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Dynamic Analysis and Optimal Control of a Multi-stage Innovation Diffusion Model Incorporating Time Delay in Awareness and Adoption Process

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Abstract
This research presents a new multi-stage innovation diffusion model that incorporates the delay effect due to the complexity and variability of actual diffusion processes as an evaluation time delay parameter, efficiently using the mathematical framework to provide insight into the progression of awareness of the product, decision-making stage, and finally the adoption of the product. In contrast to traditional diffusion models, this model considers the behavioral delay in the transition from awareness to adoption, therefore, adding realism and applicability to the diffusion process. Initially, a set of differential equations that incorporate the three stages with a single time delay parameter is formulated. The rigorous structure and well-defined nature of the model system is ensured by the positivity and boundedness of the solution to the model system. A unique equilibrium point is established, and the local stability conditions of the equilibrium point are acquired on analyzing the characteristic equation. Existence of Hopf bifurcation has been proven, and the direction and stability of bifurcating periodic solutions have also been obtained using the normal form and center manifold theorem. The global stability conditions for the equilibrium point are obtained on constructing a suitable Lyapunov function. Further, the research contributes by formulating an optimal control problem to maximize the number of adopters while minimizing promotional costs, linking theoretical results to marketing strategy. Numerical simulations, including parameter estimation with real-world data, are carried out to further analyze the model's behavior, support the theoretical results, and estimate the rate of adopters. The findings indicate that accounting for the adoption delay plays a crucial role in shaping the system stability and predicting long-term adoption dynamics.

Keywords- Multi-stage innovation diffusion model, Evaluation time delay, Equilibrium point, Hopf bifurcation, Sensitivity analysis.

1. Introduction

In today's dynamic economic environment, companies across the globe are in the process of developing new technology and ideas to prevent extinction. The primary driving factors behind this competition of innovation are technological advancements, global competition, enhanced communication channels, increased customer purchasing power, and shorter lifespans of the products. The development and introduction of a new product or service necessitate a significant investment in resources and planning. Successful products lead to the growth of the firm. Nevertheless, there is a significant risk in terms of financial loss associated with developing and advertising a new product. The scenario entails thorough planning, extensive preparation, risk analysis, market research, and logical decision-making. Mathematical models have proven to be extremely beneficial in this context.

Diffusion theory is widely employed for determining the life cycle of innovative products, from market introduction until obsolescence, and has been extensively studied in the fields of technology management, marketing, and sociology. Since its inception in the early 1960s, substantial research has been performed to construct analytical frameworks for forecasting and illustrating new product adoption among its prospective customers. Bass (1969) model is one of the earliest and most commonly used in the area of diffusion theory. Traditional innovation diffusion models, like Bass model and its extensions, provide a framework for understanding how new innovations spread through a population. These models often presume a uniform process of adoption in which individuals proceed through a single adoption stage that is highly influenced by factors such as communication and social influence.

Griliches (1957) and Mansfield (1961) made significant contributions to the study of economic impact on technological innovation and diffusion. They developed diffusion models in order to understand and explain how new technologies propagate throughout economies and industries. Rogers (1962) defined the term “innovation” as “an idea, practice, or object that is perceived as new by an individual or other unit of adoption” and defined “diffusion” as “a process via which an innovation disseminates among people in a social system” (Rogers and Shoemaker, 1971). Drawing upon these foundational principles and insights, the Bass diffusion model (Bass, 1969) emerged within the marketing literature to model consumer behavior and market dynamics. It describes the process by which the adoption of a new product spreads over time through a population of potential adopters. A diffusion model in general aims at measuring the magnitude of innovation diffusion within a specified group of potential adopters. Time corresponds to the pace at which the innovation spreads or the comparative swiftness of its adoption among members of the social system. Diffusion models with S-shaped curves have been widely used by researchers in various fields in order to study the changing flow of cumulative buyers or adopters over time.

Diffusion processes in the real world, however, are frequently more intricate. Innovations are rarely accepted in a homogeneous manner; instead, they expand via a number of stages, each with its own adopter dynamics and behaviors. Adoption barriers can fluctuate greatly amongst different demographic groups, and early adopters may be affected by different variables than late adopters. The introduction of the multi-stage innovation diffusion model is designed to address these complexities. The early diffusion models primarily comprise of two-stage or binary models integrating the flow from potential adopters to existing adopters. Dodson and Muller (1978) suggested the three-stage diffusion model comprising of uninformed members of the social system, potential adopters, and adopters, wherein a mixed-influence approach has been adopted with positive word of mouth and advertising as an important factor in communicating the innovation. Similar and more advanced multi-stage diffusion models have been studied Wang et al. (2006), Chanda and Das (2015), and Singhal et al. (2019), with mixed combinations of various marketing variables. It is unambiguous that multi-stage innovation diffusion models are an effective tool for comprehending and promoting the spread of innovative principles, commodities, and technological advancements due to the fact that these models capture the complexity of the diffusion process.

The proposed study aims at achieving a comprehensive approach to modeling the diffusion of a recent product, incorporating a multi-stage diffusion model wherein it integrates a single time delay parameter from aware to adopter class, leading to complex dynamic behaviors such as Hopf bifurcation, analyzing the stability and direction of Hopf bifurcation, and also the local and global stability of the proposed model. Furthermore, actual data set has been used to estimate the rate of adopters. This approach offers significant advancements in understanding and predicting the diffusion of recent innovations.

The paper is structured as follows. The model formulation, positivity, and boundedness of the model system are discussed in Section 3. Section 4 determines the stability analysis (Section 4.2), along with Hopf bifurcation (Section 4.3), direction and stability of Hopf bifurcation (Section 4.4), and then global stability analysis of the system is provided in Section 4.5. Optimal control analysis on the application of Pontryagin's maximum principle is outlined in Section 5. Section 6 provides the numerical simulation to validate our theoretical findings. Simulation-Based Estimation of Adoption Rates Using Real-World Data is conducted in Section 7. The final Section 8 gives a brief conclusion of the model system and findings.

2. Literature Review

Since the introduction of innovation diffusion theory to marketing in the 1960s, it has resulted in extensive literature. Everett Rogers in 1962 wrote a book on the diffusion of innovations, a widely recognized framework for understanding how new ideas, technologies, products, or behaviors spread through societies or social systems. The theory provides insights into the factors influencing the adoption and diffusion process, as well as the various stages that individuals or groups typically go through when adopting a product. The Bass Model (Bass, 1969) serves as a broadly endorsed and most commonly adopted model in marketing for evaluating the growth of sales of new products. Its value as a demand model is additionally acknowledged in contexts involving production, inventory, and resource allocation. The Bass Model, which illustrates the dynamics of the progressive adoption of innovation in terms of both internal and external factors, assumes that potential adopters of an innovation are influenced by external factors, which he refers to as "innovators", and internal factors, i.e., "imitators" (Mahajan et al., 1990). External influences typically pertain to the impact of mass media communications on the diffusion process, whereas internal influences encompass the social interactions among previous and potential adopters within the social system. The Bass Model (Bass, 1969) exhibits significant alignment with consumer durable data and provides insightful predictions for long-term sales forecasting, especially regarding peak timing and size as well as new technology diffusion. Despite the strong theoretical foundation of the Bass Model, the model assumes that the maximum number of potential adopters is fixed and that those buyers will not repurchase the product; that is, repeat purchases are not considered; hence, real market dynamics are not being fully analyzed. Several academics in marketing have sought ways to extend the Bass model to incorporate various marketing dynamics and environments. Mahajan and Peterson (1978) highlight the significance of considering adopter diversity and demographic factors in the study of innovation diffusion and market expansion. According to their model, there is a positive correlation between market growth and population size, meaning that as the population grows, so does the market potential for new products. Mahajan et al. (1990) offer additional Bass model modifications that include pricing and advertising.

Several mathematical models have been proposed to address and accomplish various tasks, such as sales forecasting and assisting firms in selecting appropriate strategies that yield optimal expected returns, among many other purposes. For further insights, refer to Lilien et al. (1992) for a comprehensive review. Dodson and Muller (1978) introduced a three-stage model of new product diffusion that took into account advertising, positive word-of-mouth, and repeat purchasing behavior. Furthermore, in order to predict these repeat purchases, there have been similar proposals of diffusion models that make use of initial diffusion data and include word-of-mouth communication (Lilien et al., 1981; Mahajan and Muller, 1982). Kalish (1985) introduced a comprehensive framework for modeling innovation diffusion that expanded upon earlier models by incorporating additional factors such as price, advertising, and uncertainty, with advertising remaining an essential component in Kalish's framework, similar to Dodson and Muller's model. In their study, Krishnan and Jain (2006) likely developed or utilized a diffusion demand function to quantitatively depict the correlation between advertising expenditures and the rate of adoption of a new product or innovation. Their research likely provided a rigorous empirical basis for comprehending the influence of advertising on driving adoption and diffusion processes. Wang et al. (2006) introduced a model with stage structures to illustrate the process of awareness, evaluation, and decision-making for successful innovation diffusion whilst introducing a single time delay parameter that represents the evaluation stage. A multi-stage diffusion model developed by Chanda and Das (2015) takes into account both new purchasers and repeat buyers, addressing the dynamics of market withdrawal, thus providing insights into the dynamics of multi-stage diffusion in high-technology products. Singhal et al. (2019) then proposed a multi-stage innovation diffusion model that accounts for the role of informed and disinterested potential adopters, as well as the impact of dynamic pricing on adoption patterns by introducing the impact of potential customers' attrition behavior.

Stability analysis has been recognized as one of the most vital components of marketing diffusion models in recent research, yet it remains inadequately examined in the literature. Both local and global stability are crucial to know if one is to predict long-run system behavior and optimize marketing efforts. Recent studies, such as works of Chaudhary et al. (2022) and Yu et al. (2022), have applied Lyapunov-based methods and graph-theoretical approaches to analyze stability and design useful interventions. Additionally, Friedrich et al. (2024) researched nonlinear effects in information diffusion, focusing on the role of social reinforcement. These improvements highlight the significance of stability analysis in marketing dynamics determination and decision-making.

As a result of the discussion outlined above, it becomes evident that researchers have made significant strides in understanding the diffusion of innovation. Most of the existing models either disregard the behavioral delay caused by the decision-making process or consider it a mere external effect while neglecting to influence the system dynamics. This paper addresses these limitations and focuses on incorporating multi-stage structures into the Bass Model, taking into account the delay in adopting the recent product to better understand the process of innovation diffusion dynamics. Furthermore, by introducing a control function, we have broadened the model to encompass an optimal control problem in pursuit of enhancing the quantity of adopters while concurrently reducing the expenses linked to promotional efforts within our framework. The key contributions of this study are as highlighted:

- Analyzes a multi-stage innovation diffusion model with a time delay in the adoption of the recent product, which captures the behavioral decision lag.
- Examines how time delay affects the number of adopters, as well as the influence of each individual parameter and the combined effect of all parameters on the number of adopters.
- Analyzes the stability of the system and determines the Hopf bifurcation, along with the direction and stability of the resulting periodic solutions.
- Establishes an optimization framework for promotional efforts to obtain optimal policies to accelerate the diffusion process and uses actual data to estimate the rate of adopters while also demonstrating the practicality of the developed product to fit real marketing data.

3. Model Formulation

In this section, we propose and formulate a mathematical model of innovation diffusion for understanding the spread mechanism of recent product launched in the market. To account for the delay in the adoption of the recent product by the informed individual, a time delay parameter (τ) has been incorporated into the model.

We consider a firm that introduces a product to the market. Diffusion modeling is often an intricate process since it depends on a number of factors. Thus, in order to formulate a mathematical model, suitable assumptions must be made. Accordingly, the following assumptions form the basis of the proposed diffusion model:

i. The total population at any time t is considered to be a constant, say, $\bar{N}$, and is classified into three groups, namely:

- $N(t)$: Number of individuals who are unaware of the recent product at time t.
- $I(\mathrm{t})$: Number of individuals who are aware of the recent product at time t but still have not adopted the recent product.
- $A(t)$: Number of adopters of the recent product at time t.

Therefore,
$\bar{N} = N(t) + I(t) + A(t)$.

ii. Those individuals who are unaware of the existence of the recent product become aware through interaction with the aware class at the rate of β. Thus, βIN represents the number of individuals who become aware of the recent product on interaction with the aware class. Through promotional efforts, individuals can move from the unaware class to the aware class at the rate of μ. The term μuN represents the number of individuals moving from the unaware class to the aware class, where u gives the rate of promotional efforts.

iii. The rate at which aware individuals become adopters of the recent product is δ. The time gap in transitioning from aware individual groups to adopters is taken as the time delay parameter τ. Thus, $\delta I(t-\tau)$ indicates the number of people who transition from the aware class to the adopter class after time delay τ.

iv. Due to product/service issues and service discontinuation or due to misinformation regarding the product/services, the adopters join the unaware class at the rate of α.

The schematic representation of the proposed model with the transitions among unaware, aware, and adopter populations is illustrated in the **Figure 1**.

The proposed delayed differential equation model for recent product adoption which takes into account the factors mentioned above, can be summarized as follows:

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= -\beta IN - \mu uN + \alpha A, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta IN + \mu uN - \delta I(\mathrm{t}-\mathrm{\tau}), and \\ \frac{\mathrm{dA}}{\mathrm{d}t} &= \delta I(t-\tau) - \alpha A. \end{aligned} \tag{1}$$

with the initial conditions,
$N(\varpi) = \psi_1(\varpi),\ I(\varpi) = \psi_2(\varpi),\ A(\varpi) = \psi_3(\varpi),\ \psi_i(\varpi) \geq 0,\ \psi_i(0) > 0,\ i = 1,2,3, -\tau \leq \varpi \leq 0,$

where, $(\psi_1(\varpi), \psi_2(\varpi), \psi_3(\varpi)) \in C([-\tau, 0], \{R\}^3_+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_+, here $R^3_+ = \{(m_1, m_2, m_3): m_i > 0, \ i = 1,2,3\}$.

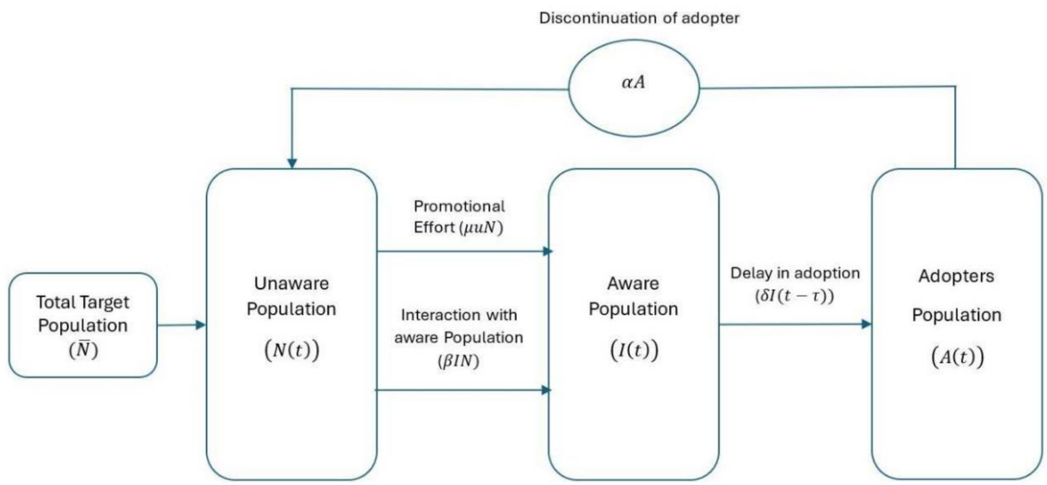


Figure 1. Flow diagram of model system.

Remark 1. The condition $-\tau \leq \varpi \leq 0$ represents the initial history window of the delayed system. Since our proposed model contains the delayed term $I(t-\tau)$, the system dynamics at time $t \geq 0$ depend not only on the current state variables but also on the past states of the system. Therefore, instead of just prescribing initial values only at $t = 0$, it is necessary to specify continuous history functions on $[-\tau, 0]$, which will in turn describe the past behavior of the unaware, aware, and adopter populations and are required to evaluate the delayed terms for early times $t \in [0, \tau]$. (Hale and Lunel, 2013).

Notations:

- $N(t)$: Number of unaware individuals at time t.
- $I(t)$: Number of aware individuals at time t who have not yet adopted the recent product.
- $A(t)$: Number of adopters of the recent product at time t.
- $\bar{N}$: Total population size.
- β : Interaction rate between unaware and aware individuals.
- μ : Awareness rate due to promotional efforts.
- u : Intensity of promotional effort.
- δ : Adoption rate of aware individuals.
- α : Rate at which adopters revert to the unaware class.
- τ : Time delay in the adoption process.
- $\psi_i(\varpi)$: Initial history functions, $i = 1,2,3$, defined on $[-\tau, 0]$.

3.1 Preliminaries

This section focuses on determining the positivity and boundedness of the solutions to system (1).
From the first equation of system (1), we have

$$\frac{dN}{dt} = -\beta I N - \mu u N + \alpha A,$$

$$\Rightarrow N(t) = N(0) e^{\int_0^t (-\mu u - \beta I) d\xi} + e^{\int_0^t (-\mu u - \beta I) d\xi} \int_0^t e^{\int_0^x (\mu u + \beta I) d\xi} \big(\alpha A(x)\big) dx. \quad (2)$$

The above Equation (2) does not imply the positivity of $N(t)$ as the integral term in Equation (2) can become negative. It can be seen that as and when the trajectory $N(t)$ touches the boundary at any time, say, t_1, such that $N(t_1) = 0$, then Equation (1) becomes $\frac{\mathrm{d}N}{\mathrm{d}t_1} = \alpha A(t_1)$. If $\alpha A(t_1) < 0$, we have $\frac{\mathrm{d}N(t_1)}{\mathrm{d}t} < 0$ which results in the statement that the trajectory $N(t)$ can enter the negative region. We state the following condition:
$\aleph_1 : \alpha A(t_1) \geq 0$ at any boundary $N(t_1) = 0$.

If $\aleph_1$ holds, then $N(\mathrm{t}) \geq 0$.
Now from the second equation of system (1), we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} - \beta IN = \mu uN - \delta I(t-\tau),$$

$$\Rightarrow I(t) = e^{\int_0^t \beta N d\xi} I(0) + e^{\int_0^t \beta N d\xi} \int_0^t \big(\mu uN(x) - \delta I(x-\tau)\big) e^{-\int_0^x \beta N d\xi} dx. \tag{3}$$

As argued in the case of positivity of $N(\mathrm{t})$, the positivity of $I(t)$ is not guaranteed as the integral term in Equation (3) can become negative. At time, say, t_2, when the trajectory $I(t)$ touches the boundary such that $I(t_2) = 0$, if we get $\mu uN(t_2) < \delta I(t_2 - \tau)$, then the trajectory $I(t)$ can enter the negative region. We state the following condition:
$\aleph_2 : \mu uN(t_2) \geq \delta I(t_2 - \tau)$ at any boundary $I(t_2) = 0$.

The above condition $\aleph_2$ implies that $I(t) \geq 0$.
For the third equation of system (1), we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \alpha A = \delta I(t-\tau),$$

$$\Rightarrow A(t) = A(0)e^{-\alpha t} + e^{-\alpha t} \int_0^t \delta I(x-\tau) e^{\alpha x} dx. \tag{4}$$

As argued in the previous two equations of Equation (1), we choose the system inputs so as to obtain positive values of $A(t)$, so in this case satisfying the following condition implies positivity of $A(t)$ for all $t \geq 0$,
$\aleph_3 : \delta I(t_3 - \tau) \geq 0$ at any boundary $A(t_3) = 0$.

Now, for the boundedness of the solution,
We know that $\overline{N} = N(t) + I(t) + A(t)$ gives the total number of individuals at time t.
Then,

$$\frac{\mathrm{d}\overline{N}}{\mathrm{d}t} = \frac{\mathrm{d}(N+I+A)}{\mathrm{d}t} = 0,$$

$$\Rightarrow \overline{N} = N + I + A = \text{ constant}.$$

Define the set $\mho$ as follows:

$$\mho = \left\{ \big(N(t), I(t), A(t)\big) \in R_+^3 : \ N(t) + I(t) + A(t) = \overline{N} \right\}$$

From the above analysis, we get that the model system's solution is bounded, and the result is stated as follows:

Lemma 1. If conditions $\aleph_1, \aleph_2$, and $\aleph_3$ are satisfied, then all the solutions of the system (1) are in R_+^3 for all $t \geq 0$ and are also bounded. Also, the set $\mho$ is positively invariant for system (1).

Now, substituting $N = \overline{N} - I - A$ in system (1) we see that the first equation does not contain the variable N, therefore system (1) is equivalent to the following reduced system:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta I\left(\overline{N} - (I + A)\right) + \mu u\left(\overline{N} - (I + A)\right) - \delta I(t - \tau) \tag{5}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \delta I(t - \tau) - \alpha A.$$

The initial conditions of system (5) above are given as $I(\varpi) = \psi_2(\varpi)$, $A(\varpi) = \psi_3(\varpi)$, $\psi_i(\varpi) \geq 0$, $\psi_i(0) \geq 0,\ i = 2, 3,\ -\tau \leq \varpi \leq 0$, where $\left(\psi_2(\varpi), \psi_3(\varpi)\right) \in C([-\tau, 0], R_+^2)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R_+^2, here $R_+^2 = \{(m_2, m_3): m_i > 0, i = 2,3\}$.

4. Rigorous Analysis of System (5)

4.1 Existence of Equilibrium Points

In this section, we obtain the equilibrium point of the system (5) under study. Consider $E^* = (I^*, A^*)$ to be a positive (adopter) equilibrium point for the system (5). The steady-state solution of the system is established, wherein we put delay $\tau = 0$. The equilibrium solutions of a time-delayed model are equivalent to the corresponding model with zero delays (Tipsri and Chinviriyasit, 2014; Kumar and Nilam, 2018). Equating both equations of the system (5) to zero and further solving it, we get the following:

$$A^* = \frac{\delta I^*}{\alpha},$$

we assume that $\alpha > 0$ throughout the analysis,

and I is the solution of the following equation:

$$\beta I\left(\overline{N} - (I + A)\right) + \mu u\left(\overline{N} - (I + A)\right) - \delta I = 0,$$

$$\Rightarrow I = \frac{(\alpha\beta\overline{N} - \alpha\mu u - \mu u\delta - \alpha\delta) \pm \sqrt{(\alpha\beta\overline{N} - \alpha\mu u - \mu u\delta - \alpha\delta)^2 + 4(\alpha\beta + \beta\delta)\alpha\mu u\overline{N}}}{2(\alpha\beta + \beta\delta)}.$$

For the positive value of I, we take

$$\Rightarrow I^* = \frac{(\alpha\beta\overline{N} - \alpha\mu u - \mu u\delta - \alpha\delta) + \sqrt{(\alpha\beta\overline{N} - \alpha\mu u - \mu u\delta - \alpha\delta)^2 + 4(\alpha\beta + \beta\delta)\alpha\mu u\overline{N}}}{2(\alpha\beta + \beta\delta)} n.$$

Thus, we obtain the unique equilibrium point $E^* = (I^*, A^*)$ of the system (5).

4.2 Stability Analysis of Equilibrium Points

The characteristic equation of the system (5) at E^* is given by $|J - \lambda I| = 0$, where J is as follows:

$$J = \begin{bmatrix} \beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u & -\beta I^* - \mu u \\ 0 & -\alpha \end{bmatrix} + e^{-\lambda\tau}\begin{bmatrix} -\delta & 0 \\ \delta & 0 \end{bmatrix}$$

The characteristic equation at E^* is as follows:

$$\lambda^2 + \left(\alpha - \beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u\right)\lambda - \alpha\left(\beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u\right) + e^{-\lambda\tau}\left(\delta\lambda + \alpha\delta + \delta(\beta I^* + \mu u)\right) = 0 \tag{6}$$

Let $a_0 = \alpha - \left(\beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u\right),\ a_1 = -\alpha\left(\beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u\right)$

$b_0 = \alpha\delta + \delta(\beta I^* + \mu u)$. Therefore, the reduced characteristic equation is as follows:

$$\lambda^2 + a_0\lambda + a_1 + e^{-\lambda\tau}(\delta\lambda + b_0) = 0 \tag{7}$$

Now, for $\tau = 0$,

$$\lambda^2 + (a_0 + \delta)\lambda + a_1 + b_0 = 0.$$

For the above equation, we will get roots with negative real parts if the Routh-Hurwitz criterion holds true. Thus,

Theorem 1. For $\tau = 0$, the unique equilibrium point E^* is locally asymptotically stable if it satisfies the following conditions:

i. $(a_0 + \delta) > 0$
ii. $(a_1 + b_0) > 0$

Now, we obtain the conditions for local stability of the equilibrium point E^* of the delayed system (5). In order to access the stability of the equilibrium point E^* of the system with delay, we substitute $\lambda = i\omega,\ \omega \geq 0$ in Equation (7), we get

$$-\omega^2 + a_1 + b_0 \cos\omega\tau + \omega\delta \sin\omega\tau + (a_0\omega + \omega\delta\cos\omega\tau - b_0 \sin\omega\tau)i = 0 \tag{8}$$

Separating the real and imaginary parts of the above Equation (8), we obtain the following equations,

$$b_0 \cos\omega\tau + \omega\delta\sin\omega\tau = \omega^2 - a_1 \tag{9}$$

$$\omega\delta\cos\omega\tau - b_0\sin\omega\tau = -a_0 \tag{10}$$

From the above two Equations (9) and (10) we get

$$w^4 + (a_0^2 - 2a_1 - \delta^2)\omega^2 + (a_1^2 - b_0^2) = 0 \tag{11}$$

Let $\omega^2 = z_0$, then from the above Equation (11) we get,

$$z_0^2 + (a_0^2 - 2a_1 - \delta^2)z_0 + (a_1^2 - b_0^2) = 0 \tag{12}$$

Let $A_1 = a_0^2 - 2a_1 - \delta^2,\ A_2 = a_1^2 - b_0^2$, then Equation (12) is reduced to the following equation:

$$z_0^2 + A_1 z_0 + A_2 = 0.$$

Here,

$$\begin{aligned} A_1 &= a_0^2 - 2a_1 - \delta^2, \\ &= \left(\alpha - \left(\beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u\right)\right)^2 - 2\left(\alpha\left(\beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u\right)\right) - \delta^2. \\ A_2 &= a_1^2 - b_0^2, \\ &= \left(\alpha\left(\beta\left(\overline{N} - (I^* + A^*)\right) - \beta I^* - \mu u\right)\right)^2 - \left(\alpha\delta + \delta(\beta I^* + \mu u)\right)^2. \end{aligned}$$

If we get both conditions $A_1 > 0$ and $A_2 > 0$, then we get that the roots of Equation (7) will have negative real parts by Routh-Hurwitz Criterion. Thus, for all $\tau \geq 0$, the equilibrium point obtained is locally asymptotically stable.

4.3 Hopf Bifurcation Analysis

In this subsection, we establish the existence of Hopf bifurcation when the equilibrium E^* loses its stability. We consider the delay parameter τ to be a bifurcation parameter. If $A_2 < 0$, then Equation (12) has a positive root, thereby implying that the characteristic Equation (7) has at least one root with a positive real part. WLOG, assume that Equation (12) has two positive roots z_{01}, z_{02}. Let $w_{0i} = \sqrt{z_{0i}}$ $(i = 1,2)$.
From Equations (9) and (10), we obtain

$$\tau_j^i = \frac{1}{\omega_{0i}} arccos \frac{(b_0 - a_0\delta)\omega_{0i}^2 - a_1 b_0}{(\omega_{0i}\delta)^2 + b_0^2} + \frac{2j\pi}{\omega_{0i}} \tag{13}$$

Let $\tau^* = min\{\tau_0^i, i = 1,2, ...\}$ and let $\lambda = \pm i\omega_0$ with the corresponding root of Equation (7). Now taking derivative of Equation (7) with respect to τ, we get,

$$\frac{2\lambda + a_0}{e^{-\lambda\tau}(\delta\lambda + b_0)\lambda} + \frac{\delta}{(\delta\lambda + b_0)\lambda} - \frac{\tau}{\lambda} = \left(\frac{d\lambda}{d\tau}\right)^{-1}.$$

Therefore,

$$\frac{2\lambda + a_0}{-\lambda(\lambda^2 + a_0\lambda + a_1)} + \frac{\delta}{(\delta\lambda + b_0)\lambda} - \frac{\tau}{\lambda} = \left(\frac{d\lambda}{d\tau}\right)^{-1} \tag{14}$$

Now substituting $\lambda = i\omega_0$ in Equation (14), we get

$$\begin{aligned}
Re\left[\frac{d\lambda}{d\tau}\right]^{-1} &= Re\left(\frac{2i\omega_0 + a_0}{-i\omega_0(-\omega_0^2 + a_0 i\omega_0 + a_1)} + \frac{\delta}{(i\omega_0\delta + b_0)i\omega_0} - \frac{\tau}{i\omega_0}\right),\\
&= Re\left(\frac{1}{\omega_0}\left(\frac{2i\omega_0 + a_0}{(\omega_0^2 - a_1)i + a_0\omega_0} + \frac{\delta}{(-\omega_0\delta + ib_0)} - \frac{\tau}{i}\right)\right),\\
&= Re\left(\frac{1}{\omega_0}\left(\frac{2i\omega_0 + a_0}{(\omega_0^2 - a_1)i + a_0\omega_0} \times \frac{(\omega_0^2 - a_1)i - a_0\omega_0}{(\omega_0^2 - a_1)i - a_0\omega_0}\right) + \left(\frac{\delta}{(-\omega_0\delta + ib_0)} \times \frac{(-\omega_0\delta - ib_0)}{(-\omega_0\delta - ib_0)} + i\tau\right)\right),\\
&= \frac{1}{\omega_0}\left(\frac{2\omega_0(\omega_0^2 - a_1) + a_0^2\omega_0}{(\omega_0^2 - a_1)^2 + (a_0\omega_0)^2} - \frac{\delta^2\omega_0}{(\delta\omega_0)^2 + b_0^2}\right),\\
&= \frac{2(\omega_0^2 - a_1) + a_0^2}{(\omega_0^2 - a_1)^2 + (a_0\omega_0)^2} - \frac{\delta^2}{(\delta\omega_0)^2 + b_0^2}.
\end{aligned}$$

Now, using Equations (9) and (10) in the above result, we get

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2\omega_0^2 - 2a_1 + a_0^2 - \delta^2}{b_0^2 + \omega_0^2\delta^2}.$$

Hence, $\frac{d(Re(\lambda))}{d\tau} > 0$ if $H_{\mathbb{1}}: (2\omega_0^2 - 2a_1 + a_0^2 - \delta^2) > 0$ and the transversality condition is satisfied. Thus, the following result holds.

Theorem 2. If the assumption $A_2 < 0$ and the condition $H_{\mathbb{1}}$ holds, then the positive equilibrium point E^* of system (5) is locally asymptotically stable for $\tau \in [0, \tau^*)$ and unstable when $\tau > \tau^*$. Also, the system experiences Hopf bifurcation at $\tau = \tau^*$.

Remark 2. If the value of delay is less than the critical value, then the equilibrium point is stable, which means that the number of adopters can be predicted, which will aid in the launch of the appropriate number of recent products in the market so that the company will not incur any losses. If the value of the delay goes beyond the critical value, then due to the oscillatory nature of the solution, the number of adopters cannot be predicted accurately. This will make it difficult for the company to assess how many recent products to launch in the market. As a result, there should be enhanced promotional initiatives that highlight the benefits of adopting the recent product.

4.4 Direction and Stability of Hopf Bifurcation

Proceeding further from what we have established from the above analysis, that the system (5) undergoes a Hopf bifurcation when $\tau = \tau^*$, we will examine the direction of Hopf bifurcation around the equilibrium point E^* in this subsection and subsequently investigate the conditions for stability of the bifurcating periodic solutions. For this, we have used the normal form and center manifold theorem discussed in Hassard et al. (1981).

Let $y(t) = (I(t) - I^*,\ A(t) - A^*)^T \in R^2,\ \tau = \tau^* + \xi,\ \xi \in R$. Then the system (5) can be transformed into the following functional differential equation of the form $\left(\text{in } C = C\left([-\tau, 0], R^2\right)\right)$:

$$\dot{y}(t) = L_\xi(y_t) + F(\xi, y_t) \tag{15}$$

where, $L_\xi : C \to R^2,\ F: R \times C \to R^2$ are defined, respectively, by

$$L_\xi(\psi) = M_1\psi(0) + M_2\psi(-\tau) \tag{16}$$

$M_1 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, M_2 = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix}$, and

$F(\psi, \xi) = \begin{pmatrix} f_1(\psi, \xi) \\ f_2(\psi, \xi) \end{pmatrix}$,

where, $f_1 = \rho_1\psi_1^2(0) + \rho_2\psi_1(0)\psi_2(0)$,

$f_2 = 0$,

Here, $\rho_1 = -\beta,\ \rho_2 = -\beta$,

$a_{11} = \beta\left(\overline{N} - 2I^* - A^*\right) - \mu u,\ a_{12} = -\beta I^* - \mu u$,

$a_{22} = -\alpha,\ \ b_{11} = -\delta,\ b_{21} = \delta$.

According to Riesz representation theorem, there exists a 2×2 matrix function $\eta(\varpi, \xi): [-\tau, 0] \to R^{2\times 2}$ whose elements are of bounded variations such that

$$L_\xi(\psi) = \int_{-\tau}^{0} d\eta(\varpi, \xi)\psi(\varpi) \tag{17}$$

for all $\psi \in C$. Here, we can choose

$\eta(\varpi, \xi) = M_1\delta(\varpi) - M_2\delta(\varpi + \tau)$,

where, δ is the Dirac delta function.

For any $\psi \in C' = C'\left([-\tau, 0],\ R^2\right)$, define

$$G(\xi)\psi(\varpi) = \begin{cases} \dfrac{d\psi(\varpi)}{d\varpi},\ \varpi \in [-\tau, 0) \\ \int_{-\tau}^{0}[d\eta(\varpi, \xi)]\psi(\varpi),\ \varpi = 0 \end{cases} \tag{18}$$

and

$$R(\xi)\psi(\varpi)=\begin{cases}0, & \varpi\in[-\tau,0)\\ F(\psi,\xi), & \varpi=0\end{cases} \tag{19}$$

Then system (15) can be rewritten as the following operator equation:

$$\dot{y_t}=G(\xi)y_t+R(\xi)y_t \tag{20}$$

where, $y_t=y(t+\varpi),\ \varpi\in[-\tau,0]$.

Next, for $\phi\in C'\left([-\tau,0],\left(R^{2}\right)^*\right)$, the adjoint operator G^* of G is defined as

$$G^*(0)\phi(s)=\begin{cases}-\frac{d\phi(s)}{ds}, & s\in(0,\tau],\\ \int_{-\tau}^{0}d\eta^T(s,0)\phi(-s), & s=0,\end{cases} \tag{21}$$

We now define a bilinear inner product for $\psi\in C'\left([-\tau,0],R^{2}\right)$ and $\phi\in C'\left([-\tau,0],\left(R^{2}\right)^*\right)$ as follows:

$$<\phi(s),\psi(\varpi)\geq\overline{\phi}^T(0)\psi(0)-\int_{\varpi=-\tau}^{0}\int_{s=0}^{\varpi}\overline{\phi}^T(s-\varpi)d\eta(\varpi)\psi(s)\,ds \tag{22}$$

where, $\eta(\varpi)=\eta(\varpi,0)$ and $G=G(0)$ and G^* are adjoint operators with respect to the bilinear product. We note that $\pm i\omega_0$ are eigenvalues of $G(0)$ as well as G^*.

Now, suppose we have $r(\varpi)=(1,d_1)^Te^{i\omega_0\varpi}$ as the eigenvector of $G(0)$ corresponding to eigenvalue $i\omega_0$. Then, $G(0)r(\varpi)=i\omega_0r(\varpi)$ for all $\varpi\in[-\tau,0]$. For $\varpi=0$, we use Equations (16), (17) and (18) to obtain the following equation:

$$M_1r(0)+M_2r(-\tau^*)=i\omega_0r(0).$$

which gives $d_1=\frac{i\omega_0-a_{11}-b_{11}e^{-i\omega_0\tau^*}}{a_{12}}$.

In a similar manner, we suppose $r^*(s)=\frac{1}{B}(1,d_1^*)^Te^{i\omega_0s}$ as the eigenvector of G^* corresponding to eigenvalue $-i\omega_0$ and we have

$$M_1^Tr^*(0)+M_2^Tr^*(\tau^*)=-i\omega_0r^*(0) \tag{23}$$

On solving the above Equation (23), we get the value of d_1^* as

$$d_1^*=\frac{-i\omega_0-a_{11}-b_{11}e^{-i\omega_0\tau^*}}{b_{21}e^{-i\omega_0\tau^*}}.$$

Here, to ensure that $\langle r^*,r\rangle=1$, we will determine the value of $\overline{B}$ using Equation (22),

$$\langle r^*,r\rangle=\overline{r^*}^T(0)r(0)-\int_{\varpi=-\tau^*}^{0}\int_{s=0}^{\varpi}\overline{r^*}^T(s-\varpi)d\eta(\varpi)r(s)ds,$$

$$=\frac{1}{\overline{B}}\left(1,\overline{d_1^*}\right)(1,d_1)^T-\int_{\varpi=-\tau^*}^{0}\int_{s=0}^{\varpi}\frac{1}{\overline{B}}\left(1,\overline{d_1^*}\right)e^{-i\omega_0(s-\varpi)}\,d\eta(\varpi)(1,d_1)^Te^{i\omega_0s}ds,$$

$$=\frac{1}{\overline{B}}\left[\left(1+d_1\overline{d_1^*}\right)-\int_{\varpi=-\tau^*}^{0}\left(1,\overline{d_1^*}\right)\varpi e^{i\omega_0\varpi}\,d\eta(\varpi)(1,d_1)^T\right],$$

$$= \frac{1}{\overline{B}}\left[\left(1 + d_1\overline{d_1^*}\right) + \tau^*\left(b_{11} + b_{21}\overline{d_1^*}\right)e^{-i\omega_0\tau^*}\right].$$

Thus, by taking $\overline{B} = 1 + d_1\overline{d_1^*} + \tau^*\left(b_{11} + b_{21}\overline{d_1^*}\right)e^{-i\omega_0\tau^*}$, we obtain $\langle r^*(s), r(\varpi)\rangle = 1$.

Next, we will use the same notations as given in Hassard et al. (1981) to compute the coordinates of the center manifold C_0 at $\xi = 0$. Let y_t be the solution of Equation (20) when $\xi = 0$. Define

$$c(t) = \langle r^*, y_t\rangle \tag{24}$$

and

$$W(t,\varpi) = y_t - cr - \overline{c}\overline{r} = y_t - 2Re\left(c(t)r(\varpi)\right) \tag{25}$$

On the center manifold, we have $W(t,\varpi) = W(c(t),\overline{c}(t),\varpi)$, where,

$$W(c(t),\overline{c}(t),\varpi) = W_{20}(\varpi)\frac{c^2}{2} + W_{11}(\varpi)c\overline{c} + W_{02}(\varpi)\frac{\overline{c}^2}{2} + \cdots \tag{26}$$

From the above, c and $\overline{c}$ are local coordinates for the center manifold C_0 in the direction of r^* and $\overline{r^*}$ respectively. We observe that $W(t,\varpi)$ is real if y_t is real. So, we may consider only real solutions. From Equations (24) and (25), we have

$$\langle r^*, W\rangle = \langle r^*, y_t\rangle - c(t)\langle r^*, r\rangle - \overline{c}(t)\langle r^*, \overline{r}\rangle = 0.$$

For a solution $y_t \in C_0$ of Equation (20), with $\xi = 0$, Equations (18), and (21) yield

$$\begin{aligned} \dot{c}(t) &= \langle r^*, \dot{y}_t\rangle = \langle r^*, G(0)y_t + R(0)y_t\rangle, \\ &= \langle G^*r^*, y_t\rangle + \overline{r^*}^T(0)F(0,y_t), \\ &= i\omega_0 c(t) + \overline{r^*}^T(0)f_0\left(c(t),\overline{c}(t)\right), \end{aligned} \tag{27}$$

where, the abbreviated form of the above equation is given as

$$\dot{c}(t) = i\omega_0 c(t) + g(c,\overline{c}) \tag{28}$$

and

$$g(c,\overline{c}) = g_{20}\frac{c^2}{2} + g_{11}c\overline{c} + g_{02}\frac{\overline{c}^2}{2} + g_{21}\frac{c^2\overline{c}}{2} + \cdots \tag{29}$$

On substituting Equations (20) and (27) in $\dot{W} = \dot{y_t} - \dot{c}r - \dot{\overline{c}}\overline{r}$, we get

$$\dot{W} = \begin{cases} G(0)W - 2Re\left(\overline{r^*}(0)f_0(c,\overline{c})r(\varpi)\right), & \varpi \in [-\tau, 0); \\ G(0)W - 2Re\left(\overline{r^*}(0)f_0(c,\overline{c})r(\varpi)\right) + f_0(c,\overline{c}), & \varpi = 0. \end{cases}$$

$$= G(0)W + Q(c,\overline{c},\varpi) \tag{30}$$

where,

$$Q(c,\overline{c},\varpi) = Q_{20}(\varpi)\frac{c^2}{2} + Q_{11}(\varpi)c\overline{c} + Q_{02}(\varpi)\frac{\overline{c}^2}{2} + \cdots \tag{31}$$

On the center manifold C_0, we have

$$\dot{W} = \dot{W}_c\dot{c} + \dot{W}_{\overline{c}}\dot{\overline{c}} \tag{32}$$

Substituting Equations (24), (25), and (28) in Equation (32) and comparing the coefficients with Equation (30), we get

$$(G(0)-2i\omega_0)W_{20}(\varpi)=-Q_{20}(\varpi),$$
$$G(0)W_{11}=-Q_{11}(\varpi) \quad (33)$$
$$(G(0)+2i\omega_0)W_{02}=-Q_{02}(\varpi).$$

Since $y_t=y(t+\varpi)=W(c,\overline{c},\varpi)+cr+\overline{c}\,\overline{r}$, we have

$$y_1(t+\varpi)=W^{(1)}(t+\varpi)+ce^{i\omega_0\varpi}+\overline{c}e^{-i\omega_0\varpi},$$
$$y_2(t+\varpi)=W^{(2)}(t+\varpi)+cd_1e^{i\omega_0\varpi}+\overline{c}d_1^*e^{-i\omega_0\varpi} \quad (34)$$

where,

$$f_1(0,y_t)=\rho_1I_t^2(0)+\rho_2I_t(0)A_t(0),$$
$$f_2(0,y_t)=0.$$

Now, we have

$$f_0(c,\overline{c})=\begin{pmatrix}\mathcal{V}_{11}c^2+\mathcal{V}_{12}c\overline{c}+\mathcal{V}_{13}\overline{c}^2+\mathcal{V}_{14}c^2\overline{c}\\ \mathcal{V}_{21}c^2+\mathcal{V}_{22}c\overline{c}+\mathcal{V}_{23}\overline{c}^2+\mathcal{V}_{24}c^2\overline{c}\end{pmatrix}+\cdots,$$

where,

$$\mathcal{V}_{11}=\rho_1+\rho_2 d_1,$$
$$\mathcal{V}_{12}=2\rho_1+\rho_2\left(d_1+\overline{d_1}\right),$$
$$\mathcal{V}_{13}=\rho_1+\rho_2\overline{d_1},$$
$$\mathcal{V}_{14}=2\rho_1W_{11}^{(1)}(0)+\rho_2\left(W_{11}^{(1)}d_1+W_{11}^{(2)}(0)\right).$$

Since, $\overline{r^*}(0)=\frac{1}{\overline{B}}\left(1,\overline{d_1^*}\right)^T$, we get

$$\begin{aligned}g(c,\overline{c})&=\overline{r^*}^T(0)f_0(c,\overline{c}),\\ &=\frac{1}{\overline{B}}\left(1,\overline{d_1^*}\right)\begin{pmatrix}\mathcal{V}_{11}c^2+\mathcal{V}_{12}c\overline{c}+\mathcal{V}_{13}\overline{c}^2+\mathcal{V}_{14}c^2\overline{c}\\ \mathcal{V}_{21}c^2+\mathcal{V}_{22}c\overline{c}+\mathcal{V}_{23}\overline{c}^2+\mathcal{V}_{24}c^2\overline{c}\end{pmatrix},\\ &=\frac{1}{\overline{B}}\left[\left(\mathcal{V}_{11}+\mathcal{V}_{21}\overline{d_1^*}\right)c^2+\left(\mathcal{V}_{12}+\mathcal{V}_{22}\overline{d_1^*}\right)c\overline{c}+\left(\mathcal{V}_{13}+\mathcal{V}_{23}\overline{d_1^*}\right)\overline{c}^2+\left(\mathcal{V}_{14}+\mathcal{V}_{24}\overline{d_1^*}\right)c^2\overline{c}\right].\end{aligned}$$

Comparing the coefficients of the above equation with those in Equation (29), we get

$$g_{20}=\frac{2}{\overline{B}}\mathcal{V}_{11},\ g_{11}=\frac{1}{\overline{B}}\mathcal{V}_{12},\ g_{02}\frac{2}{\overline{B}}\mathcal{V}_{13},\ g_{21}=\frac{2}{\overline{B}}\mathcal{V}_{14}.$$

On following the same procedures as in Guo et al. (2010), we obtain

$$W_{20}(\varpi)=\frac{ig_{20}}{\omega_0}r(0)e^{i\omega_0\varpi}+\frac{i\overline{g}_{02}}{3\omega_0}\overline{r}(0)e^{-i\omega_0\varpi}+\mathcal{E}_1e^{2i\omega_0\varpi},$$

and

$W_{11}(\varpi) = -\frac{ig_{11}}{\omega_0} r(0)e^{i\omega_0\varpi} + \frac{i\overline{g}_{11}}{\omega_0}\overline{r}(0)e^{-i\omega_0\varpi} + \mathcal{E}_2.$

where, $\mathcal{E}_1 = \left(\mathcal{E}_1^{(1)}, \mathcal{E}_1^{(2)}\right)$ and $\mathcal{E}_2 = \left(\mathcal{E}_2^{(1)}, \mathcal{E}_2^{(2)}\right)$ are two-dimensional vectors such that

$\left(2i\omega_0 I_2 - \int_{-\tau^*}^{0} e^{2i\omega_0\varpi}\, d\eta(\varpi)\right)\mathcal{E}_1 = (\mathcal{V}_{11}, 0)^T,$

$\left(\int_{-\tau^*}^{0} d\eta(\varpi)\right)\mathcal{E}_2 = -(\mathcal{V}_{12}, 0)^T.$

Hence, we get the following

$$\begin{pmatrix} 2i\omega_0 - a_{11} - b_{11}e^{-2i\omega_0\tau^*} & -a_{12} \\ -b_{21}e^{-2i\omega_0\tau^*} & 2i\omega_0 - a_{22} \end{pmatrix}\begin{pmatrix} \mathcal{E}_1^{(1)} \\ \mathcal{E}_1^{(2)} \end{pmatrix} = \begin{pmatrix} \mathcal{V}_{11} \\ 0 \end{pmatrix} - \begin{pmatrix} a_{11} + b_{11} & a_{12} \\ b_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \mathcal{E}_2^{(1)} \\ \mathcal{E}_2^{(2)} \end{pmatrix} = \begin{pmatrix} \mathcal{V}_{12} \\ 0 \end{pmatrix}.$$

Based on the above linear systems and from our analysis, we can obtain each $g_{20}, g_{11}, g_{02}, g_{21}$ in terms of the parameters and delay in system (5). Therefore, we compute the following quantities:

$$C_1(0) = \frac{1}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 + \frac{g_{21}}{2},$$

$$v_2 = -\frac{Re[C_1(0)]}{Re[\lambda'(\tau^*)]},$$

$$\beta_2 = 2Re[C_1(0)] \tag{35}$$

$$T_2 = -\frac{Im[C_1(0)] + v_2 Im[\lambda'(\tau^*)]}{\omega_0}.$$

From the above results, we get the following theorem:

Theorem 3. For the system (5), according to Equation (35), we form the conclusions,

i. The sign of v_2 determines the direction of Hopf bifurcation, i.e., if $v_2 > 0 (v_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical), and the bifurcating periodic solutions exist for $\tau > \tau^* (\tau < \tau^*)$.
ii. The sign of β_2 determines the stability of the bifurcating periodic solutions, i.e., if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable).
iii. If $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increases (decreases), as the sign of T_2 determines the period of the bifurcating periodic solutions.

4.5 Global Stability of the Equilibrium Point E^*

In this section, we will construct a suitable Lyapunov function to establish certain conditions required for ensuring the global stability of the unique equilibrium point E^* of the system (5).

Firstly, we consider a positive definite function V_1 given by

$$V_1(I, A) = \frac{(I - I^*)^2}{2} + \frac{(A - A^*)^2}{2} \tag{36}$$

It can be noted that $V_1(I^*, A^*) = 0$ and it is positive otherwise.

Differentiating Equation (36) along the solution of system (5), we get

$$\frac{dV_1}{dt} = (I - I^*)\frac{dI}{dt} + (A - A^*)\frac{dA}{dt},$$

$$= \beta\overline{N}(I-I^*)^2 + \beta(I+I^*)(I^*-I)(I-I^*) + \beta I(A^*-A)(I-I^*) + \beta A^*(I^*-I)(I-I^*) - \mu u(I-I^*)^2 - \mu u(A-A^*)(I-I^*) - \delta(I-I^*)^2 + (A-A^*)\delta(I-I^*) - \alpha(A-A^*)^2.$$

equivalently, we have

$\leq \frac{7}{2}\beta\overline{N}(I-I^*)^2 + \beta A^*(I-I^*)^2 + \frac{\beta\overline{N}}{2}(A-A^*)^2 - \frac{\mu u}{2}(I-I^*)^2 + \frac{\mu u}{2}(A-A^*)^2 - \delta(I-I^*)^2 + \frac{\delta}{2}(I-I^*)^2 + \frac{\delta}{2}(A-A^*)^2 - \alpha(A-A^*)^2.$

Now, on applying Cauchy-Schwartz inequality and then rearranging the similar terms, we get

$\frac{dV_1}{dt} \leq -\left[\frac{\mu u}{2} + \frac{\delta}{2} - \frac{7}{2}\beta\overline{N} - \beta A^*\right](I-I^*)^2 - \left[\alpha - \frac{\beta\overline{N}}{2} - \frac{\mu u}{2} - \frac{\delta}{2}\right](A-A^*)^2.$

$\aleph_4: \frac{\mu u}{2} + \frac{\delta}{2} > \frac{7}{2}\beta\overline{N} + \beta A^*$ and $\alpha > \frac{\beta\overline{N}}{2} + \frac{\mu u}{2} + \frac{\delta}{2}$.

Thus, we have the following theorem:

Theorem 4. For $\tau = 0$, if the condition $\aleph_4$ holds, then the equilibrium point E^* of system (5) is globally asymptotically stable.

Now, in order to take into account the effects of the past states that arise due to the presence of time delay, we will construct an appropriate Lyapunov functional and obtain the conditions for global stability of the equilibrium point E^* of system (5) for $\tau \neq 0$.

We choose a Lyapunov functional of the form

$$V_2(I,A) = \frac{(I-I^*)^2}{2} + \frac{(A-A^*)^2}{2} + r_1\int_{t-\tau^*}^{t}(I(v)-I^*)^2\,dv \tag{37}$$

or

$$V_2(I,A) = V_1 + r_1\int_{t-\tau^*}^{t}(I(v)-I^*)^2\,dv \tag{38}$$

where, $r_1 > 0$ is a positive constant selected and given a suitable value later.

Differentiating Equation (37) along the solution of system (5), we have

$\frac{dV_2}{dt} = \dot{V}_1 + r_1(I(t)-I^*)^2 - r_1(I(t-\tau^*)-I^*)^2,$

$\leq \left[\beta\overline{N} - \beta A^* - \mu u\right](I-I^*)^2 + \left[\beta\overline{N} + \mu u\right](A^*-A)(I-I^*) - \delta(I(t-\tau^*)-I^*)(A^*-A) - \alpha(A^*-A)^2 + r_1(I-I^*)^2 - r_1(I(t-\tau^*)-I^*)^2.$

By taking $r_1 = \mu u$, and using Cauchy-Schwartz inequality, we get

$\frac{dV_2}{dt} \leq -\left(\beta A^* - \frac{3\beta\overline{N}}{2} - \frac{\mu u}{2} - \frac{\delta}{2}\right)(I-I^*)^2 - \left(\alpha - \frac{\beta\overline{N}}{2} - \frac{\mu u}{2} - \frac{\delta}{2}\right)(A-A^*)^2 - (\mu u - \delta)(I(t-\tau)-I^*)^2.$

$\aleph_5: \beta A^* > \frac{3\beta\overline{N}}{2} + \frac{\mu u}{2} + \frac{\delta}{2}, \alpha > \frac{\beta\overline{N}}{2} + \frac{\mu u}{2} + \frac{\delta}{2}$ and $\mu u > \delta$.

Then, we have the following theorem:

Theorem 5. For $\tau \neq 0$, if the condition $\aleph_5$ holds, then the equilibrium point E^* of system (5) is globally asymptotically stable.

Remark 3. If the system is globally asymptotically stable, irrespective of wherever you begin with in the domain of potential values of population classes (unaware, aware, and adopters), the system always will reach a common stable state, that is, this guarantees marketers that, no matter what the initial rate of adoption or level of awareness or in presence of delay, the system will stabilize and produce a certain number of adopters.

5. Optimal Control Analysis

Optimal control theory in marketing can be utilized to facilitate decisions that enhance the efficacy of marketing strategies, maximizing key performance metrics like revenues, and cost minimization of resource utilization. Continuous optimal control theory provides a powerful tool for understanding the behavior of marketing systems where dynamic aspects play an important role (Sethi and Thompson, 2000).

In this section, we extend our model (1) to an optimal control problem concerning how information about the innovation or product might be disseminated over time among the population with the aim of maximizing the number of adopters and minimizing the cost associated with promotional efforts in our system. Hence in this problem, we use Pontryagin's maximum principle (Pontryagin, 2018) in order to identify necessary conditions for the optimal control of the innovation diffusion model we have mentioned in Equation (1). For this optimal control formulation, we have considered the proposed model (1) without delay. This modelling choice is commonly adopted in modern literatures (see the works of (Wang et al., 2006) for further reference), and the non-delayed control framework allows us to derive explicit optimality conditions and obtain clear managerial insights, while the effects of time delay on system dynamics has been analysed separately in the stability and bifurcation results.

With the aim to 1). maximize the number of adopters i.e. $C_1A(t)$ and 2). minimize the promotional effort cost i.e. $\frac{C_2}{2}u^2(t)$ in planing horizon $[0,T]$. The Objective functional J which we have accounted for is given below:

$$J(u) = \int_0^T \{ - C_1A(t) + \frac{C_2}{2}u^2(t)\}dt \tag{39}$$

subject to the system of equations

$$\begin{cases} \frac{dN}{dt} = -\beta I\, N - \mu u(t)\, N + \alpha A, \\ \frac{dI}{dt} = \beta I\, N + \mu u(t)\, N - \delta I, \\ \frac{dA}{dt} = \delta I - \alpha A. \end{cases} \tag{40}$$

We have to acquire an optimal control u^* such that

$$J(u^*) = \min \{J(u), u \in \mathfrak{U}\} \tag{41}$$

wherein the control set $\mathfrak{U}$ is defined as

$$\mathfrak{U} = \{u | 0 \leq u(t) \leq 1,\ t \in [0,T]\} \tag{42}$$

The control function $u(t)$ is a Lebesgue integrable function.

As stated initially, the optimal control problem is solved by using Pontragyin's maximum principle (Pontragyin, 2018) and the following theorems below provide the derivation of optimality system and establishes the existence and the characteristics of the optimal control.

5.1 Existence of Deterministic Optimal Control

Now, drawing upon the results from Fleming and Rishel's work (Fleming and Rishel, 2012), we can provide proof for the existence of optimal control, refer to the theorem and proof stated below.

Theorem 6. Consider the optimal control problem (39) subject to the control system of Equation (40) with the initial conditions at $t = 0$. Then there is existence of an optimal control $u^* \in \mathfrak{U}$ on a fixed interval $[0, T]$ and optimal states (N^*, I^*, A^*) that minimizes the objective function $J(u)$, such that, $J(u^*) = \min\{J(u), u \in \mathfrak{U}\}$.

Proof. The existence of solution of the control system is ensured by the boundedness of the solution to system (40). It is evident that the state variables are positive and by definition, the control set $\mathfrak{U}$ is non-empty, closed, and convex. Therefore, it follows that the optimal system (39)-(40) is bounded, thus ensuring the compactness necessary for the existence of an optimal control. The integrand function J is a convex function of the control $u(t)$. Hence, it has been demonstrated that all the conditions for the optimal control to exist have been satisfied.

5.2 Characterization of Optimal Control

The optimal control problem (39)-(40) is characterized by a Hamiltonian function $\mathcal{H}$ which is formulated in the following ways:

$$\mathcal{H} = -C_1 A + \frac{C_2}{2} u^2(t) + \lambda_1[-\beta I N - \mu u(t) N + \alpha A] + \lambda_2[\beta I N + \mu u(t) N - \delta I] + \lambda_3[\delta I - \alpha A] \quad (43)$$

where, $\lambda_i = 1,2,3, \ldots$ are the adjoint variables.

Theorem 7. For the given the optimal state solutions N, I, and A, and the optimal control u^* of the corresponding state system (40) that minimizes $J(u)$ over $\mathfrak{U}$, there exists adjoint variables λ_1, λ_2, and λ_3 satisfying

$$\frac{d\lambda_i}{dt} = -\frac{\partial \mathcal{H}}{\partial Y}, \text{where}, Y = N, I, A \quad (44)$$

with the conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0 \quad (45)$$

and optimal control u^* is given by

$$u^* = \max\left\{0, \min\left(1, \frac{1}{C_2}(\lambda_1 - \lambda_2)\mu N\right)\right\} \quad (46)$$

Proof. The existence of the optimal control problem following the results of Fleming and Rishel's work (Fleming and Rishel, 2012) has been established as the integrand function J is a convex function of control u(t). On differentiating the Hamiltonian function $\mathcal{H}$, the adjoint system can be outlined as below:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial \mathcal{H}}{\partial N} = (\lambda_1 - \lambda_2)\big(\beta I + \mu u(t)\big), \\
\frac{d\lambda_2}{dt} &= -\frac{\partial \mathcal{H}}{\partial I} = (\lambda_1 - \lambda_2)\beta N + (\lambda_2 - \lambda_3)\delta, \\
\frac{d\lambda_3}{dt} &= -\frac{\partial \mathcal{H}}{\partial A} = (\lambda_3 - \lambda_1)\alpha + C_1.
\end{aligned} \quad (47)$$

with transversality conditions $\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0$.

The uniqueness of the optimal control for small values of T is obtained in accordance with the antecedent established boundedness of the state system, adjoint system, and the resulting Lipschitz structure of the ordinary differential equations. The uniqueness of the optimal control is derived from the uniqueness of the optimality system (40), which includes Equations (44), (45), and (46). Next, for optimality, $\frac{\partial \mathcal{H}}{\partial u} = 0$. We then differentiate $\mathcal{H}$ with respect to control u and equate to zero as follows:

$$\frac{\partial \mathcal{H}}{\partial u} = C_2 u(t) - \lambda_1 \mu N + \lambda_2 \mu N = 0 \tag{48}$$

Now, we obtain the solution,

$$u^*(t) = \frac{1}{C_2}(\lambda_1 - \lambda_2)\mu N.$$

Consequently, through optimal control problem with constraints on the optimal control u^*, it is verified that u^* satisfies the conditions as follows:

$$u^* = \begin{cases} 0, & \frac{1}{C_2}(\lambda_1 - \lambda_2)\mu N \leq 0 \\ \frac{1}{C_2}(\lambda_1 - \lambda_2)\mu N, & 0 < \frac{1}{C_2}(\lambda_1 - \lambda_2)\mu N < 1 \\ 1, & \frac{1}{C_2}(\lambda_1 - \lambda_2)\mu N \geq 1 \end{cases} \tag{49}$$

Thus, the optimal control is characterized in the compact form as

$$u^* = \max\left\{0\left(1, \frac{1}{C_2}(\lambda_1 - \lambda_2)\mu N\right)\right\}.$$

6. Numerical Simulation

In this section, we analyze the behavior of system (5) by plotting graphs using MATLAB to corroborate our analytical outcomes. The initial conditions taken are $(N_0, I_0, A_0) = (10, 5, 5)$ and the value of the parameters taken are listed in **Table 1** below. For the value of parameters listed in **Table 1**, the equilibrium point E^* of system (5) is $(0.58, 0.0726, 19.347)$.

Table 1. Parametric values.

Parameter	Value	Units
$\bar{N}$	20	person
α	0.003	$(\text{day})^{-1}$
β	0.001	$(\text{person})^{-1}(\text{day})^{-1}$
δ	0.8	$(\text{day})^{-1}$
μ	0.1	$(\text{person})(\text{day})^{-1}$
u	1	$(\text{person})^{-1}$

For non-delayed system, where $\tau = 0$, the system (5) satisfies Routh-Hurwitz criteria with conditions $a_0 + \delta = 0.902 > 0$ and $a_1 + b_0 = 0.083 > 0$, and therefore the equilibrium point E^* of system (5) is stable (**Figure 2**). For the delayed system, that is, when $\tau \neq 0$ we get $A_2 = a_1^2 - b_0^2 = -0.0067 < 0$, then we have a unique positive root of Equation (11) as $\omega = 0.8004$. For this value of ω and using Equation (13), the critical value of delay evaluated is $\tau^* = 1.96$. Now, for $\tau = 1.8 < \tau^* = 1.96$, the equilibrium points of system (5) is stable (**Figure 3**). For $\tau = \tau^* = 1.96$, the system loses its stability and a family of periodic solutions arises (**Figures 3** and **4**). Thus, the system (5) experiences Hopf bifurcation around equilibrium point E^*.

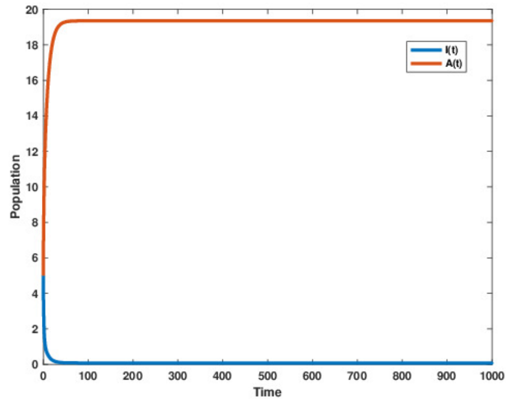
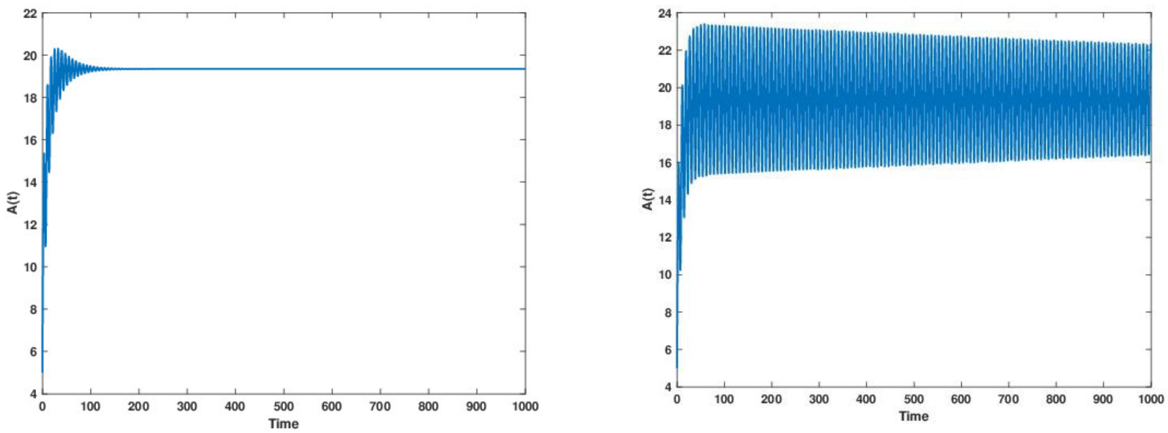


Figure 2. System behavior when delay is not present ($\boldsymbol{\tau = 0}$).



$\tau = 1.8 < \tau^* = 1.96$ $\tau = \tau^* = 1.96$

Figure 3. System behavior when delay is present.

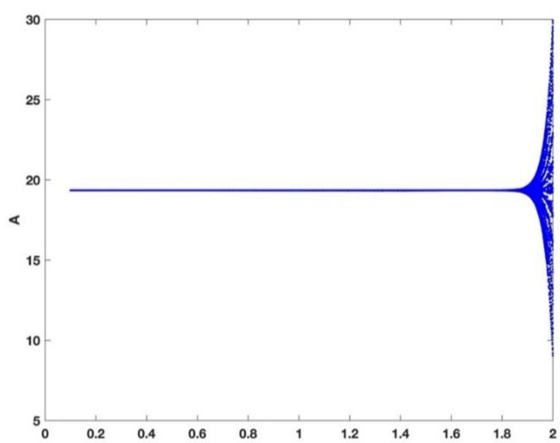


Figure 4. Plot of bifurcation diagram.

6.1 Sensitivity Analysis of A^*

In this subsection, we perform a sensitivity analysis on the number of adopters at equilibrium level, we have already obtained A^* in the previous subsection as

$$A^* = \frac{\delta I^*}{\alpha},$$

$$A^* = \frac{\delta\left(\alpha\beta\overline{N} - \alpha\mu u - \mu u\delta - \alpha\delta\right) + \sqrt{\left(\alpha\beta\overline{N} - \alpha\mu u - \mu u\delta - \alpha\delta\right)^2 + 4(\alpha\beta + \beta\delta)\alpha\mu u\overline{N}}}{2\alpha(\alpha\beta + \beta\delta)}.$$

On substituting I^* in the above equation, we note that A^* is a function of six parameters, namely, $\alpha,\ \beta,\ \delta,$ μ, u and $\overline{N}$. Sensitivity analysis aids in observing how changes in predictor values affect the system's dynamic behavior, and in this case, we study the impact of each parameter on the number of adopters; that is, we are able to see the sensitiveness of the number of adopters to the alteration of the mentioned parameters. The computation of sensitivity indices of A^* with respect to the mentioned parameters is obtained using the formulas given in (Samsuzzoha et al., 2013) and which are mentioned below. $\chi_\alpha = \frac{\partial A^*}{\partial \alpha}\frac{\alpha}{A^*};\ \chi_\beta = \frac{\partial A^*}{\partial \beta}\frac{\beta}{A^*};\ \chi_\delta = \frac{\partial A^*}{\partial \delta}\frac{\delta}{A^*};\ \chi_\mu = \frac{\partial A^*}{\partial \mu}\frac{\mu}{A^*};\ \chi_u = \frac{\partial A^*}{\partial u}\frac{u}{A^*}.$

The sign of the indices indicates whether the number of adopters increases or decreases in response to changing parameters, while the value of the indices represents the extent of this change. The values of the sensitivity indices of A^* with respect to different parameters are provided in (**Figure 5**). From **Figure 5**, we see that the value of the number of adopters is directly proportional to the parameters, $\overline{N}, u, \mu, \beta,$ and inversely proportional to the parameter α. Thus, it can be concluded that increasing promotional efforts can help in increase in the number of adopters. Also, decreasing the value of α can aid in the increase in the number of adopters. This can be achieved by providing timely and good service to the customers of the recent product.

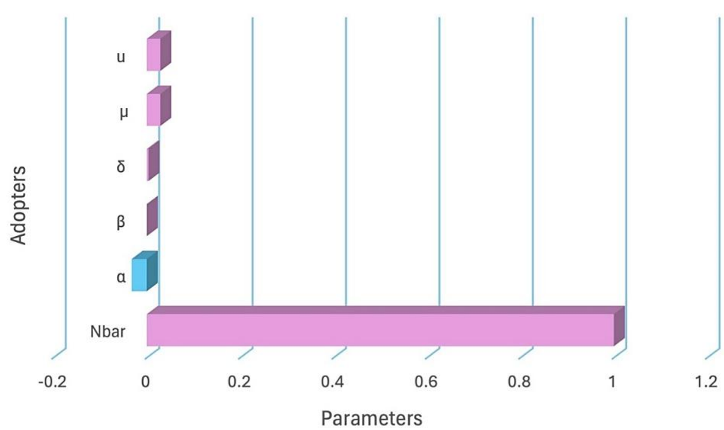
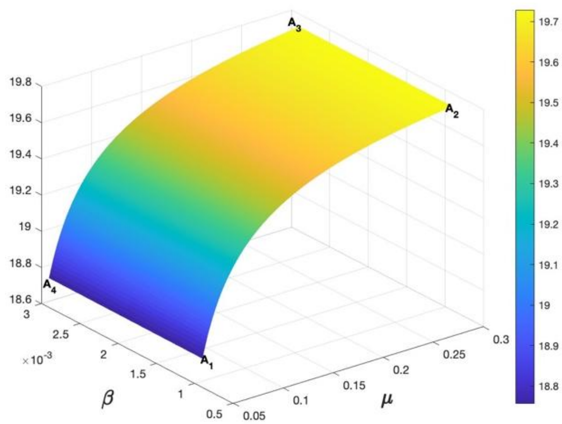


Figure 5. Sensitivity index in response to change in parameters.

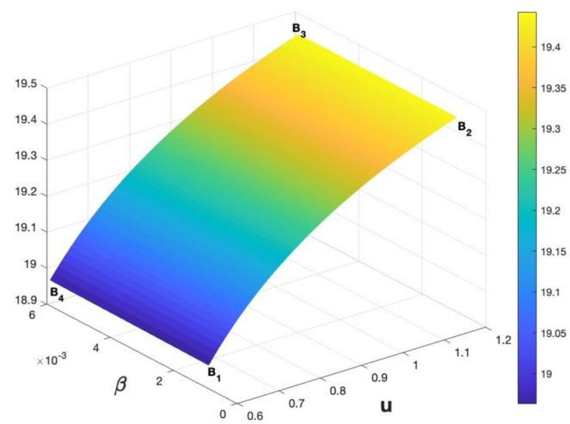
6.2 Effect of Variables

In this subsection, we will study the combined effect of all the model parameters $\mu, u, \delta, \alpha, \beta$ on the number of adopters at equilibrium level. The impact of various combinations of the parameters is displayed in the **Figure 6** below and the corner points of **Figure 6** are presented in **Table 2**. From **Figure 6** ((μ and β), (μ

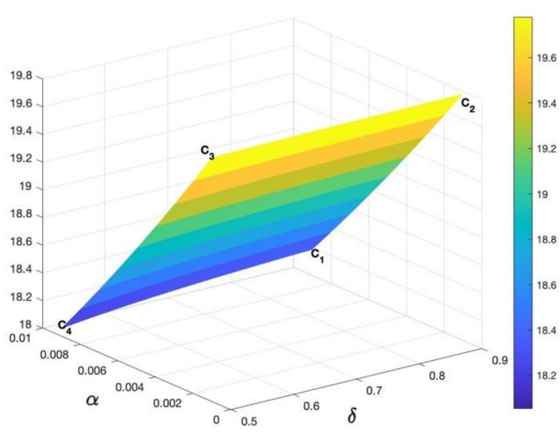
and α), (μ and u), and (μ and δ)), we observe that the number of adopters increases significantly with the increase in the value of the parameter μ. Thus, it can be concluded that if the rate at which unaware individuals become aware increases, then the number of adopters increases. From **Figure 6** ((u and β), (u and δ), (u and α), and (μ and u)), it can be deduced that an increase in the value of parameter u positively effects the number of adopters. Therefore, it is advised that companies invest in making promotional efforts to increase the sale of the recent product. It can be observed from **Figure 6** ((μ and β), (u and β), and (β and α)) that an increase in the value of β positively impacts the number of adopters. Therefore, an increase in the interaction rate between unaware and aware individuals leads to an increase in the number of adopters. Similar observation can be made for the parameter δ from **Figure 6** ((δ and α), (u and δ), and (μ and δ)). As can be seen from **Figure 6** ((δ and α), (β and α), (μ and α), and (u and α)), an increase in the value of α leads to a decrease in the value of number of adopters. The increase in the value of α can be avoided by providing suitable timely and good service to the customers of the recent product.



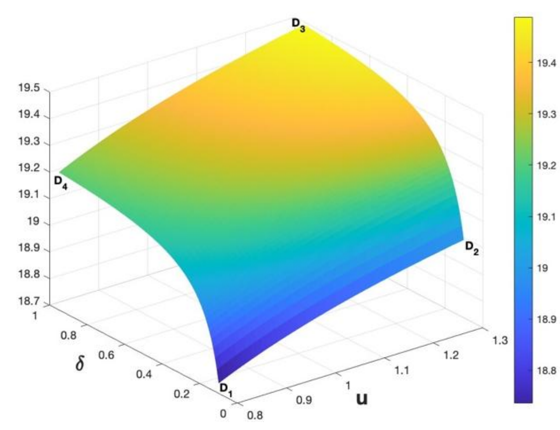
(a) μ and β



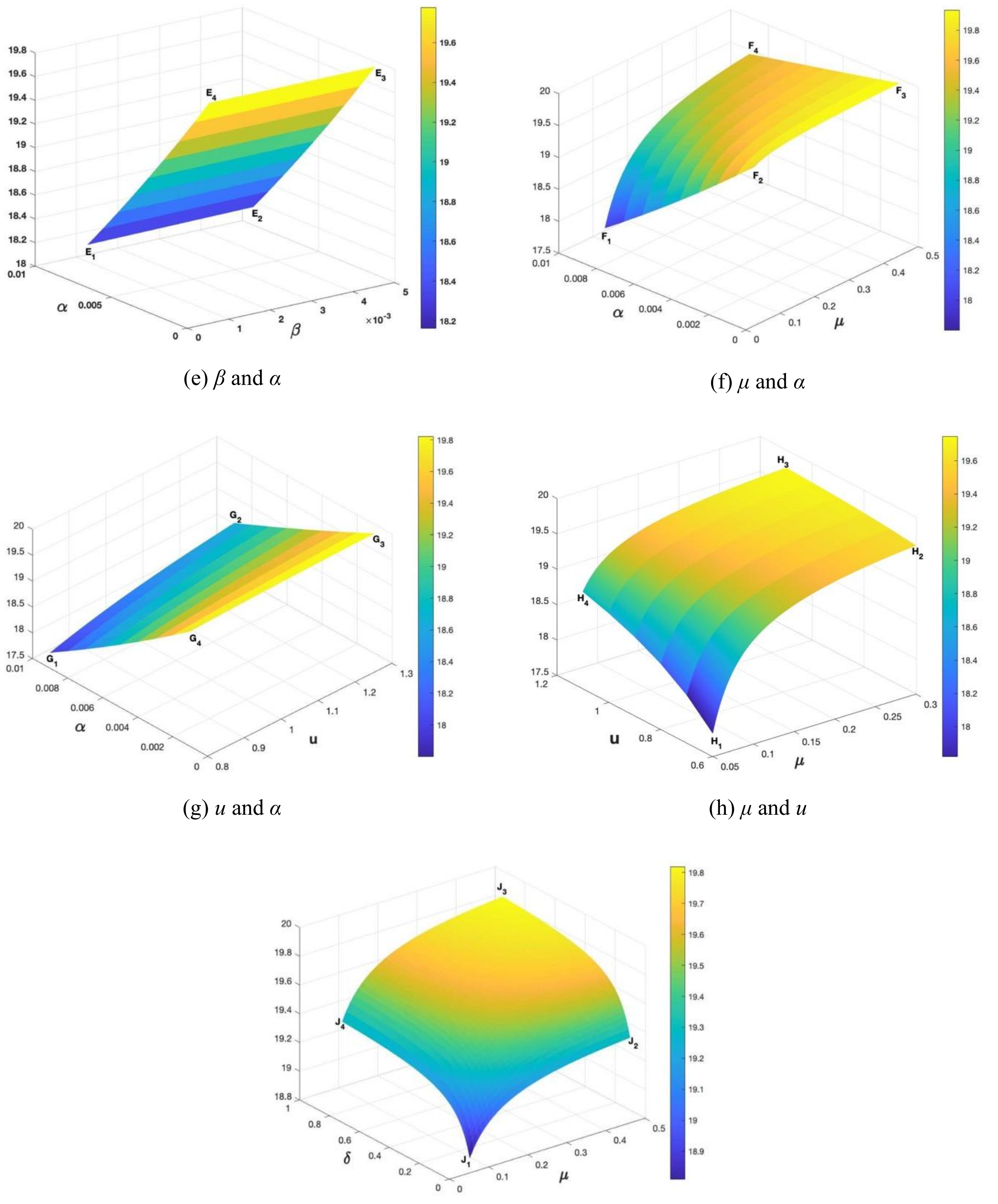
(b) u and β



(c) δ and α



(d) u and δ



(e) β and α

(f) μ and α

(g) u and α

(h) μ and u

(i) μ and δ

Figure 6. System behavior with respect to change in parameters.

Table 2. Corner points of **Figures 6 (a-i).**

Corner point (a)	μ	β	A	**Corner point** (b)	u	β	A
A_1	0.05	0.0009	18.7577	B_1	0.6	0.0009	18.9638
A_2	0.299	0.0009	19.7281	B_2	1.2	0.0009	19.4413
A_3	0.299	0.0029	19.7282	B_3	1.2	0.0059	19.4427
A_4	0.05	0.0029	18.7629	B_4	0.6	0.0059	18.9712
Corner point (c)	δ	α	A	**Corner point** (d)	u	δ	A
C_1	0.5	0.009	18.0552	D_1	0.8	0.1	18.7364
C_2	0.9	0.009	18.1844	D_2	1.3	0.1	18.9935
C_3	0.9	0.001	19.7798	D_3	1.3	0.95	19.4889
C_4	0.5	0.001	19.7625	D_4	0.8	0.95	19.2168
Corner point (e)	β	α	A	**Corner point** (f)	μ	α	A
E_1	0.0009	0.009	18.1638	F_1	0.08	0.009	17.801
E_2	0.0049	0.009	18.1759	F_2	0.08	0.001	19.7257
E_3	0.0049	0.001	19.7773	F_3	0.5	0.001	19.9352
E_4	0.0009	0.001	18.7771	F_4	0.5	0.009	19.4318
Corner point (g)	u	α	A	**Corner point** (h)	μ	u	A
G_1	0.8	0.009	17.801	H_1	0.05	0.6	17.8213
G_2	1.3	0.009	18.5122	H_2	0.3	0.6	19.6
G_3	1.3	0.001	19.8227	H_3	0.3	1.1	19.7465
G_4	0.8	0.001	19.7257	H_4	0.05	1.1	18.8719
Corner point (j)	μ	δ	A	**Corner point** (j)	μ	δ	A
J_1	0.09	0.1	18.8109	J_3	0.5	0.95	19.8185
J_2	0.5	0.1	19.305	J_4	0.09	0.95	19.2954

6.3 Optimal Control Strategy

In this subsection, numerical simulations for the optimal control problem (39) with the state system (40) along with the adjoint system (47) are performed with the setting parameters listed in **Table 1.** Parametric values and initial conditions taken to be $(N_0, I_0, A_0) = (20,10,1)$ together with the transversality constraints and the weighting factors, $C_1 = 10\ and\ C_2 = 100$.

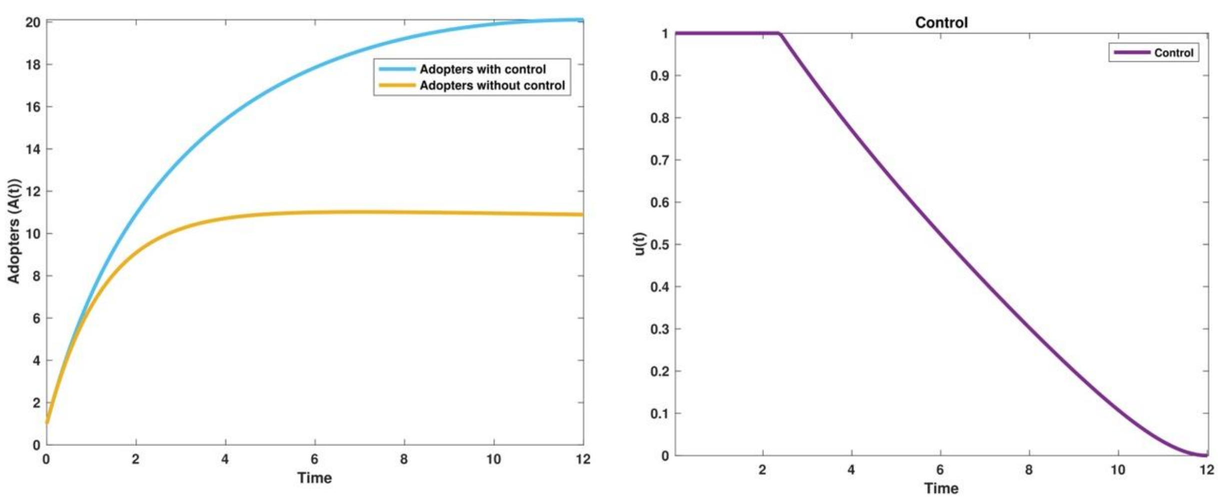


Figure 7. Simulation result with optimal control.

The simulation is implemented in MATLAB and in order to facilitate numerical simulation of the state system, the state variables are then solved from the dynamics (40) using the forward fourth-order Runge-Kutta method, setting out with an initial guess for the optimal control. Subsequently, the backward Runge-Kutta method is implemented to solve the adjoint system with the state variables and initial estimates for the control. From our simulation, it is evident that on implementing a promotional control strategy, the number of adopters of the product increases, as given in **Figure 7**. The optimal control analysis provides a comprehensive understanding of the strategic planning required for promotional activities by the decision-maker. **Figure 7** (control figure) shows the optimal control strategy that recommends the firm with heavy promotional effort to give momentum to the diffusion process for adopters initially, but as the word-of-mouth process gets momentum, the demand for promotional efforts diminishes. Therefore, it is advisable that the decision maker should lower the promotional effort at the end of the planning period.

7. Simulation-Based Estimation of Adoption Rates Using Real-World Data

In this section, using real data obtained from Net Sales data (In Rs. Crores) for the companies Blue Star (Data Set-I), Colgate Palmolive (Data Set-II) and Tata Motors (Data Set-III), we conduct a parameter estimation for obtaining the rate of adopters, that is taken as the parameter δ in the proposed model. The data used here has been collected for the period March 2007 to March 2021 (Mehta and Chaudhary, 2022) as shown in **Table 3**.

Table 3. Data description.

Data set I (Bluestar)		Data set II (Colgate Palmolive)		Data set III (Tata Motors)	
Mar'07	1,601.28	Mar'07	1,295.14	Mar'13	44,373.04
Mar'08	3,834.32	Mar'08	2,768.52	Mar'14	78,280.01
Mar'09	6,403.41	Mar'09	4,539.34	Mar'15	114,170.50
Mar'10	8,952.84	Mar'10	6,574.54	Mar'16	157,016.00
Mar'11	11,841.41	Mar'11	8,871.40	Mar'17	201,379.60
Mar'12	14,541.69	Mar'12	11,495.25	Mar'18	260,211.00
Mar'13	17,282.24	Mar'13	14,579.36	Mar'19	328,975.90
Mar'14	20,033.44	Mar'14	18,124.24	Mar'20	372,461.60
Mar'15	23,072.92	Mar'15	22,079.01	Mar'21	419,021.00
Mar'16	26,567.73	Mar'16	25,917.07		
Mar'17	30,679.82	Mar'17	29,868.54		
Mar'18	35,095.79	Mar'18	34,027.98		
Mar'19	39,879.49	Mar'19	38,460.42		
Mar'20	44,665.98	Mar'20	42,947.99		
Mar'21	48,508.21	Mar'21	47,758.47		

In this parameter estimation, we have taken the initial conditions to be $N(0) = 200,\ I(0) = 120,\ A(0) = 100$ and the parameter values as listed in **Table 1.** Parametric values and then using ordinary least squares (OLS), we fit our model, where the cost function calculates the sum of squared differences between the observed data $(\mathrm{Data_i})$ and the model's predicted values (y_i) , i.e., cost function $= \sum_{\mathrm{i}}(\mathrm{Data_i} - y_{\mathrm{i}})^2$ with $y = A$. For each Data Set-I, Data Set-II, and Data Set-III, we have obtained the rate at which aware individuals become adopters of the recent product (δ) to be 0.00068155, 0.00012435, and 0.04872530, respectively. In addition to this, a comparison table for the three Data Sets have been given in **Table 4** where we have calculated Coefficient of multiple determination (R^2), Mean square error (MSE), Variation, and Root mean square prediction error (RMSPE) as the comparison norms regarding goodness of fit for each data set.

i. **MSE:** The MSE compares and measures the difference between the actual data and the predicted values as follows:

$$MSE = \frac{\sum_{\mathrm{i=1}}^{\mathrm{n}}(\mathrm{Data_i} - y_{\mathrm{i}})^2}{\mathrm{n}},$$

where, n denotes the number of observations. Lower value of MSE indicates a better goodness of fit.

ii. **R^2:** This is defined as the ratio of the sum of squares from the trend model to that from constant model, subtracted from 1. Ranging from 0 to 1, the higher value of R^2 indicates a better goodness of fit.

$$R^2 = 1 - \frac{\text{Residual SS}}{\text{Corrected SS}}.$$

iii. **Variation:** We calculate the Prediction errors (PE_i) and BIAS, that is., the mean prediction error and then computed the BIAS-Corrected Variance. Lower the variation shows that the estimates are more consistent with the actual data, that is, better is the goodness of fit.

$$\text{Variation} = \sqrt{\frac{\sum(\text{PE}_i - \text{BIAS})^2}{(n-1)}},$$

where, BIAS $\frac{\sum_{i=1}^{n} \text{PE}_i}{n,}$

$$\text{Prediction Error}(\text{PE}_i) = \text{OE}_i - \text{PE}_i.$$

iv. **RMSPE:** This measures the average relative prediction error in terms of percentage. Lower values of RMSPE means the predictions closely match with the actual data, thus a better goodness of fit.

$$\text{RMSPE} = \sqrt{(\text{BIAS})^2 + (\text{Variation})^2}.$$

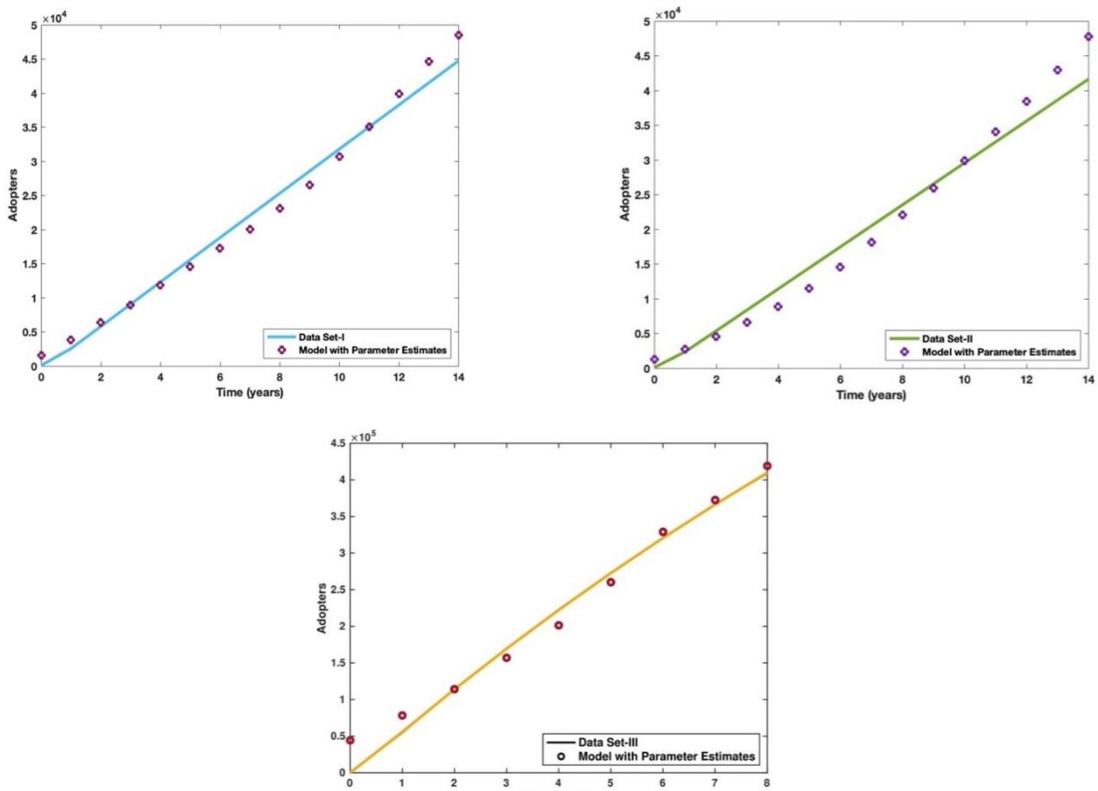


Figure 8. Model with parameter estimates and real data. (Data Set-I, Data Set-II, and Data Set-III).

From **Table 4**, we can see that Data Set-I has the highest R^2 and shows that the fitted model explains 98.44% of the total variation in the data. From the rest of the comparison criteria used, it is evident that the proposed model produces the best fit and prediction for the provided data sets (refer to **Figure 8**).

Table 4. Comparison criteria.

Data set	**Comparison criteria**			
	R²	**MSE**	**Variation**	**RMSPE**
Data set I	0.98442	3,270,705.714	1,868.964375	1,871.783705
Data set II	0.96803	6,950,553.475	2,727.785639	2,728.847334
Data set III	0.97610	379,134,230.2	19,981.73337	20,579.05553

8. Conclusion

In this paper, we have considered a mathematical model of innovation diffusion with a single time delay to provide insight into the effect of adoption delay due to decision-making on the adoption of recent product. We have proved the existence of a unique adopter equilibrium point $E^*(I^*, A^*)$. By comprehensive mathematical analysis of the system without and with delay, we have obtained the local stability conditions for E^* as given in Theorem 1 and Theorem 2. It has also been observed that there is an alteration in the system dynamics as the delay exceeds a critical value wherein the equilibrium point E^* loses its stability, that is, periodic solutions arise as a result of Hopf bifurcations. In a marketing scenario, when this occurs, companies are unable to accurately forecast the number of adopters in the market, resulting in potential losses within the company. Therefore, keeping the delay parameter close to its permissible limit can aid in ensuring system stability so that the number of adopters can be predicted accurately. Furthermore, the properties of Hopf bifurcation have been studied and conditions given in Theorem 3. On constructing a suitable Lyapunov function, we have determined that the equilibrium point E^* of the system is globally asymptotically stable with the conditions given in Theorem 4 and Theorem 5. If the system is globally stable, this assists marketers in realizing that after an initial fluctuation, the adoption process will stabilize at a known level and offer useful insight into developing long-term strategy and resource allocation. Numerical simulations have been carried out to validate the analytical results obtained. On performing sensitivity analysis on the number of adopters at equilibrium level, we have concluded that increased promotional efforts may result in a higher number of adopters, and that lowering α can also contribute to the rise in adopters. This can possibly be accomplished by offering prompt and quality service to the recent product customers. Additionally, we have examined the combined effect of the system parameters and it is clear that with increase in the value of parameters $\mu, u, \beta, and\ \delta$, that is, on increasing promotional efforts and increasing the interaction rate between unaware and aware individuals significantly results in higher number of individuals adopting the recent product. The optimal control model is described with proof of existence and characterization of the optimal control followed by numerical analysis which points to the conclusion that with the control strategy, the number of adopters of the recent product increases. It shows the most effective control strategy which advocates for the firm to implement substantial promotional endeavors to facilitate the diffusion process among early adopters; however, as the word-of-mouth mechanism gains traction, the necessity for continued promotional efforts diminishes. As a result, it is recommended that the individuals responsible for decision-making should reduce the intensity of promotional activities towards the conclusion of the planning phase. Lastly, using parameter estimation to estimate the rate at which individuals join the adopter class, we have estimated δ with actual data obtained from Net Sales data (In Rs. Crores) for the companies Blue Star, Colgate Palmolive, and Tata Motors and provided a comparison criteria table using R^2, MSE, Variation and RMSPE as the comparison norms. This simulation clearly indicates that the proposed model provides the best possible fit to the data provided.

Despite providing useful analytical and numerical insights, there are some limitations to the proposed model approach. The proposed model assumes that its population is homogeneous and well mixed and does not take into consideration any heterogeneity at the individual level. To model the time lag involved in adoption, one discrete time delay is introduced, which in actuality could either involve a distributed form or consist of multiple delays. Simulations are conducted with deterministic parameters and do not involve the stochastic influences of randomness in consumer behavior.

In light of the aforementioned limitations, several more directions could be pursued in extending the proposed framework of modeling. Future research may extend the optimal control formulation to incorporate the concept of a delay, resulting in a delayed optimal control model. Additionally, the stochastic extension of the proposed model could be investigated. Furthermore, and in a more practical light, the proposed framework could help companies develop appropriate promotional tools, predict the adoption rate, and allocate marketing efforts, taking into consideration more advertising strategies that target different customer categories.

Conflicts of Interest
The authors confirm that there is no conflict of interest to declare for this publication. All the authors contribute to this work and read and approved the final manuscript.

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AI Disclosure
During the preparation of this work the author(s) used generative AI in order to improve the language of the article. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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