mathcal{B}^q_n$ and $g(\boldsymbol{r}) = f_1(\boldsymbol{r}) + \cdots + f_p(\boldsymbol{r})$. Then the generalized nega-convolution of g at $\boldsymbol{c} \in \mathbb{Z}_2^n$ is

$$\mathcal{N}^q_g(\boldsymbol{c}) = 2^{-\frac{m}{2}}\sum_{\boldsymbol{u}\in\mathbb{Z}_2^m}\mathcal{N}^q_{f_1}(\boldsymbol{u}_1)\prod_{i=2}^{p}\mathcal{T}^q_{f_i}(\boldsymbol{u}_i + \boldsymbol{u}_{i-1}).$$

where, $m = (p-1)n$, $\boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_{p-1})$, $\boldsymbol{u}_p = \boldsymbol{c}$ with each $\boldsymbol{u}_i \in \mathbb{Z}_2^n$.

Proof. We shall prove this result by applying induction on p. The result is clearly true for $p = 2$. Suppose the result is true for $p > 2$. Now, we apply the nega-convolution theorem on $f_p(\boldsymbol{r})$ and $f(\boldsymbol{r}) = f_1(\boldsymbol{r}) + \cdots + f_{p-1}(\boldsymbol{r})$. We get,

$$\begin{aligned}\mathcal{N}^q_g(\boldsymbol{c}) &= 2^{-\frac{n}{2}}\sum_{\boldsymbol{u}_{p-1}\in\mathbb{Z}_2^n}\mathcal{N}^q_f(\boldsymbol{u}_{p-1})\mathcal{T}^q_{f_p}(\boldsymbol{c} + \boldsymbol{u}_{p-1})\\ &= 2^{-\frac{n}{2}}\sum_{\boldsymbol{u}_{p-1}\in\mathbb{Z}_2^n}\mathcal{N}^q_f(\boldsymbol{u}_{p-1})\mathcal{T}^q_{f_p}(\boldsymbol{u}_p + \boldsymbol{u}_{p-1}).\end{aligned}$$

Now by using induction hypothesis for $p-1$ on the function f, we get

$$\begin{aligned}\mathcal{N}^q_g(\boldsymbol{c}) &= 2^{-\frac{n}{2}}\mathcal{T}^q_{f_p}(\boldsymbol{c} + \boldsymbol{u}_{p-1}) \times \left(2^{-\frac{(p-2)n}{2}}\sum_{(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{p-1})\in\mathbb{Z}_2^{m-1}}\mathcal{N}_{\mathrm{f}_1}(\boldsymbol{u}_1)\prod_{i=2}^{p-1}\mathcal{T}^q_{f_i}(\boldsymbol{u}_i + \boldsymbol{u}_{i-1})\right)\\ &= 2^{-\frac{m}{2}}\sum_{\boldsymbol{u}\in\mathbb{Z}_2^m}\mathcal{N}^q_{f_1}(\boldsymbol{u}_1)\prod_{i=2}^{p}\mathcal{T}^q_{f_i}(\boldsymbol{u}_i + \boldsymbol{u}_{i-1}).\end{aligned}$$

3. Conclusion and Future Scope

In the construction of secure cryptographic algorithms, bent and negabent functions have been considered. The functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$ have been extensively studied by Schmidt (2009) and established several important results in this setup. In this paper, we have explored some more properties of these functions and provided some novel results on the GNT of generalized Boolean functions like inverse of generalized nega-Hadamard transform, generalized nega-cross correlation, generalized nega-

Parseval's identity, relationship between GNT and generalized nega-cross correlation. We obtained the GNT for the derivative of generalized Boolean functions and establish the connection of GNT and GWT for the derivatives of generalized Boolean functions. Also, we present the GNT for the composition of generalized Boolean function and a vectorial Boolean function. Further, the generalized nega-convolution theorem for the generalized Boolean function is presented. The results provided in this article will be helpful in analyzing the security of various wireless communications. The functions studied in this article will be helpful in construction of secure cryptographic algorithms for various CDMA communications. In future, our aim is to apply these results to construct of some efficient encryption and decryption techniques for smooth digital communication.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication

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