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Statistical Theory with Applications for the Multicomponent Stress-Strength Reliability for a Unified Set of Distributions

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Abstract
In this work we focus on the stress-strength reliability for a multicomponent system for a general set of distributions. The set is proposed to unify under the same umbrella, several of the classical distributions frequently encountered in reliability theory. The multicomponent stress-strength reliability is defined and evaluated for the case of the proposed unified set of distributions. The theoretical results explore inferential statistics including point and interval estimation, the relevant asymptotic theory and properties for some special multicomponent systems. Examples and real case applications are provided for illustrative purposes.

Keywords- Unified set of distributions, s-out-of-k system, Multicomponent system, Stress-strength reliability, Reliability theory.

1. Introduction

The statistical inference, including parameter estimation in the context of reliability modeling, is of vital importance. In addition to the classical problem of the identification of a proper distribution for modeling purposes, one is often interested in evaluating the performance of the reliability system involved. Indeed, for instance, the problem of performance of a system is of great importance in mechanical engineering and refers to a component of strength X, which is subject to stress Y. The system stays in operation as long as the stress is less than the strength, so the system performance is associated with the probability of exceedance, usually denoted by R. The quantity of interest in such cases is the *stress-strength reliability* or *reliability parameter* which is a measure of reliability defined as the probability that the random strength X exceeds the stress Y:

$$R = P(Y < X).$$

The concept which has been introduced by Birnbaum (1956) and developed by Birnbaum and McCarty (1958) plays an important role in many scientific fields. Classical applications abound in areas like aeronautical, civil, mechanical, and nuclear engineering. Such examples include for instance, the strength Y of a rocket chamber, and X, the operating pressure, which is the stress the chamber can withstand (Guttman et al., 1988) or measuring shear strength for spot welds for two different gauges of steel (Duncan, 1986). Although R is of great interest in reliability studies, it is an equally important concept in fields like biomedical sciences, quality control, finance or the comparison between two populations. For example, R can be used for measuring the treatment effect in a case where Y is the response of a control group and X represents a treatment group or for comparing the strength of two types of steel (Guttman et al., 1988) or the lifetimes of two devices (see for instance, Kotz et al., 2003). Finally, one could mention applications in textile science and chemistry where one may consider the breaking strength of jute fiber at different gauge lengths (Xia et al., 2009).

Thus, multicomponent reliability inference is one of the crucial research problems with wide important implications in the field of modern industrial systems, where in system reliability becomes crucial either to maintain continuous operation or to ensure safety. Examples include aircraft engines, power grids, or cloud storage servers, which may have redundant or parallel structures to prevent failures that can extend their operational life. Estimating the reliability of such systems is difficult, since the stress level often varies, and the strength of the system degrades. The multicomponent stress-strength model is thus central in the quantification of such a reliability, since it assesses the probability that system components will bear stress without failure. Recent developments in this regard include not only Classical but also Bayesian inference which cope with the intricacy of real-case scenarios. As a result, such problems are of immense relevance to fields like aerospace, manufacturing, computing and many others where the improvement of reliability is directly reflected to the reduction of operational risks or/and maintenance costs.

In the literature, there are several works that investigate the reliability parameter for numerous different distributional assumptions imposed on X and Y. Indeed, the reliability parameter R has been evaluated for distributions frequently encountered in reliability like Exponential, Gamma, Weibull, Burr, Marshall-Olkin extended Lomax distribution and inverse Rayleigh distribution, etc. (see Awad et al., 1981; Church and Harris, 1970; Constantine et al., 1986; Weerahandi and Johnson, 1992). For recent readings please see Barbu et al. (2021), Basirat et al. (2015), Dey et al. (2017), Gunasekera (2015), Hussian (2013), Kizilaslan (2018), Kundu and Gupta (2005), Nadar and Kizilaslan (2014), and Rao et al. (2017) for the generalized exponential, generalized inverted exponential, Kumaraswamy, Pareto distributions and the proportional hazard models. Some recent works related to the Bayesian approach include among others Akgül (2019), Kohansal and Shoaee (2021), and Zhang et al. (2022). For a comprehensive review the interested reader may refer to Kotz et al. (2003).

Due to the great applicability and importance of this concept it is vital to attempt to unify the methodology for a unified set of distributions that covers most if not all distributions popular in reliability theory and at the same time investigate and furnish under this common umbrella, the statistical inference for the reliability parameter and the asymptotic theory concerning its estimator. This unification, which in addition to the scientific value will be advantageous to practitioners across fields, provided the motivation and the special reasoning for undertaking this study.

At the same time, it is of practical interest to explore the above unification under a multicomponent system which could be a system in series, in parallel, a combination of both or a bridge structure. The first two types of systems are considered to be special cases of the general class of s-out-of-k systems. An s-out-of-k:G system requires the operation of at least s components for the operation of the entire system consisting of k components. By contrast, a s-out-of-k:F system refers to a system that fails if and only if at least s out of k components fail. For details about such systems see e.g. Dafnis et al. (2019), Ross (1979), Triantafyllou (2020) and Triantafyllou and Koutras (2014). Typical examples of such systems include a power system powering a manufacturing unit that needs at least s out of k fuses cut-outs for the unit to be operative. According to the above discussion, the key contribution and the novelty of this work is the investigation of the reliability parameter of multicomponent stress-strength systems when both the stress and the strength variables have distributions which belong to a unified set of distributions.

Suppose a system with k identical components functions if s $(1 \leq s \leq k)$ or more of the components operate simultaneously. In its operating environment, the system is subjected to a common stress Y. Assume that the strengths of the components are iid random variables with cdf $F_X(\cdot)$ and subjected to the common random stress Y having the cdf $F_Y(\cdot)$. In this work, we focus on the evaluation and the statistical inference for the multicomponent reliability parameter of an s-out-of-k:G system when both the stress and strength variables follow distributions belonging to the unified set of distributions which is presented and discussed in the following section (Section 2). In the same section, the multicomponent stress-strength reliability is defined, and the main result (Theorem 1) is established for the unified set of distributions under consideration. (Section 3) provides some properties and (Section 4) the inferential statistics (estimation, asymptotic theory and confidence intervals) for the stress-strength reliability. In (Section 4) some examples, applications and two real data applications are explored while in (Section 6) the main conclusions are stated.

2. Reliability Parameter

In the concept of multi states systems it is of great interest to investigate the multicomponent stress-strength models, in order to capture the reliability of models consisting of k components, when at least s components simultaneously survive a common random stress Y which acts independently of the strength. The reliability parameter or stress-strength parameter of an s-out-of-k system has been defined by Bhattacharyya and Johnson (1974), is denoted by $R_{s,k}$ and it is given by

$$\begin{aligned} R_{s,k} &= \mathbb{P}(\mathit{at\ \ least\ \ s\ \ of\ \ the\ \ } X_1, \dots, X_k \ \ \mathit{exceed\ \ } Y) \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} (1 - F_X(x))^i (F_X(x))^{k-i} dF_Y(x) \end{aligned} \quad (1)$$

where, the k components $X_1, \dots, X_k$ have a common distribution $F_X(\cdot)$ and the stress variable Y, which is independent of the $X's$, has a cdf $F_Y(\cdot)$.

It is of great interest to investigate the reliability parameter for a unified set of classical well-known distributions which at the same time share some special theoretical properties, one of which is that the distribution of the minimum order statistic is of the same distributional form and thus the proposed unified set of distributions is closed under minima. For this purpose, let X_j be a random variable with cdf denoted by $F(x_j; a_j)$ that belongs to the set of distributions

$$\mathcal{F} = \{F \mid F(\cdot; a_j) = 1 - (1 - F(\cdot; 1))^{a_j},\ a_j > 0\} \quad (2)$$

Let us assume that $F(\cdot; a_j)$ is absolutely continuous w.r.t. the Lebesgue measure and let us denote its density by $f(\cdot; a_j)$ which according to the property in Equation (2) is given by

$$f(\cdot; a_j) := a_j (1 - F(\cdot; 1))^{a_j - 1} f(\cdot; 1) \quad (3)$$

Note that for an $F \in \mathcal{F}$, $F(\cdot; 1)$ is the obvious standard or baseline distribution associated with F, when $a_j \equiv 1$. The set F consists of several distributions satisfying the property in Equation (2) and in that sense it provides a unification of several classical distributions (i.e. Geometric, Exponential, Weibull, Pareto, Erlang truncated Exponential, Kumaraswamy etc.) all of which satisfy Equation (2). Indeed, for instance, $Expression(1) \in \mathcal{F}$ is the baseline distribution associated with $Exp(a_j) \in \mathcal{F}$ and Pareto$(m_0, 1) \in \mathcal{F}$ is the baseline of Pareto$(m_0, a_j) \in \mathcal{F}$, for $m_0 > 0$.

It is important to point out that the set $\mathscr{F}$ is not by definition, offering a mechanism for constructing new distributions but rather for unifying under a common umbrella, existing ones, all of which satisfy the special property presented in Equation (2). It should be also pointed out that the property in Equation (2) has an interest from the theoretical point of view, consequence see (Section 3.1) according to which the cdf of the minimum of a random sample from an $F \in \mathscr{F}$ belongs also to F and thus the set $\mathscr{F}$ is closed under minima.

Returning now to the reliability parameter, we turn our attention to the case where both F_X and F_Y in Equation (1), belong to the set $\mathscr{F}$. More specifically, in what follows, we assume that F_X and F_Y are of the same probabilistic structure but with different parameters, namely $F_X(\cdot) = F(\cdot; a_1)$ and $F_Y(\cdot) = F(\cdot; a_2)$, with $F \in \mathscr{F}$. The following Theorem reveals that the reliability parameter is related to the distributional scale parameters irrespectively of the distribution used, as long as it is a member of the unified set of distributions. The result provides a clear advantage since it establishes that the distributional scale parameters uniquely determine the reliability parameter R for *any* $F \in \mathscr{F}$. In fact, it can be shown that the reliability parameter is determined solely by the ratio of the two scale parameters involved. In what follows $F(x) \equiv F(x; 1)$ represents the baseline distribution of $\mathscr{F}$.

Theorem 1: *Assume that $X_1, \dots, X_k$ and Y are random variables with cdf $F \in \mathscr{F}$ with parameters a_1 for all X_i's and a_2 for Y. Then, the reliability parameter $R_{s,k}$ is given by*

$$R_{s,k} = \frac{a_2}{a_1} \sum_{i=s}^{k} \binom{k}{i} B\left(i + \frac{a_2}{a_1}, k - i + 1\right) \tag{4}$$

where, $B(a, b)$ is the usual Beta function (Euler integral of the 1st kind).

Proof: Combining expressions (2) and (1) the reliability parameter becomes

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - F(x)]^{a_1 i} [1 - (1 - F(x))^{a_1}]^{k-i} \times \\ &\quad \times a_2 (1 - F(x))^{a_2 - 1} f(x) dx \\ &= \sum_{i=s}^{k} \binom{k}{i} a_2 \int_{-\infty}^{\infty} [1 - F(x)]^{a_1 i + a_2 - 1} [1 - (1 - F(x))^{a_1}]^{k-i} f(x) dx. \end{aligned}$$

Let now $u = 1 - F(x)$ and $du = -f(x)dx$

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} a_2 \int_0^1 u^{a_1 i + a_2 - 1} [1 - u^{a_1}]^{k-i} du \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j a_2 \int_0^1 u^{a_1(i+j) + a_2 - 1} du \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k-i}{j} \frac{(-1)^j a_2}{a_1(i+j) + a_2} \\ &= \frac{a_2}{a_1} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k-i}{j} \frac{(-1)^j}{i + j + \frac{a_2}{a_1}} \\ &= \frac{a_2}{a_1} \sum_{i=s}^{k} \binom{k}{i} B\left(i + \frac{a_2}{a_1}, k - i + 1\right) \end{aligned} \tag{5}$$

where, $B(a, b)$ is the usual Bet`a function (Euler integral of the 1st kind).

According to the above result, the multicomponent reliability parameter depends only on the parameters of the aforementioned $F \in \mathscr{F}$.

Note that in case $\delta \equiv \frac{a_2}{a_1} = 1$, then, due to the Beta function properties, we have

$$R_{s,k} = (k-s+1)\binom{k}{s} B(s+\delta, k-s+1) = \frac{(k-s+1)}{k+1} = 1 - \frac{\mathrm{s}}{\mathrm{k}+1}.$$

Figure 1 shows how the reliability increases for the special case of $\delta = 1$, as the number s of working components decreases (from at least 5 to at least 1) for the system to remain in an operative condition.

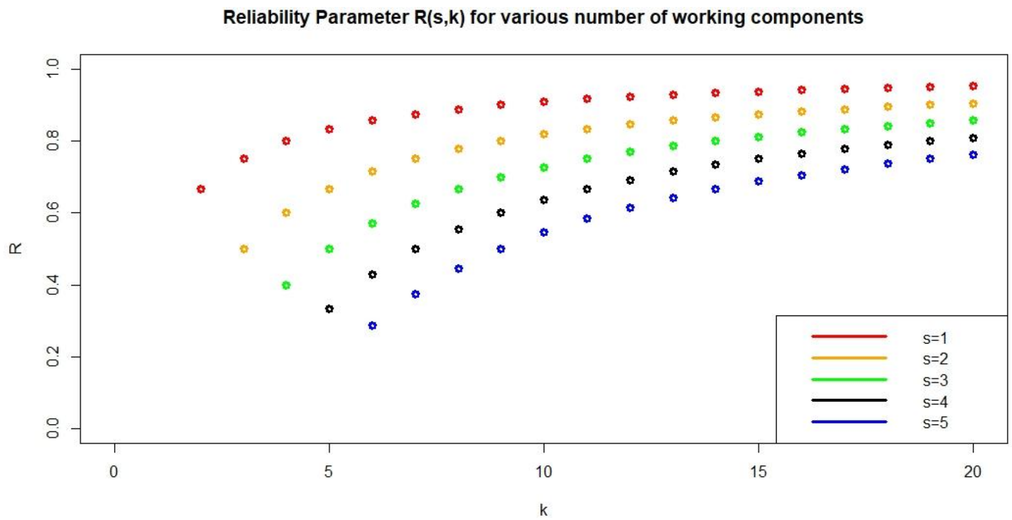

Figure 1. Reliability Parameter, s-out-of-k system, various k and s, $\delta = d = 1$.

Furthermore, observe in **Figures 2** and **3** the behavior of the Reliability parameter when at least $s = 1$ versus at least $s = 5$ components are needed for the system to be in operation. For the former case the reliability is approximately equal to 50% even for a number of components as low as $k = 1$ while for the latter case at least 12 components are needed for the reliability to be over 50% irrespectively of the δ value. All figures in this work have been produced in R and the codes are available on request.

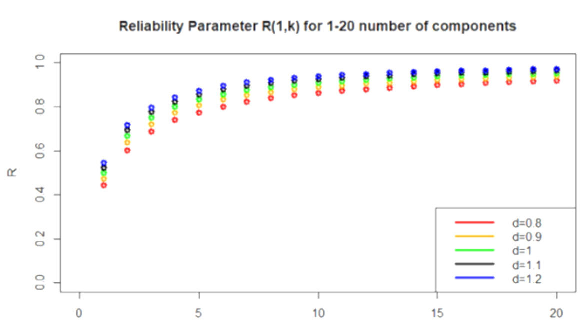

Figure 2. Reliability parameter, 1-out-of-k system, various $\delta \equiv d$.

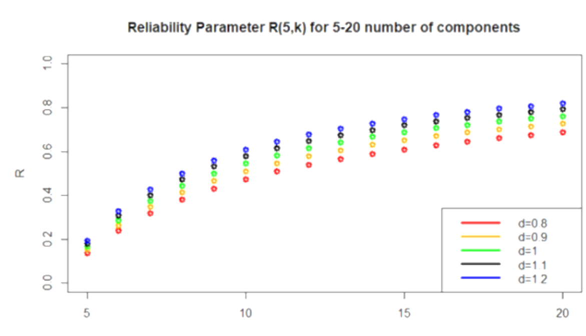

Figure 3. Reliability parameter, 5-out-of-k system, various $\delta \equiv d$.

3. Theoretical Properties and Implications

Before we proceed with the statistical inference for the parameter R, we provide in this subsection some properties of the set $\mathcal{F}$ associated with the minimum of a random sample from an $F \in \mathcal{F}$. A similar result for the maximum order statistic is also provided based on another special set of distributions to be provided below. The implications of such properties are also discussed. Furthermore, some properties directly related to the reliability parameter and the main Theorem of (Section 2) are also provided.

3.1 Minimum Order Statistic

The following result, the proof of which is straightforward, shows that the minimum order statistic from an independent but not necessarily identically distributed (*inid*) random sample from a distribution F belonging to the set $\mathscr{F}$ given in Equation (2) has a distribution of the same form. In other words, the distributions in F are closed under minima.

Lemma 1: *Let $X_1, \ldots, X_k$ be inid random variables but of the same distributional form such that $X_j \sim F(x; a_j)$ which belongs to the set* (2). *Then the distribution function $F^{(1)}$ of the minimum order statistic $X_{(1)}$ belongs also to Equation* (2) *and has the form $F^{(1)}(t; a_1, \ldots, a_k) = 1 - (1 - F(t;1))^{a_0}$, $a_0 = \sum_{j=1}^{k} a_j$.*

Note that the following result (cf. Balasubramanian et al., 1991) holds for the distribution function $F^{(r)}$ of the r^{th} order statistic $X_{(r)}$:

$$F^{(r)}(t; a_1, \ldots, a_k) = \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{k-i-1}{k-r} \binom{k}{k-i} F^{(1)}(t; a_1, \ldots, a_k)$$

which, by Lemma 1 and for $a_0 = \sum_{j=1}^{k} a_j$, takes the final form

$$F^{(r)}(t; a_1, \ldots, a_k) = \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{k-i-1}{k-r} \binom{k}{k-i} [1 - (1 - F(t;1))^{a_0}]$$

It must be noted that the above result can be extended to the case of inid random variables of completely different distributions belonging to set (2), where the distribution function $F^{(r)}$ is written as

$$F^{(r)}(t; a_1, \ldots, a_k) = \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{k-i-1}{k-r} \binom{k}{k-i} \prod_{j=1}^{k} \left[1 - \left(1 - F_j(t;1)\right)^{a_j}\right],$$

with $F_j(t;1)$ being the standard member for the r.v. X_j.

The above formula can then be used for the calculation of the expectation of functions of the r^{th} order statistic. Indeed, let $h(X_{(r)})$ be a function of $X_{(r)}$. Then, if absolute continuity is assumed for the inid with $a_0 = \sum_{j=1}^{k} a_j$, we have

$$E\left[h(X_{(r)})\right] = \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{k-i-1}{k-r} \binom{k}{k-i} \int h(t) dF(t; a_0) \tag{6}$$

Using $h(u) = u^m$ and $h(u) = \exp(uz)$, for some z one can obtain the m^{th} order moment and the moment generating function of $X_{(r)}$.

***Example*:** As a general example for demonstration purposes, we consider here the Weibull distribution. Let $X_1, \ldots, X_k$ be inid random variables such that $X_j \sim F(x; r; a_j)$ which is a Weibull distribution with scale parameter a_j and shape parameter r namely,

$$F(t; r; a_j) = 1 - e^{-(t/a_j)^r}, \quad t \geq 0, \quad a_j,\ r > 0 \tag{7}$$

If $X_{(1)} < X_{(2)} < \cdots < X_{(k)}$ is the ordered sample, then the cdf of $X_{(1)}$ is given by

$$F^{(1)}(t) = 1 - e^{-t^r/\left(\left(\sum_{j=1}^{k}\frac{1}{a_j^r}\right)^{-1/c}\right)^c} \tag{8}$$

which is a Weibull distribution function with scale parameter $\gamma = \left(\sum_{j=1}^{k}\frac{1}{a_j^r}\right)^{-1/c}$. Observe that for $r = 1$ we obtain the results for the exponential distribution and for $r = 2$ and $a_j = (2\sigma^2)^{-1}$ for the Rayleigh distribution. The moment generating function and the r^{th} order moment of any order statistic can be easily obtained via expression (6).

3.2 Maximum Order Statistic

In the field of reliability engineering, the maximum order statistic and the stress-strength parameter are crucial concepts for assessing the performance and longevity of systems composed of multiple components. The maximum order statistic represents the time until the failure of the last surviving component in a series of identical, independent components, highlighting the system's lifespan under worst-case scenarios. By combining these concepts, engineers can gain a comprehensive understanding of both the extreme lifespan and the reliability of the system under varying stress conditions. This integrated approach enables the design of robust systems, optimized maintenance schedules, and enhanced safety and reliability, ultimately leading to products that can meet or exceed their required operational lifetimes.

The need to consider a set of distributions having the property that the distribution of the maximum order statistic has the same distributional form (see Lemma 2) is of high importance. Such distributions like Bernoulli, power function, extreme value Type I distribution belong to the set

$$\mathcal{F}_{max} = \{F \mid F(\cdot; a_j) := (F(\cdot;1))^{a_j}\},\ a_j > 0 \tag{9}$$

with each $F \in \mathcal{F}_{max}$ having the density

$$f(\cdot; a_j) := a_j (F(\cdot;1))^{a_j - 1} f(\cdot;1) \tag{10}$$

Lemma 2: *Let $X_1, \dots, X_k$ be inid random variables such that $X_j \sim F(x; a_j)$ which belongs to the set* (9). *Then the distribution function $F^{(k)}$ of the maximum order statistic $X_{(k)}$ belongs also to Equation* (9) *and has the form $F^{(1)}(t; a_1, \dots, a_k) = (F(t;1))^{a_0}$, $a_0 = \sum_{j=1}^{k} a_j$.*

Similar results to those in Section 3.1 concerning the distribution function $F^{(r)}$ of the r^{th} order statistic $X_{(r)}$, the m^{th} order moment and the moment generating function of $X_{(r)}$ can arise (see Balasubramanian et. al (1991)). Indeed for $a_0 = \sum_{j=1}^{k} a_j$ and using the previous Lemma, $F^{(r)}$ is written as,

$$F^{(r)}(t) = \sum_{i=0}^{k-r} (-1)^{k-r-i} \binom{k-i-1}{k-r} \binom{k}{k-i} (F(t;1))^{a_0}.$$

which can be used for the evaluation of the quantities of interest under the set $\mathcal{F}_{\max}$.

The reliability parameter can be evaluated under $\mathcal{F}_{max}$. Indeed, consider the random variables $X_1, \dots, X_k$ and Y following a distribution $F \in \mathcal{F}_{\max}$ given in Equation (9) with parameters a_1 for all X_i's and a_2 for Y respectively. To that end, the reliability parameter is expressed with respect to parameters of the distributions involved:

Theorem 2: *Assume that $X_1, \dots, X_k$ and Y are independent random variables with $F \in \mathcal{F}_{max}$ with parameters a_1 for all X_i's and a_2 for Y. Then, the reliability parameter $R_{s,k}$ is given by*

$$R_{s,k} = \frac{a_2}{a_1}\sum_{i=s}^{k} \binom{k}{i} B\left(i+1, k-i+\frac{a_2}{a_1}\right) \tag{11}$$

The proof follows along the lines of the proof of the Theorem of the previous section, for $F \in \mathcal{F}$.

Remark 1: *As expected, if $a_1 = a_2$, i.e. for $\delta = 1$, the expressions of the reliability parameter under the sets of distributions F and $\mathcal{F}_{max}$ are identical due to the fact that the two terms in the integrand in Equation* (1), *are complementary.*

3.3 Implications

In multi-state systems, the interest of analysis is often concentrated on the determination and estimation of time instant and the corresponding probability of occurrence of critical events with the focus primarily on the first failure or event occurrence. Such events are very important for system behavior prediction and risk mitigation strategy formulation. Thus, the required distribution of the time until the first occurrence is provided by Lemma 1 and in fact it is a member of the set $\mathcal{F}$ as long as the underlying distribution is one of those belonging to $\mathcal{F}$. Therefore, the set of distributions discussed in this work together with the properties associated with it, provide the path to make inferences on system reliability and the occurrence of early failures, which are applicable to conducting preventive maintenance and mitigating risks.

An analysis of equal importance is the one focusing on the last failure or final event which a system experiences before it enters a (total) failure state. The maximum occurrence is the endpoint of a system's functional life and is crucial for assessing a system's total duration of operation and pattern of failures. Lemma 2 provides a framework for assessing the distribution of such maximum occurrences and, hence, for obtaining a complete view of the system's behavior over time. The latter is clearly valuable in planning system overhauls, maintenance scheduling and resource optimization.

Examples that provide valuable insights for understanding the distribution of first events, include natural hazards such as earthquakes and floods. For example, predicting when the next earthquake will happen, or when the next flood will take place, is much more critical for public safety, urban planning, and disaster management than simply data analysis (see Makrides, 2022) These events are viewed as critical benchmarks for understanding the statistical properties of extreme events, while developing early warning systems so that their impact on communities and infrastructure is minimal.

Scenarios dealing with the distribution of the maximum occurrence are relevant in the development of a system at the point just before it reaches a critical threshold or failure point. A representative example comes from structural engineering where engineers are interested in stress and strain diagrams that graphically display the basic material characteristics when designing various types of constructions like bridges, highways or building. In such cases the investigator is looking for the maximum time that represents the moment the material ruptures, fails or cracks (cf. Barbu et al., 2019; Courtney, 2005). Other relevant examples include, among others, machine lifespan and network failures in telecommunication systems where the interest lies on the last successful transmission prior to the networks' failure.

3.4 Reliability Parameter Properties

In this section we provide general expressions for the reliability parameter for cases where at most one and at most two components fail.

If a system consists of a single strength variable ($k = 1$) then the reliability parameter can be easily

evaluated as follows using expression (5) with $k = s = 1$ and for $\delta = \frac{a_2}{a_1}$:
$R_{1,1} = \delta B(1+\delta, 1) \equiv 1/(\delta+1)$.

Observe that for $a_2 < a_1$, $R_{1,1} > 0.5$, otherwise $R_{1,1} \leq 0.5$ with equality holding for $a_2 = a_1$, i.e. if the strength and stress variables are identically distributed.

Theorem 3: *Consider a system with k components with corresponding strengths $X_1, \dots, X_k$ from the set $\mathcal{F}$ with $a = a_1$ and one stress random variable Y from the same set $\mathcal{F}$ with $a = a_2$. Assume that $k \geq 2$ and assume that the system is in operation if at least $s = k-1$ components are in operation, i.e. $k - s = 1$. Then,*

$$R_{s,k} \equiv R_{k-1,k} = \frac{\delta}{k+\delta}\left(\frac{k}{s+\delta} + \frac{1}{k+\delta+1}\right) \tag{12}$$

Proof: Assume that $k = \gamma$ so that $s = \gamma - 1$. By the definition of the Beta function, we have that

$$\begin{aligned} R_{s,k} \equiv R_{k-1,k} &= \delta\gamma B(\gamma-1+\delta, 2) + \delta B(\gamma+\delta, 2) \\ &= \delta\gamma \int_0^1 u^{\gamma+\delta-2}(1-u)du + \delta \int_0^1 u^{\gamma+\delta-1}(1-u)du \\ &= \delta\gamma\left(\frac{1}{\gamma+\delta-1} - \frac{1}{\gamma+\delta}\right) + \delta\left(\frac{1}{\gamma+\delta} - \frac{1}{\gamma+\delta+1}\right) \end{aligned} \tag{13}$$

and the result is immediate.

Theorem 4: *Consider a system with k components with corresponding strengths $X_1, \dots, X_k$ from the set $\mathcal{F}$ in Equation (2) with $a = a_1$ and one stress random variable Y from the same set $\mathcal{F}$ with $a = a_2$. Assume that $k \geq 3$ and assume that the system is in operation if at least $s = k-2$ components are in operation, i.e. $k - s = 2$. Then,*

$$R_{s,k} \equiv R_{k-2,k} = \frac{\delta(k-1)}{2}\left(\frac{k}{s+\delta} + \frac{s}{k+\delta}\right) - \frac{\delta ks}{k+\delta-1} \tag{14}$$

Proof: Setting $s = k-2$ and using Theorem 1, we have

$$\begin{aligned} R_{s,k} = R_{k-2,k} &= \delta k(k-1)/2B(k-2+\delta, 3) + \delta k B(k-1+\delta, 2) + \delta B(k+\delta, 1) \\ &= \frac{\delta k(k-1)}{2}\int_0^1 u^{k+\delta-3}(1-u)^2 du + \delta k \int_0^1 u^{k+\delta-2}(1-u)du + \delta \int_0^1 u^{k+\delta-1}du \\ &= \frac{\delta k(k-1)}{2}\left(\int_0^1 u^{k+\delta-3}du - 2\int_0^1 u^{k+\delta-2}du + \int_0^1 u^{k+\delta-1}du\right) \\ &\quad + \delta k \int_0^1 u^{k+\delta-2}du - \delta k \int_0^1 u^{k+\delta-1}du + \delta \int_0^1 u^{k+\delta-1}du \end{aligned} \tag{15}$$

and the result is immediate.

A visualization of the theoretical results presented in Theorems 4 and 5 are provided in **Figures 4** to **7**. The behavior of the reliability parameter for different values (0.8, 0.9, 1.0, 1.1, 1.2) of $\delta = a_2/a_1$ is depicted in **Figures 4** and **5** for a number of components ranging from $k = 2$ to $k = 30$. **Figure 4** shows the behavior of the reliability parameter if at least $k-1$ out of k components are required to be in operation (i.e. at most one fails) for the entire system to be operative while in **Figure 5** at least $k-2$ of the components need to be operative (i.e. at most 2 fail).

Observe that in **Figure 5**, a higher reliability parameter is naturally observed since more components are allowed to fail as compared with **Figure 4** where at most one component is allowed to fail. Furthermore, as expected (results not shown), $R_{s,k}$ increases as the difference $k - s$ increases.

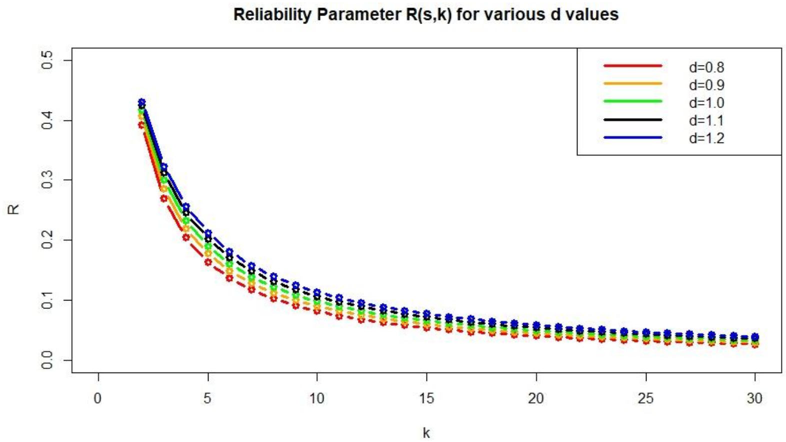

Figure 4. Reliability parameter $R_{k-1,k}$ for various $\delta \equiv d$ values, $k = 2, \dots, 30$.

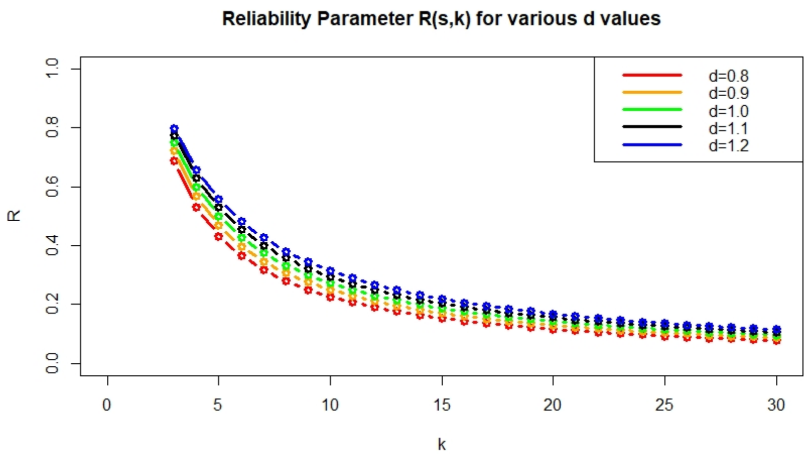

Figure 5. Reliability parameter $R_{k-2,k}$ for various $\delta \equiv d$ values, $k = 3, \dots, 30$.

Figures 6 and **7** provide the behavior of the reliability parameter as a function of the parameter $\delta = a_2/a_1$. We observe that $R_{s,k}$ increases as δ increases, for fixed k, with $k = 10,20,30,40$ and 50.

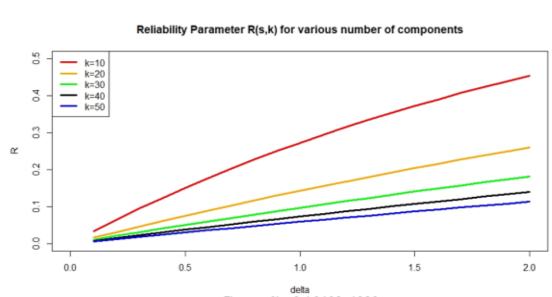

Figure 6. Reliability parameter $R_{k-1,k}$, various k, $\delta =$ 0.1… (0.1)…2.0.

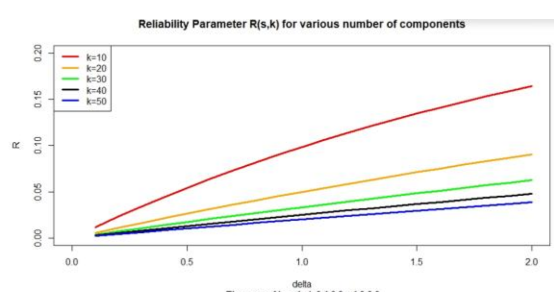

Figure 7. Reliability parameter $R_{k-2,k}$, various k, $\delta =$ 0.1… (0.1)…2.0.

4. Statistical Inference

The following Theorem provides the expressions for the maximum likelihood estimators (MLE) of the parameters a_1 and a_2 based on a random sample of size n from the distribution $F(\cdot; a_1)$ and a random sample of size m from the distribution $F(\cdot; a_2)$, with $F \in \mathcal{F}$. A similar result is also provided for $F \in \mathcal{F}_{\max}$. The asymptotic theory for the estimator of the reliability parameter is provided in (Section 4.1).

Theorem 5: *Let $X_1, \ldots, X_n$ a random sample from $F(\cdot; a_1)$ and $Y_1, \ldots, Y_m$ a random sample from $F(\cdot; a_2)$ with $F \in \mathcal{F}$ given in (2), with $X's$ and $Y's$ independent. Then, the maximum likelihood estimators of a_1 and a_2 are given respectively by*

$$\hat{a}_1 = -n \,/ \sum_{i=1}^{n} \ln(1 - F(x_i; 1)) \tag{16}$$

and

$$\hat{a}_2 = -m \,/ \sum_{j=1}^{m} \ln\left(1 - F(y_j; 1)\right) \tag{17}$$

with variances given respectively by

$$Var(\hat{a}_1) = \frac{a_1^2}{n} \quad \text{and} \quad Var(\hat{a}_2) = \frac{a_2^2}{m}.$$

***Proof*:** The likelihood function for the above setting is given by

$$\begin{aligned}\mathcal{L}(a_1, a_2) &= \prod_{i=1}^{n} f(x_i; a_1) \prod_{j=1}^{m} f(y_j; a_2) \\ &= \prod_{i=1}^{n} a_1(1 - F(x_i; 1))^{a_1 - 1} f(x_i; 1) \prod_{j=1}^{m} a_2(1 - (y_j; 1))^{a_2 - 1} f(y_j; 1)\end{aligned}$$

which applying the log, takes the form

$$\log\mathcal{L}(a_1, a_2) = n\ln a_1 + m\ln a_2 + \sum_{i=1}^{n} \ln f(x_i; 1) + \sum_{j=1}^{m} \ln f(y_j; 1) + (a_1 - 1)\sum_{i=1}^{n} \ln[1 - F(x_i; 1)] + (a_2 - 1)\sum_{j=1}^{m} \ln\left[1 - F(y_j; 1)\right] \tag{18}$$

Taking the derivatives w.r.t. a_1 and a_2 and setting to 0 we obtain the desired results. Using the expression

$$Var\left(\hat{a}_j\right) = \left[E\left(-\frac{\vartheta^2 log\mathcal{L}(a_1, a_2)}{\vartheta a_j^2}\right)\right]^{-1}, j = 1,2.$$

the results for the variances are immediate.

Theorem 6: *Let $X_1, \ldots, X_n$ a random sample from $F(\cdot; a_1)$ and $Y_1, \ldots, Y_m$ a random sample from $F(\cdot; a_2)$ with $F \in \mathcal{F}_{max}$ given in (9), with $X's$ and $Y's$ independent. Then, the maximum likelihood estimators of a_1 and a_2 are given respectively by*

$$\hat{a}_1 = -n \,/ \sum_{i=1}^{n} \ln(F(x_i; 1)) \tag{19}$$

and

$$\hat{a}_2 = -m \,/ \sum_{j=1}^{m} \ln\left(F(y_j; 1)\right) \tag{20}$$

with variances given respectively by

$$Var(\hat{a}_1) = a_1^2 \,/ n \quad \text{and} \quad Var(\hat{a}_2) = a_2^2 \,/ m.$$

The proof is omitted as it is similar to the proof of Theorem 5.

Remark 2: In case some or all of the underlying distributions involve additional (possibly nuisance) parameters, their estimation can be easily performed by differentiating accordingly the log Likelihood in Equation (18). In case of an extra common parameter for both the strength and the stress distributions, both random samples will be involved in the expression for the estimation. For further discussion, please see the Examples subsection. Furthermore, the interested researcher could consider the Bayesian approach with a proper distribution, like the Gamma, the Uniform etc. (see e.g. Dey et al., 2017). In Real Case Application 2 in (Section 5.3), we will see that even if different estimating methods are used the ratios of the various estimators of the scale parameters are practically identical resulting in identical estimators for the reliability parameter R.

4.1 Statistical Inference for the Reliability Parameter

Theorem 7: *Let $X_1, \ldots, X_n$ a random sample from $F(\cdot; a_1)$ and $Y_1, \ldots, Y_m$ a random sample from $F(\cdot; a_2)$ with $F \in \mathcal{F}$ given in Equation* (2), *with $X's$ and $Y's$ independent. Then, the distribution of the estimate $\hat{R}_{s,k}$ of the reliability parameter is asymptotically Normal with asymptotic variance given by*

$$Var(\hat{R}_{s,k}) = a_1^2 /n \left(\frac{\vartheta R_{s,k}}{\vartheta a_1}\right)^2 + a_1^2 /m \left(\frac{\vartheta R_{s,k}}{\vartheta a_2}\right)^2.$$

Proof: The result comes immediately by implementing the Delta method for variance evaluation (see e.g Wolter, 2007).

$$Var(\hat{R}_{s,k}) = Var(\hat{a}_1)\left(\frac{\vartheta R_{s,k}}{\vartheta a_1}\right)^2 + Var(\hat{a}_2)\left(\frac{\vartheta R_{s,k}}{\vartheta a_2}\right)^2.$$

For the asymptotic normality by Rao (2009) and the fact that the MLE's $\hat{a}_1$ and $\hat{a}_2$ are asymptotically Normal, we immediately have

$$\frac{\hat{R}_{s,k} - R_{s,k}}{\sqrt{Var(\hat{R}_{s,k})}} \sim N(0,1),$$

by Theorem 1,

$$\hat{R}_{s,k} = \hat{\delta} \sum_{i=s}^{k} \binom{k}{i} B\left(i + \hat{\delta}, k - i + 1\right), \quad \hat{\delta} = \frac{\hat{a}_2}{\hat{a}_1}.$$

As a natural consequence of the above results, a $100(1-\alpha)\%$ confidence interval for the reliability parameter $R_{s,k}$ is given by

$$\hat{R}_{s,k} \pm z_{\alpha/2}\sqrt{\hat{a}_1^2 n \left(\frac{\widehat{\vartheta R_{s,k}}}{\vartheta a_1}\right)^2 + \hat{a}_2^2 m \frac{\widehat{\vartheta R_{s,k}}}{\vartheta a_2}^2}.$$

The result of this section holds true if the $F \in \mathcal{F}$ in Theorem 7 is replaced by a distribution $F \in \mathcal{F}_{\max}$.

5. Examples and Real Data Applications

In this section we provide a few examples of distributions that belong to the set $\mathcal{F}$. The examples presented show the extent, the flexibility and the diversity of the set $\mathcal{F}$ of distributions considered in this work. Furthermore, the Power function is used as a representative example of the distributions belonging to $\mathcal{F}_{max}$. The accuracy of the theoretical results of this work is also investigated. Finally, the applicability of the proposed methodology is discussed via a number of applications. The section ends with two real data applications.

5.1 Examples

•Assume that $X_1, \dots, X_k$ are iid Weibull r.v.'s with
$F(x; r; a_1) = 1 - exp(-(xa_1)^r) \in \mathscr{F}$

and Y the stress r.v. with $F(y; r; a_2) = 1 - exp(-(ya_2)^r)$. Applying Theorem 5, we get
$\hat{a}_1 = \widehat{m}_{r,x}^{-1}$ and $\hat{a}_2 = \widehat{m}_{r,y}^{-1}$,

where, $\widehat{m}_{r,w} = \frac{1}{n}\sum_{i=1}^{n} w_i^r = \overline{w}^r$ represents the sample moment of order r based on a sample $w_1, \dots, w_n$. Observe that for $r = 1$ the above expressions refer to the Exponential distribution while for $r = 2$ and $a_j = (2\sigma_j)^{-1/2}$, $j = 1,2$, we obtain the Rayleigh distribution.

Note that the common extra parameter r involved in both the distributions of strength and stress can be easily estimated by differentiating Equation (18) w.r.t. r using both random samples from F and G.

• Assume that $X_1, \dots, X_k$ are iid Pareto r.v.'s with
$F(x; m_0; a_1) = 1 - \left(\frac{m_0}{x}\right)^{a_1} \in \mathscr{F}, \quad x \geq m_0$

and Y the stress r.v. with $F(y; m_0; a_2) = 1 - \left(\frac{m_0}{y}\right)^{a_2}$, $y \geq m_0$. Then, by Theorem 5, it is easy to obtain the estimators
$\hat{a}_1 = \frac{n}{\sum_{i=1}^{n} \ln(x_i m_0)}$ and $\hat{a}_2 = \frac{m}{\sum_{i=1}^{m} \ln(y_i m_0)}$.

• Assume that $X_1, \dots, X_k$ are iid Kumaraswamy Generalized r.v's (Kw-G, Cordeiro and de Castro, 2011 and Kumaraswamy, 1980) with $F(x; c; a_1) = 1 - (1 - (G(x))^c)^{a_1}$ and Y the stress r.v. with $F(y; c; a_2) = 1 - (1 - (G(y))^c)^{a_2}$, for any cumulative distribution function $G(\cdot)$. It is easy to see that the Kw-G distribution belongs to the set F since for any a it satisfies the property of $\mathscr{F}$:
$F(x; c; a) = 1 - (1 - F(x; c; 1))^a = 1 - (1 - [1 - (1 - (G(t))^c)^1])^a \in \mathscr{F}$.

Then, by Theorem 5 and for $1 - F(\cdot; c; 1) = 1 - (G(\cdot))^c$, we get

$$\hat{a}_1 = -\frac{n}{\sum_{i=1}^{n} \ln(1-(G(x_i))^c)},$$

$$\text{and} \quad \hat{a}_2 = -\frac{m}{\sum_{i=1}^{m} \ln(1-(G(y_i))^c)}.$$

Observe that if $G(\cdot)$ is chosen to be the Uniform distribution in [0,1], i.e. for the identity $G(t) = t$, then we obtain the classical Kumaraswamy distribution which is well-known in hydrological applications (see for instance, Kumaraswamy, 1980). Furthermore, if $G(\cdot)^c$ is chosen to be the Inverse Rayleigh distribution, i.e. for $G(t)^c = \exp^{-(t/\lambda)^{-2}}$, then we obtain the Exponentiated Inverse Rayleigh distribution (Fatima et. al, 2018). Finally, note that the 3-parameter Burr type XII distribution also belongs to this class for an appropriately chosen function $G(\cdot)$ (see e.g. AL-Hussaini and Ahsanullah, 2015). For a thorough discussion on Kw-G, please see Barbu et al. (2021).

•Assume that $X_1, \dots, X_k$ are iid Power function r.v.'s with
$F(x; c; a_1) = (xc)^{a_1} \in \mathcal{F}_{\max}, x \leq c$

and Y the stress r.v. with $F(y; c; a_2) = (yc)^{a_2}$, $y \leq c$. The estimators arise from Theorem 6

$$\hat{a}_1 = -\frac{n}{\sum_{i=1}^{n} \ln(x_i c)}, \text{ and}$$

$$\hat{a}_2 = -\frac{m}{\sum_{i=1}^{m} \ln(y_i c)}.$$

5.2 Accuracy Evaluation

In this section we wish to check the degree of accuracy of the theoretical results presented in this work using a s-out-of-k:F system. A simulation study is initially performed to investigate the accuracy of the proposed parameter estimators. As a representative example, the variances of the estimators of the parameters of interest a_1 and a_2 are evaluated based on samples of size n $(= 200, 400, 600, 800, 1000)$ from two Exponential distributions ($\in \mathcal{F}$) with parameters $a_1 = 1.0$ and $a_2 = 1.2$ respectively. **Figure 8** shows an extremely good performance with the variances of the parameter estimates (red for a_2 and blue for a_1) being at most equal to 0.005 and getting smaller and smaller as the sample size increases.

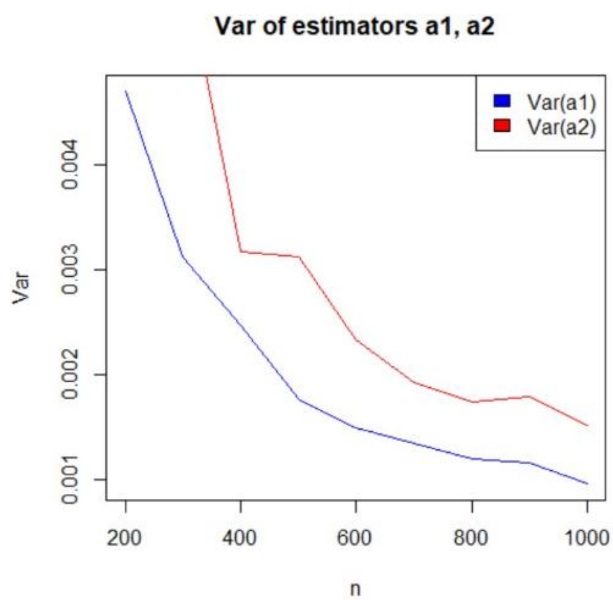

Figure 8. Variance of parameter estimates - the exponential case.

Figures 9 and **10** show the behavior of the reliability parameter for different values of k and values of s= $1 \dots (1) \dots k$. Figure 9 provides the values $R_{s,k}$for small values of k, namely $5, 10, 15, 20, 25, 30, 40, 50, 75$ and 100 while **Figure 10** provides the results for large values of k, namely $200, 400, 600, 800$ and 1000. These extremely large values are used to show the extent of the accuracy of the proposed methodology. It is easily seen that the value of $R_{s,k}$ decreases (a different color is used for each value of k) as s gets closer and closer to the number of components k. The accuracy of the reliability parameter estimators is demonstrated by the low Mean Squared Error (MSE) values shown in **Figure 11**. As anticipated, the MSE decreases as the value of k increases, with the error consistently staying below 5%.

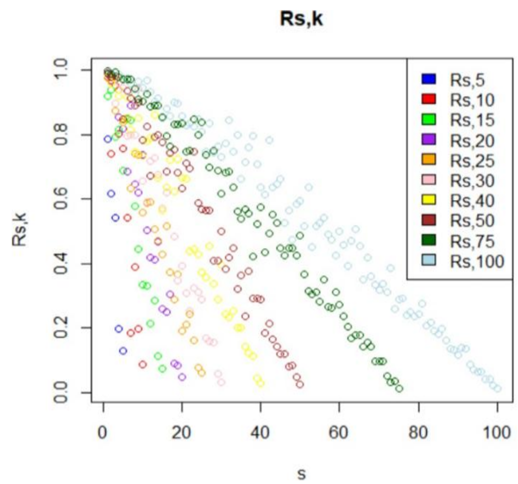

Figure 9. $R_{s,k}$ behavior.

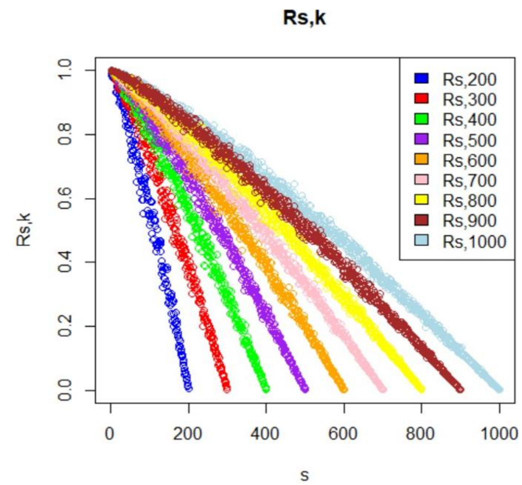

Figure 10. $R_{s,k}$ behavior.

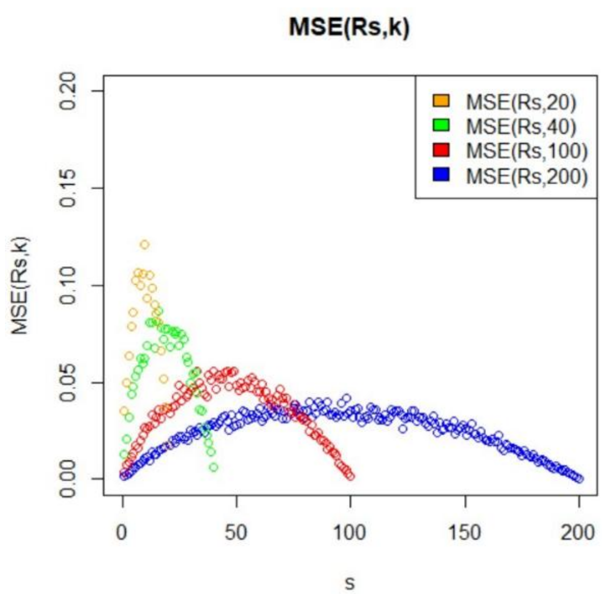

Figure 11. MSE of $R_{s,k}, k = 20, 40, 100, 200$.

To further evaluate the accuracy of the theoretical results the Weibull distribution is considered with shape parameters $a_1 = 2$ and $a_2 = 3$ and the same scale parameter equal to 0.5. The variances of the parameter estimates presented in **Figure 12** show the good performance of the proposed methodology.

An illustration of the behavior of the reliability parameter in the case of the Weibull distribution, for various values of k, with s ranging from 1 to k is reflected in **Figures 13** and **14** which are similar to those presented earlier for the Exponential distribution. As before, it becomes evident that as s nears the total number of components, k, the value of $R_{s,k}$ steadily declines. The low MSE values are also evident in the Weibull case (see **Figure 15**), further highlighting the high performance of the reliability parameter estimators.

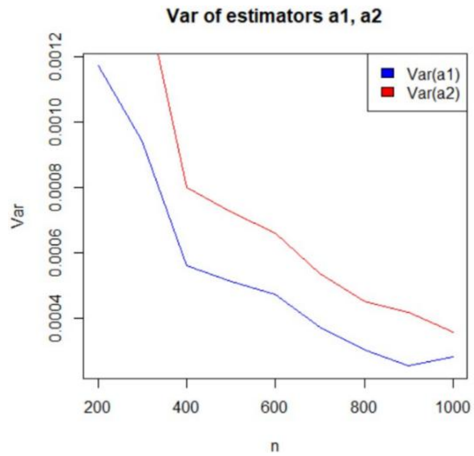

Figure 12. Variance of parameter estimates - the Weibull case.

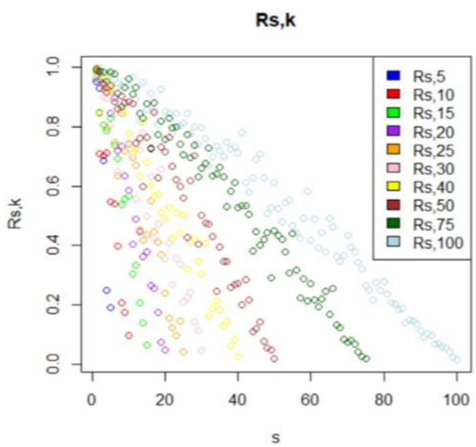

Figure 13. $R_{s,k}$ behavior, k =small, $s = \{1, \ldots, k\}$.

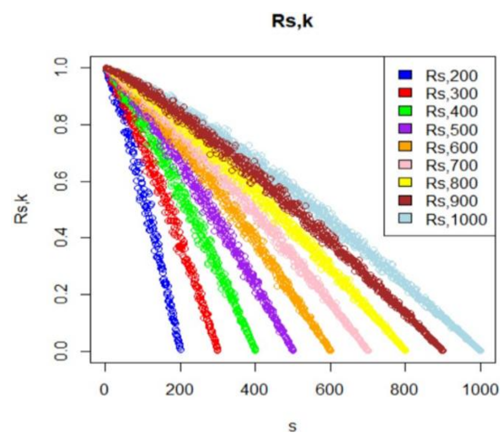

Figure 14. $R_{s,k}$ behavior, k =large, $s = \{1, \ldots, k\}$.

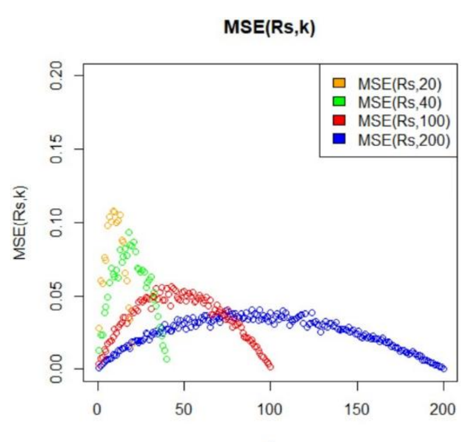

Figure 15. MSE of $R_{s,k}, k = 20, 40, 100, 200$.

5.3 Applicability and Real Data

In addition to engineering applications, the above methodology could be used for the evaluation of the reliability performance in various fields including finance, sports science, meteorology, forestry etc. For instance, applications arise naturally in fields like athletics, hydrology or meteorology where record data are available. In such cases, one may use the stress-strength reliability for comparing rainfall records in different time periods or extreme hydrological and/or meteorological phenomena like flood levels, heat waves etc. or even time records for two different athletic events. Finally, one could mention applications in forest ecosystems where one may consider the forestry loss due to various factors including the deterioration of climate quality (Rao et al., 2019). Kindly note that the distributions mentioned above, being members of the set $\mathscr{F}$ considered in this work, are representative distributions that often describe failure, life, repair times or time-to-event. This shows the great applicability of the proposed methodology that unifies all these and many more distributions, under the unified set $\mathscr{F}$.

For illustrative purposes two real cases are presented below for $k = 6$ and $k = 2$ components respectively. The reliability parameter is estimated via the proposed methodology for each possible value of s and appropriate confidence limits are provided.

Real Data Application 1. The first data set deals with a life test problem which has been extracted from Lawless (2011). The data set consists of a number of cycles, (divided by 1,000) up to the failure, for 60 electrical appliances. Many authors have analyzed the data, including Mota et al. (2021), Reed (2011) and Khan (2018). Such data are known to exhibit a bathtub-shaped hazard rate function. The data are organized in $k = 6$ components from a distribution falling into unified set (2) with scale parameter a_1 and a stress variable from the same unified set with scale parameter a_2. **Figure 16** provides the reliability parameter (green curve) for at least $s = 1,2,3,4,5$ and 6 working components, including the 95^{th} (orange and black curves) and the 99^{th} confidence limits (red and blue curves). Note that the estimator of the ratio $\delta = a_2/a_1$ according to Theorem 2, is found to be equal to 0.51594.

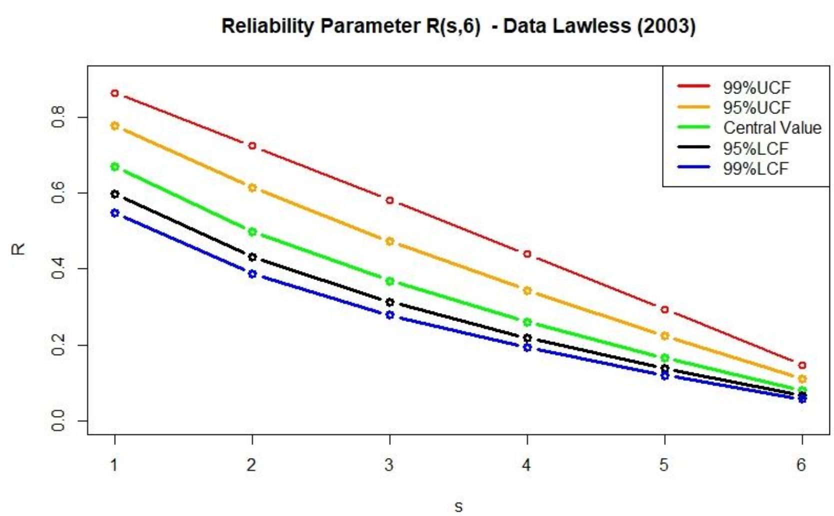

Figure 16. Reliability parameter R – Data (Lawless, 2003).

Real Data Application 2. We analyze a data set not only for evaluating the reliability parameter but also for comparing the MLE with the Bayesian estimating approach. The data set provides burr measurements (in mm) for different machines taken from Dasgupta (2011). **Figure 17** provides the reliability parameter based on the proposed methodology using the maximum likelihood estimation of Theorem 5 (green curve) as well as on Bayes estimation with Gamma and Uniform priors (for the details see Dey et al., 2017), with the results been identical (red and orange curves) to those based on MLE. The results are presented for at least one ($s = 1$) and both ($s = 2$) components, including the 95^{th} upper and lower confidence limits (blue curves). The results reveal that the ratios of the scale parameter estimators are practically identical (the estimator of the ratio δ ranges from 0.8710 to 0.8782) irrespectively of the estimating method considered, as it is evident from the results depicted in **Figure 17** where red, green and orange curves coincide with the same for the corresponding confidence limits.

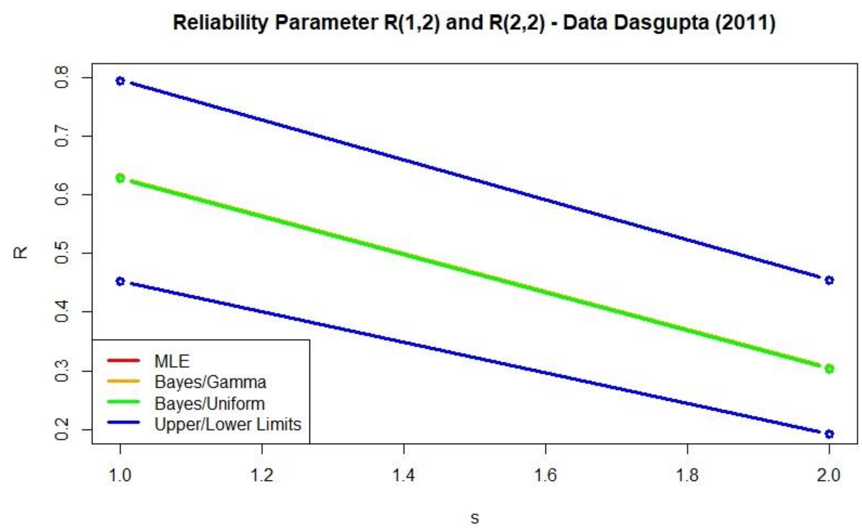

Figure 17. Reliability Parameter R - Data (Dasgupta, 2011).

6. Conclusions

In this work, we focus on the evaluation and statistical inference of the multicomponent reliability parameter of an s-out-of-k:G system when both the stress and strength variables follow distributions belonging to a unified set $\mathcal{F}$ of distributions which is closed under minima. The set $\mathcal{F}$ provides great flexibility by having as members numerous distributions that are frequently encountered in reliability theory. All results presented in this work apply to any $F \in \mathcal{F}$ given in Equation (2). The main contribution of the present work lies on the fact that to the best of our knowledge, it unifies under the same umbrella, several distributions popular in reliability theory and investigates under this common umbrella the reliability parameter and provides a general single expression applicable to all members of the unified set $\mathcal{F}$.

The results of this work clearly show that the multicomponent reliability parameter is related exclusively to the parameters a_1 and a_2 associated with the distributions of the strength and stress variables and not to the mathematical form/expression of the underlying distribution. More specifically, the main result (Theorem 1) reveals that the reliability parameter R is solely related to the distributional scale parameters irrespectively of the distribution used, as long as the underlying distribution is a member of the unified set $\mathcal{F}$. As a result, Theorem 1 provides a clear practical advantage since it establishes that the distributional scale parameters or rather the ratio of the scale parameters, uniquely determines the reliability parameter R. The main result is complemented by the statistical inference and the asymptotic theory. Indeed, in addition to the practical implications, this work provides inferential statistics including point and confidence interval estimation and the relevant asymptotic theory as well as properties of the stress-strength reliability parameter for multicomponent systems.

It should be also pointed out that the results provided in this work will serve as a foundation for optimizing maintenance decisions in multicomponent systems irrespectively of the underlying distribution, which is an important step forward not only in reliability engineering but also in numerous other fields where time-to-event is of primary interest.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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