

## Approximation of Integrable Functions by Modified Barbosu Operators

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(Received on November 7, 2025; Revised on January 13, 2026 & February 9, 2026; Accepted on February 12, 2026)

### Abstract

This paper presents an approximation method of integrable functions using a modified Barbosu operator, aimed at improving the rate of convergence in function approximation on the interval  $[0,1]$ . By introducing a suitable adjustment in the weight function, we construct a sequence of positive linear operators that better preserve the function's characteristics and demonstrate superior approximation behaviour, are studied. Theoretical error bounds are established in terms of the first and second order modulus of smoothness using their equivalence with the  $K$ -functionals. Finally, numerical experiments using test functions validate the theoretical findings and confirm that the King-type modified Barbosu operator achieve better approximation performance than the usual Kantorovich type Barbosu operator.

**Keywords-** Linear positive operators, Rate of convergence, Modulus of smoothness, Kantorovich variant.

### 1. Introduction

The theory of approximation by positive linear operators has played a central role in analysis since the pioneering work of Bernstein (1912) who introduced Bernstein polynomials to provide a constructive proof of the Weierstrass approximation theorem. Later, Kantorovich (1930) generalized Bernstein's approach by replacing point evaluations with integral means, thereby developing a new Kantorovich type variant, which significantly extended the applicability of such operators, especially to functions belonging to  $L^p$  spaces. Over the decades, numerous generalizations and modifications of these operators have been introduced in order to improve approximation properties, preserve specific test functions, and obtain sharper convergence results. For example, Stancu (1968) introduced a parameterized generalization of Bernstein operators that enhanced flexibility in approximation by controlling shape parameters. Jain (1972) extended this framework by defining new classes of positive linear operators, focusing on the preservation of linearity and convergence rate improvement. Further, Altomare & Campiti (1994) provided a comprehensive study of Korovkin-type approximation theorems which became fundamental in the analysis of operator convergence. These advances laid a solid foundation for later King-type modifications and Kantorovich-type integral generalizations (King, 2003; Gonska et al., 2009; Mursaleen et al., 2015). This modification preserves quadratic functions and improves the rate of convergence compared to the classical form. Subsequently, King-type ideas have been successfully applied to various operator families including Baskakov, Szász-Mirakyan, Bleimann-Butzer-Hahn, and others (see, Gonska et al., 2009; Mursaleen et al., 2015).

In 2004, Barbosu (2004) proposed a Stancu-type generalization of the Bernstein-Kantorovich operators

$$\widetilde{k}_{n,q}^* f(x) := (n + q + 1) \sum_{k=0}^{n+q} p_{n,k} (r_n(q, x)) \int_{\frac{k}{n+q+1}}^{\frac{k+1}{n+q+1}} f(u) du.$$

for  $f \in L \left[ \frac{\alpha}{m+\beta+1}, \frac{m+\alpha+1}{m+\beta+1} \right]$  and  $m \in \mathbb{N}$  incorporating two parameters  $\alpha, \beta$  with  $0 \leq \alpha \leq \beta$  to enhance flexibility and approximation efficiency. These Barbosu operators and their Kantorovich type variants have attracted considerable attention due to their extended applicability in approximation theory and related fields.

In recent years, much work has been devoted to combining a new Kantorovich type variant with King-type modifications. Such constructions not only preserve important functions but also lead to better error estimates and convergence behaviour. Moreover, advanced tools such as the Ditzian-Totik modulus of smoothness, Peetre’s  $K$ -functional, and Voronovskaya-type theorems have been employed to analyze the approximation properties of these operators (Ditzian & Totik, 1987; Gupta & Agarwal, 2014). For related works see (Korovkin, 1960; Rassias & Gupta 2016; Gairola et al., 2017, Gairola et al., 2022, and Gairola et al., 2024).

Motivated by these developments, the objective of this work is to propose and examine the King-type modification of a new Kantorovich type variant of the Barbosu operator. We establish direct approximation results, asymptotic behaviour, and rates of convergence using various smoothness measures. The results presented here continue the line of research connecting the classical Bernstein-Kantorovich operator with modern modifications and provide new insights into the approximation capabilities of Barbosu operators.

In Section 1, we recall basic definitions. Section 2, contains the basic properties of a new Kantorovich type variant of the Barbosu operator and the King-type modification  $\widetilde{k}_{n,q}^* f(x)$  of a new type Kantorovich variant  $\widetilde{k}_{n,q} f(x)$  of the Barbosu operator. Section 3 is devoted to direct approximation results by employing the concept of modulus of smoothness and the last of this Section 3, we discuss the theorem of Voronovskaya-type. In the Section 4, we discuss the numerical verification of  $\widetilde{k}_{n,q} f(x)$  and  $\widetilde{k}_{n,q}^* f(x)$  for various values of  $n$  and  $q$ . We also compare of these operators with the help of a suitable function.

Let  $p_{n,k}(q, x) = \binom{n+q}{k} x^k (1-x)^{n+q-k}$ ,  $x \in [0,1]$ . A new type Kantorovich variant of Barbosu operator is defined by

$$\widetilde{k}_{n,q} f(x) := (n + q + 1) \sum_{k=0}^{n+q} p_{n,k}(q, x) \int_{\frac{k}{n+q+1}}^{\frac{k+1}{n+q+1}} f(u) du \tag{1}$$

We have

$$\mu_2(q, x) = \widetilde{k}_{n,q}(e_1 - xe_0)^2(r_n(q, x)) = \frac{1-3\{(r_n(q,x) - 1)\}r_n(q,x)(n+q-1)\}}{3(n+q+1)^2}.$$

By minimizing the quantity  $\mu_2(q, x)$  we obtain

$$r_n(q, x) = \frac{x(n+q+1)-1}{(n+q-1)}.$$

The King type modification of a new Kantorovich type variant of Barbosu operator is defined as

$$\widetilde{k}_{n,q}^* f(x) := (n + q + 1) \sum_{k=0}^{n+q} p_{n,k} (r_n(q, x)) \int_{\frac{k}{n+q+1}}^{\frac{k+1}{n+q+1}} f(u) du \tag{2}$$

For  $h > 0, f \in C[a, b]$ , the usual second-order modulus is given by

$$\omega_2(f, h) := \sup_{u,v \in [a,b]} \left\{ \left| f(u) - f\left(\frac{u+v}{2}\right) + f(v) \right|, |u - v| \leq 2h \right\}.$$

For  $f \in C[a, b], \varphi(x) = \sqrt{x(1-x)}, x \in [0,1]$ , the Ditzian-Totik modulus of smoothness (see, Ditzian & Totik, 1987) of second order is given by

$$\omega_\varphi^2(f, \delta) := \sup_{0 < h \leq \delta} |f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|.$$

And the  $K$ -functional corresponding to  $\omega_\varphi^2(f, \delta)$  is defined by

$$K_{2,\varphi}(f, \delta) := \inf_{g \in W^2} \{ \|f - g\| + \delta \|\varphi^2 g''\| \}.$$

Another equivalent  $K$ -functional is defined as

$$\overline{K}_{2,\varphi}(f, \delta) := \inf_{g \in W^2} \{ \|f - g\| + \delta \|\varphi^2 g''\| + \delta^2 \|g''\| \},$$

where,

$$W^2 := \{g \in C([0,1]) : g' \in AC_{loc}(0,1), \|\varphi^2 g''\| < \infty\}.$$

and  $g' \in AC_{loc}(0,1)$  implies that  $g$  is differentiable and its derivative  $g'$  is absolutely continuous on every compact subinterval  $[a, b] \subset (0,1)$  (see, Ditzian & Totik, 1987).

The notation  $\omega_\varphi^2(f, \delta) \asymp K_{2,\varphi}(f, \delta) \asymp \overline{K}_{2,\varphi}(f, \delta)$  implies that there exists absolute constant  $M > 0$  such that

$$M^{-1} \omega_\varphi^2(f, \sqrt{\delta}) \leq K_{2,\varphi}(f, \delta) \leq M \omega_\varphi^2(f, \sqrt{\delta}) \tag{3}$$

$$M^{-1} \omega_\varphi^2(f, \sqrt{\delta}) \leq \overline{K}_{2,\varphi}(f, \delta) \leq M \omega_\varphi^2(f, \sqrt{\delta}) \tag{4}$$

where,  $M$  is independent of  $n$  and  $x$ , not necessarily same in different cases.

## 2. Preliminary Results

We establish auxiliary results in the sequel those are needed to establish the main estimations.

**Lemma 1.** For  $e_j(t) = t^j$  where  $j = 0, 1, 2, 3, \dots$ , the operator (1) verifies that

$$\begin{aligned} (1) \quad \widetilde{k}_{n,q} e_0(x) &= (n + q + 1) \sum_{k=0}^{n+q} p_{n,k}(q, x) \int_{\frac{k}{n+q+1}}^{\frac{k+1}{n+q+1}} 1 du \\ &= (n + q + 1) \sum_{k=0}^{n+q} p_{n,k}(q, x) \frac{1}{(n+q+1)} \\ &= \sum_{k=0}^{n+q} p_{n,k}(q, x) = \sum_{k=0}^{n+q} \binom{n+q}{k} x^k (1-x)^{n+q-k} \\ &= (x + (1-x))^{n+q} = 1. \end{aligned}$$

$$\begin{aligned}
 (2) \quad \tilde{k}_{n,q}e_1(x) &= (n+q+1) \sum_{k=0}^{n+q} p_{n,k}(q,x) \int \frac{\frac{k+1}{k} u^{n+q+1}}{n+q+1} du \\
 &= (n+q+1) \sum_{k=0}^{n+q} p_{n,k}(q,x) \left( \frac{1}{2} \left( \frac{(k+1)^2 - k^2}{(n+q+1)^2} \right) \right) \\
 &= \frac{1}{2(n+q+1)} \sum_{k=0}^{n+q} p_{n,k}(q,x) (1+2k) \\
 &= \frac{1}{2(n+q+1)} \left( \sum_{k=0}^{n+q} p_{n,k}(q,x) + 2 \sum_{k=0}^{n+q} k p_{n,k}(q,x) \right) \\
 &= \frac{1}{2(n+q+1)} \left( 1 + 2 \sum_{k=0}^{n+q} k \cdot \frac{(n+q)!}{k!(n+q-k)!} x^k (1-x)^{n+q-k} \right) \\
 &= \frac{1}{2(n+q+1)} \left( 1 + 2 \sum_{k=1}^{n+q} \frac{(n+q)!}{(k-1)!(n+q-k)!} x^k (1-x)^{n+q-k} \right) \\
 &= \frac{1}{2(n+q+1)} \left( 1 + 2 \sum_{s=0}^{n+q-1} \frac{(n+q)!}{s!(n+q-1-s)!} x^{s+1} (1-x)^{n+q-1-s} \right) \\
 &= \frac{1}{2(n+q+1)} \left( 1 + 2(n+q)x \sum_{s=0}^{n+q-1} \frac{(n+q-1)!}{s!(n+q-1-s)!} x^{s+1} (1-x)^{n+q-1-s} \right) \\
 &= \frac{1}{2(n+q+1)} \left( 1 + 2(n+q)x \sum_{s=0}^{n+q-1} p_{n-1,s}(q,x) \right) \\
 &= \frac{(2x(n+q)+1)}{2(n+q+1)}. \tag{by using (1) of Lemma 1}
 \end{aligned}$$

Similarly, by using (1) and (2) of Lemma 1, we obtain

$$(3) \quad \tilde{k}_{n,q}e_2(x) = \frac{3x(n+q)(x(n+q-1)+2)+1}{3(n+q+1)^2}.$$

**Lemma 2.** For  $e_j(t) = t^j$  where,  $j = 0, 1, 2, 3, \dots$ , the operator (2) verifies that

$$(1) \quad \tilde{k}_{n,q}^*e_0(x) = 1,$$

$$(2) \quad \tilde{k}_{n,q}^*e_1(x) = \frac{(2x(n+q)-1)}{2(n+q-1)},$$

$$(3) \quad \tilde{k}_{n,q}^*e_2(x) = \frac{\frac{3(n+q)(x^2(n+q+1)^2-1)}{n+q-1} + 1}{3(n+q+1)^2}.$$

**Note 1.** Linearity of the operator  $\tilde{k}_{n,q}^*$  implies that

$$\begin{aligned}
 \tilde{k}_{n,q}^*(e_1 - xe_0)^2(x) &= - \frac{3x^2(n+q+1)^2 - 3x(n+q+1) + 2n + 2q + 1}{3(n+q+1)^2(n+q-1)} \\
 &= \frac{\varphi^2(x)}{n+q-1} - \frac{1+2n+2q}{3(n+q+1)^2(n+q-1)} \\
 &= \frac{\varphi^2(x)}{n+q-1} + o(n^{-1}).
 \end{aligned}$$

We can find

$$\tilde{k}_{n,q}^*(e_1 - xe_0)^2(x) = \frac{\alpha_n^2(x)}{n+q-1},$$

where,  $\alpha_n^2(x) = \varphi^2(x) + \frac{1}{n+q-1}$ .

**Remark 1.** By standard calculations, the error bounds for these operators can be obtained as

$$\begin{aligned} |\tilde{k}_{n,q}f(x) - f(x)| &\leq M[1 + \delta^{-1} \tilde{k}_{n,q}(e_1 - xe_0)^2(x)]\omega(f, \delta) \\ &\leq M\omega\left(f, \sqrt{\frac{(n+q-1)}{(n+q+1)^2}\varphi^2(x) + \frac{1}{3(n+q+1)^2}}\right) \end{aligned} \tag{5}$$

Similarly for the king-type modified operators we have

$$|\bar{k}_{n,q}^*f(x) - f(x)| \leq M\omega\left(f, \sqrt{\frac{\varphi^2(x)}{n+q-1} - \frac{1+2n+2q}{3(n+q+1)^2(n+q-1)}}\right) \tag{6}$$

It is evident that the estimate in Equation (6) is better than Equation (5).

### 3. Main Results

**Theorem 1.** Let  $f \in C[a, b]$ , where  $0 \leq h \leq 1/2$  and  $0 \leq x \leq 1/2(n + q)$  Then

$$|\bar{k}_{n,q}^*f(x) - f(x)| \leq \frac{1}{h} \frac{|1-2x|}{(n+q-1)} \omega_1(f, h) + \left[ \frac{3}{2} + \frac{3}{4h} \frac{|1-2x|}{(n+q-1)} + \frac{3}{4h^2} \left( \frac{\varphi^2(x)}{n+q-1} - 3 \right) \right] \omega_2(f, h).$$

**Proof.** The proof follows the method of Gonska & Kovacheva (2024).

**Theorem 2.** Let  $f \in C[0,1]$ . For the sequence of operators (2), there exists a positive absolute constant  $M$  such that

$$|\bar{k}_{n,q}^*f(x) - f(x)| \leq M\omega_2\left(f, \frac{\alpha_n(x)}{\sqrt{n+q-1}}\right) + \omega\left(f, \left|\frac{2x-1}{2(n+q-1)}\right|\right).$$

**Proof.** Let  $g \in C^2[0,1]$  be chosen so that

$$\begin{aligned} |\bar{k}_{n,q}^*f(x) - f(x)| &\leq |\bar{k}_{n,q}^*(f - g)(x)| + |\bar{k}_{n,q}^*g(x) - g(x)| + |f(x) - g(x)| \\ &\leq 2\|f - g\| + \|\bar{k}_{n,q}^*g(x) - g(x)\|. \end{aligned}$$

We introduce an auxiliary operator

$$U_{n,q}f(x) = \bar{k}_{n,q}^*f(x) + f\left(\frac{2x(n+q)-1}{2(n+q-1)}\right) + f(x).$$

Then in view of the Lemma 2, we obtain

$$\begin{aligned} U_{n,q}e_0(x) &= 1, \\ U_{n,q}e_1(x) &= \bar{k}_{n,q}^*e_1(x) - \frac{2x(n+q)-1}{2(n+q-1)} + x = x. \end{aligned}$$

By the smoothness of  $g$ , we write

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - u)g''(u)du.$$

Therefore, by straightforward calculations, it follows that

$$|U_{n,q}g(x) - g(x)| = \left| U_{n,q}\left(\int_x^t (t - u)g''(u)du\right)(x) \right|$$

$$\begin{aligned} &\leq \left| \widetilde{k}_{n,q}^* \left( \int_x^t (t-u) g''(u) du, x \right) \right| + \left| \int_x^{\frac{2x(n+q)-1}{2(n+q-1)}} (t-u) g''(u) du \right| \\ &\leq \widetilde{k}_{n,q}^* (e_1 - x)^2(x) \|g''\| + \left( \frac{2x(n+q)-1}{2(n+q-1)} - x \right)^2 \|g''\| \\ &\leq M \left[ \frac{\alpha_n^2(x)}{n+q-1} + \left( \frac{2x-1}{2(n+q-1)} \right)^2 \right] \|g''\| \\ &\leq M \frac{\alpha_n^2(x)}{n+q-1} \|g''\|. \end{aligned}$$

Consequently,

$$\begin{aligned} \left| \widetilde{k}_{n,q}^* f(x) - f(x) \right| &\leq 2\|f - g\| + |U_{n,q}g(x) - g(x)| + \left| f \left( \frac{2x(n+q)-1}{2(n+q-1)} \right) - f(x) \right| \\ &\leq M \left\{ \|f - g\| + \frac{\alpha_n^2(x)}{n+q-1} \|g''\| + \omega \left( f, \left| \frac{2x-1}{2(n+q-1)} \right| \right) \right\}. \end{aligned}$$

Taking the infimum for  $g$  it follows that

$$\left| \widetilde{k}_{n,q}^* f(x) - f(x) \right| \leq MK_2 \left( f, \frac{\alpha_n^2(x)}{n+q-1} \right) + \omega \left( f, \left| \frac{2x-1}{2(n+q-1)} \right| \right).$$

From (3), the proof follows.

**Theorem 3.** (Voronovskaya-type). If  $f \in C^2[0,1]$ , then

$$\lim_{n \rightarrow \infty} (n+q-1) \left( \widetilde{k}_{n,q}^* f(x) - f(x) \right) = \left(x - \frac{1}{2}\right) f'(x) + \frac{\varphi^2(x)}{2} f''(x).$$

**Proof.** The proof can be derived using the limits  $\widetilde{k}_{n,q}^* (e_1 - x e_0)^j(x)$ ,  $j = 1, 2$ .

We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \widetilde{k}_{n,q}^* f(x) - f(x) - \widetilde{k}_{n,q}^* (u-x)(x) f'(x) - \frac{1}{2} \widetilde{k}_{n,q}^* (u-x)^2(x) f''(x) \right) \\ = \lim_{n \rightarrow \infty} \widetilde{k}_{n,q}^* (u-x) 2\mu_x(u)(x). \end{aligned}$$

Now,

$$\lim_{n \rightarrow \infty} \widetilde{k}_{n,q}^* ((\mu^2(t-x), x)) = \mu^2(x-x) = 0.$$

Therefore, by the Schwarz Inequality,

$$\lim_{n \rightarrow \infty} \widetilde{k}_{n,q}^* (t-x) 2\mu(t-x)(x) = 0.$$

Thus, by the identities

$$\widetilde{k}_{n,q}^* (u-x)(x) = \frac{2x-1}{2(n+q-1)},$$

And

$$\widetilde{k}_{n,q}^* (u-x)^2(x) = \frac{\varphi^2(x)}{n+q-1} - \frac{1+2n+2q}{3(n+q+1)^2(n+q-1)},$$

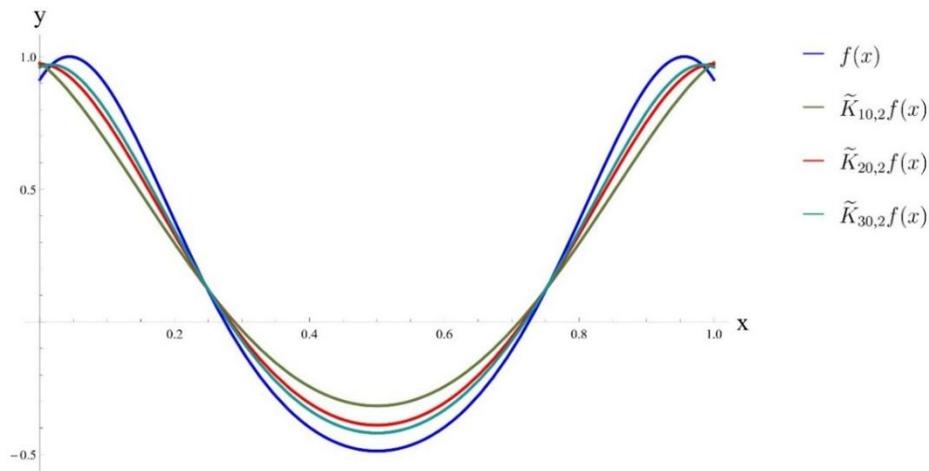
we obtain

$$\widetilde{k}_{n,q}^* f(x) - f(x) = (n+q-1)^{-1} \left( \left(x - \frac{1}{2}\right) f'(x) + \left(\frac{\varphi^2(x)}{2} + o(n)^{-1}\right) f''(x) \right).$$

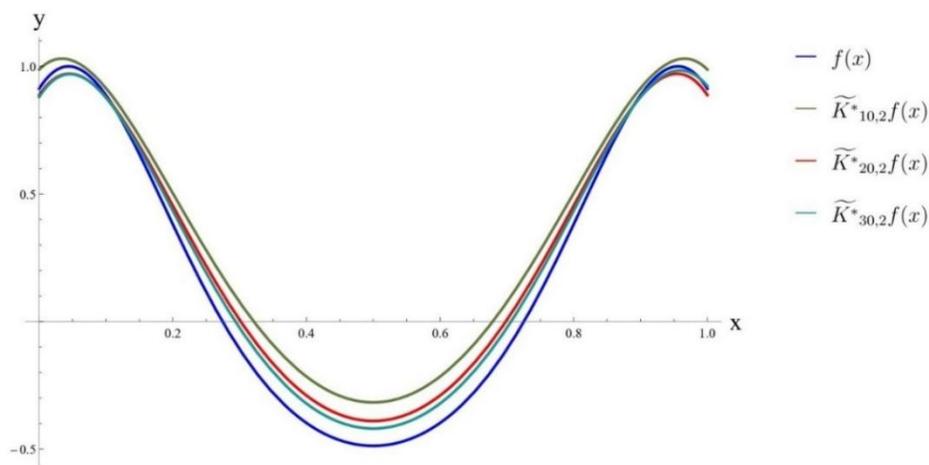
This completes the proof.

### 4. Numerical Verification

In order to demonstrate the convergence and obtain the order of approximation of the sequence  $\tilde{k}_{n,2}f(x)$ , we take the test function,  $\sin(10(1-x)x + 20), x \in [0,1]$ . In **Table 1**, we take the degree of polynomial,  $n$  as 10, 20 and 30 while keeping the parameter  $q = 2$  fixed. The situations are illustrated in **Figure 1**. The convergence of  $\tilde{k}_{n,2}f(x)$  to  $f(x)$  follows by the uniform error estimates  $E_n(f) = \sup_{x \in [0,1]} |\tilde{k}_{n,q}f(x) - f(x)|$  as we have  $E_{10}(f) = 0.2122, E_{20}(f) = 0.133203, E_{30}(f) = 0.0973073$  Thus  $E_n(f)$  approaches to 0 as claimed in Theorem 1 and 2. In order to compare the ordinary and King type modified operator, the functions  $\tilde{k}_{n,2}^*f(x)$  are shown in **Figure 2** for the same values of  $n$  and  $q$ . Subsequently, **Figure 3**, and **Figure 4**, shows the absolute error functions  $E_n(f)$  and  $E_n^*(f), n = 10, 20, 30$  and  $q = 2$  for the operators  $\tilde{k}_{n,2}f(x)$  and  $\tilde{k}_{n,2}^*f(x)$  respectively.



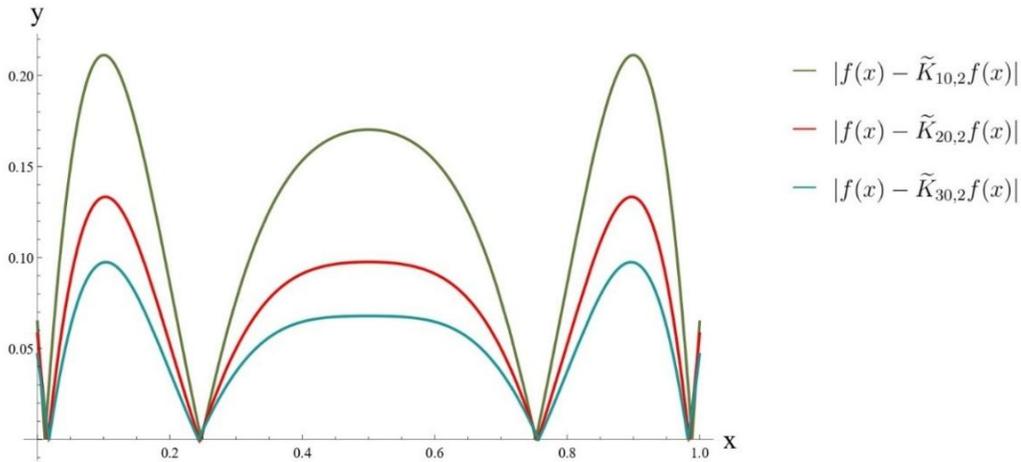
**Figure 1.** Function  $f(x)$  versus  $\tilde{k}_{n,2}f(x)$ .



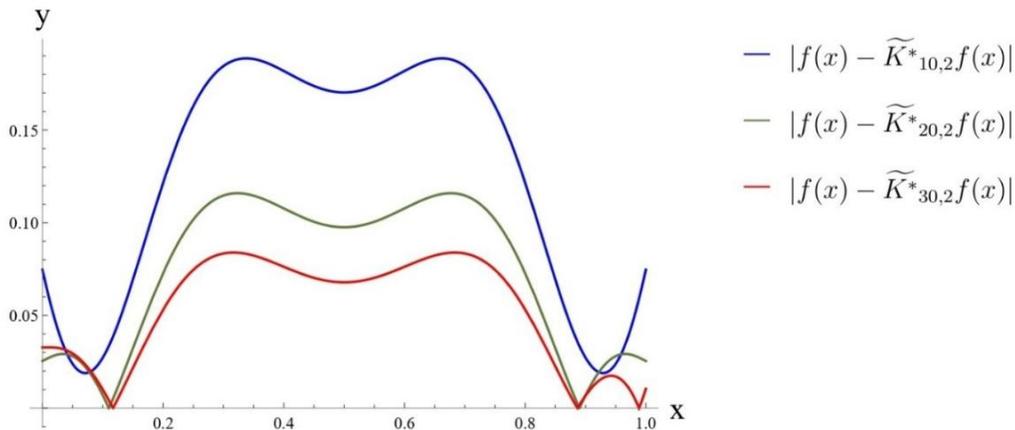
**Figure 2.** Function  $f(x)$  versus  $\tilde{k}_{n,2}^*f(x)$ .

**Table 1.** Values of  $\tilde{k}_{n,2}f(x)$  for  $n = 10, 20, 30$  with respect to  $f(x)$ .

$x$	$f(x)$	$\tilde{k}_{n,2}f(x)$			$ \tilde{k}_{n,2}f(x) - f(x) $		
		$n = 10$	$n = 20$	$n = 30$	$n = 10$	$n = 20$	$n = 30$
0	0.912945	0.977614	0.971151	0.959801	0.0646688	0.0582061	0.0468562
0.1	0.887158	0.675937	0.753955	0.78985	0.21122	0.133203	0.0973073
0.2	0.38125	0.29551	0.328894	0.343838	0.0857402	0.0523565	0.0374128
0.3	-0.108637	-0.0320988	-0.0586511	-0.0713423	0.0765378	0.0499855	0.0372942
0.4	-0.397556	-0.244267	-0.306439	-0.33289	0.153289	0.091117	0.064666
0.5	-0.487175	-0.316891	-0.389596	-0.419291	0.170284	0.0975783	0.0678839
0.6	-0.397556	-0.244267	-0.306439	-0.33289	0.153289	0.091117	0.064666
0.7	-0.108637	-0.0320988	-0.0586511	-0.0713423	0.0765378	0.0499855	0.0372942
0.8	0.38125	0.29551	0.328894	0.343838	0.0857402	0.0523565	0.0374128
0.9	0.887158	0.675937	0.753955	0.78985	0.21122	0.133203	0.0973073
1.0	0.912945	0.977614	0.971151	0.959801	0.0646688	0.0582061	0.0468562

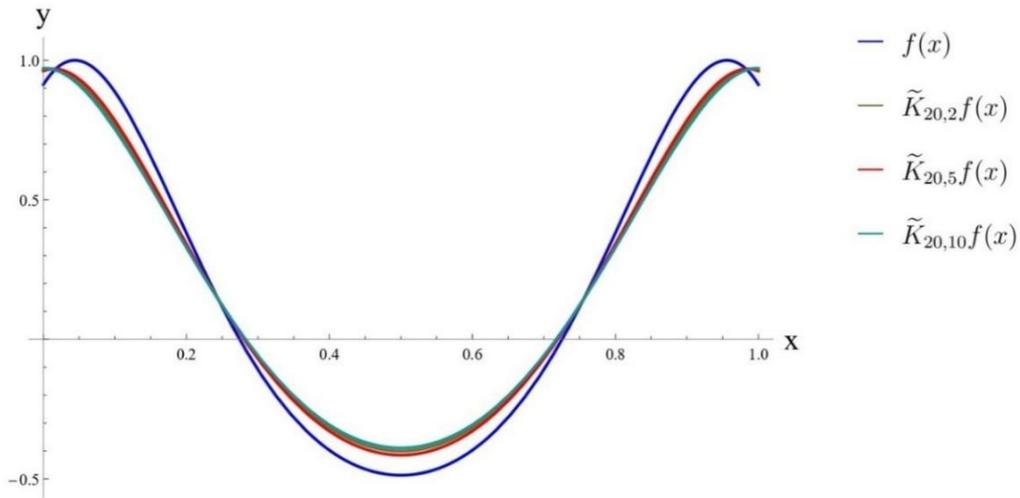


**Figure 3.** The absolute error functions  $E_n(f) = |f(x) - \tilde{k}_{n,2}f(x)|$  for  $q = 2$ .

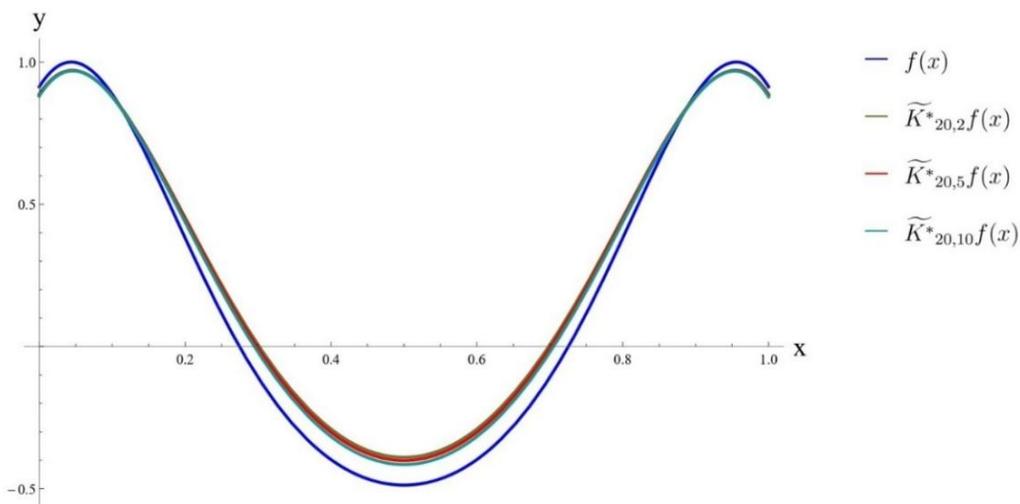


**Figure 4.** The absolute error functions  $E_n^*(f) = |f(x) - \tilde{k}_{n,2}^*f(x)|$  for  $q = 2$ .

Next, we fix the degree  $n = 20$  and consider the convergence of the sequence  $\tilde{k}_{20,q}f(x)$  for  $q = 5, 10$ , and  $20$ . It follows from **Table 2**, that the degree of approximation  $|\tilde{k}_{20,q}f(x) - f(x)|$  improves with respect to the larger values of the parameter  $q$ . The uniform error estimates  $E_{20}(f)$  are  $0.119927$ ,  $0.102849$  and  $0.0800576$  corresponding to  $q = 5, 10$  and  $q = 20$  respectively. These error bounds are smaller than  $E_{20}(f)$  for  $q = 2$ . Consequently, the larger values of  $n$  and  $q$  provide better degree of approximation. This case is illustrated in **Figure 3**, **Figure 5** and the behaviour of the absolute error functions is shown in the **Figure 7**. Also, **Figure 6**, **Figure 8** and **Table 3** shows that the degree of approximation  $|\tilde{k}^*_{20,q}f(x) - f(x)|$  improves with respect to the larger values of the parameter  $q$ .



**Figure 5.** Function  $f(x)$  versus  $\tilde{k}_{20,q}f(x)$  for  $q = 2, 5, 10$ .



**Figure 6.** Function  $f(x)$  versus  $\tilde{k}^*_{20,q}f(x)$  for  $q = 2, 5, 10$ .

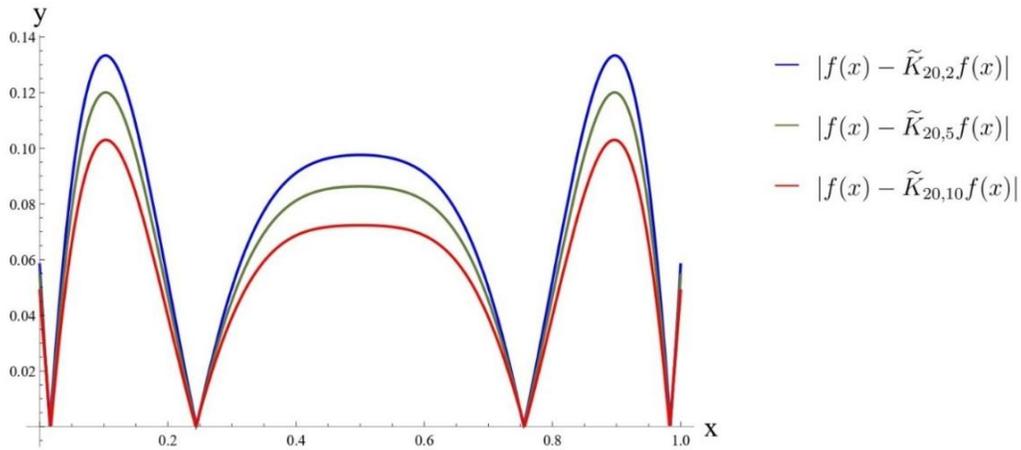


Figure 7. The absolute error functions  $E_{20}(f) = |f(x) - \tilde{k}_{20,q}f(x)|$  for  $q = 2, 5, 10$ .

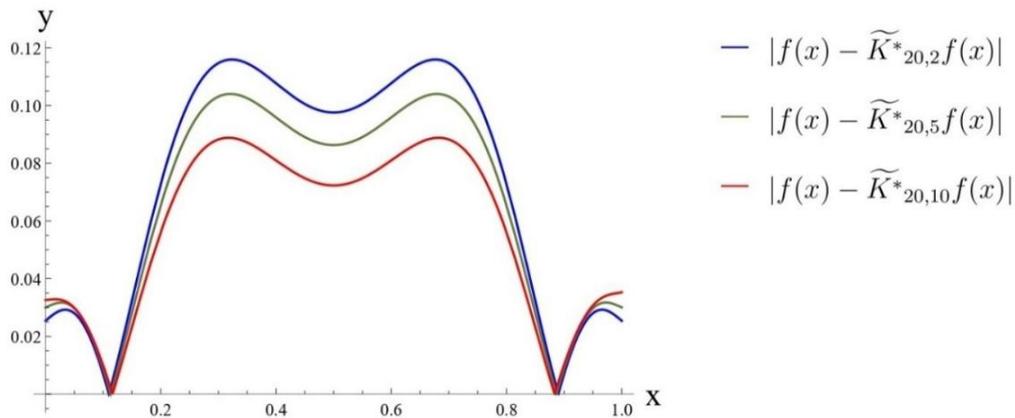
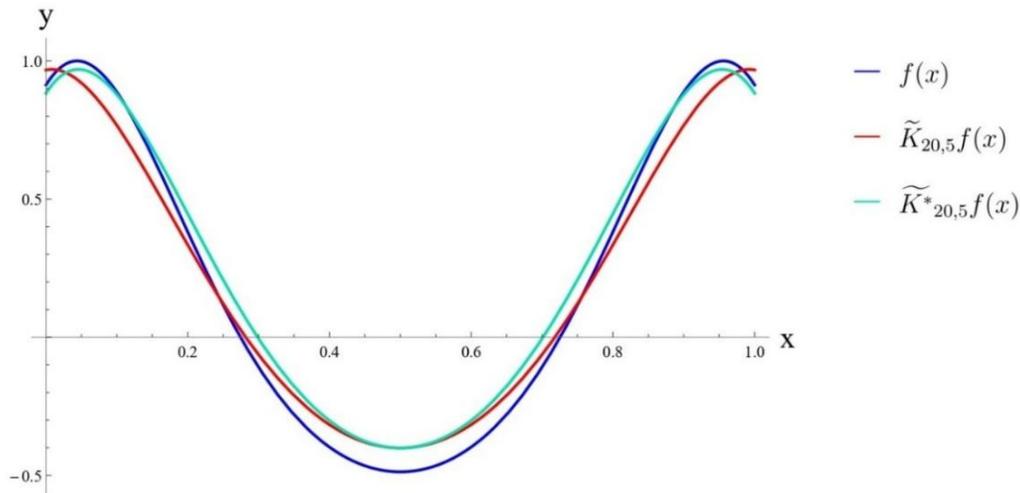


Figure 8. The absolute error functions  $E_{20}^*(f) = |f(x) - \tilde{k}_{20,q}^*f(x)|$  for  $q = 2, 5, 10$ .

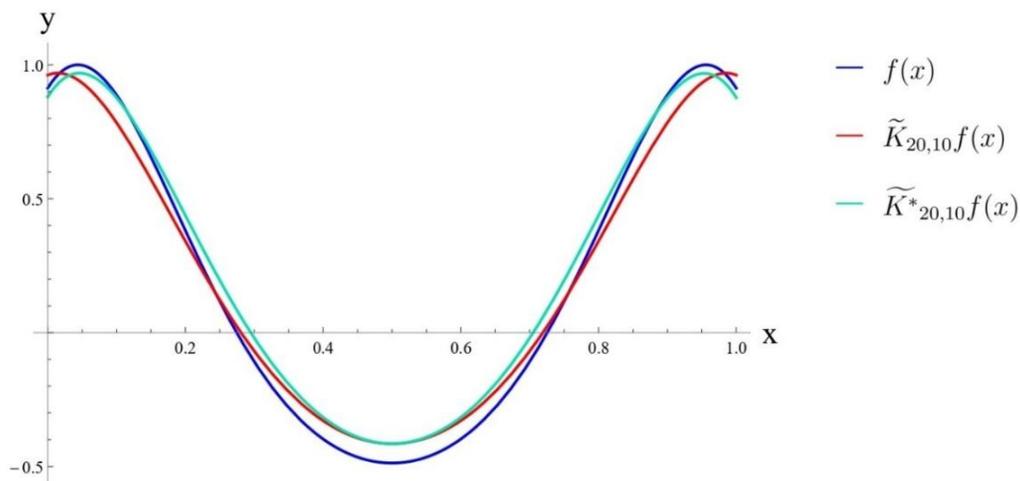
Table 2. Values of  $\tilde{k}_{20,q}f(x)$  for  $q = 5, 10, 20$  with respect to  $f(x)$ .

$x$	$f(x)$	$\tilde{k}_{20,q}f(x)$			$ \tilde{k}_{20,q}f(x) - f(x) $		
		$q = 5$	$q = 10$	$q = 20$	$q = 5$	$q = 10$	$q = 20$
0	0.912945	0.967449	0.96182	0.952971	0.0545039	0.0488749	0.0400259
0.1	0.887158	0.76723	0.784309	0.8071	0.119927	0.102849	0.0800576
0.2	0.38125	0.334472	0.341561	0.35084	0.0467783	0.0396892	0.0304108
0.3	-0.108637	-0.0632916	-0.0693517	-0.0776218	0.045345	0.0392849	0.0310148
0.4	-0.397556	-0.316382	-0.328896	-0.345108	0.0811736	0.0686593	0.0524478
0.5	-0.487175	-0.400867	-0.414869	-0.432672	0.0863079	0.072306	0.0545021
0.6	-0.397556	-0.316382	-0.328896	-0.345108	0.0811736	0.0686593	0.0524478
0.7	-0.108637	-0.0632916	-0.0693517	-0.0776218	0.045345	0.0392849	0.0310148
0.8	0.38125	0.334472	0.341561	0.35084	0.0467783	0.0396892	0.0304108
0.9	0.887158	0.76723	0.784309	0.8071	0.119927	0.102849	0.0800576
1.0	0.912945	0.967449	0.96182	0.952971	0.0545039	0.0488749	0.0400259

Finally, we compare the King type modified operator  $\widetilde{k}_{n,q}^* f(x)$  with the usual Barbosu operator  $\widetilde{k}_{n,q} f(x)$  and for the rate of approximation. Again, we choose the function  $\sin(10(1-x)x + 20)$ ,  $x \in [0,1]$  for the analysis. **Figure 9**, compares  $\widetilde{k}_{n,q}^* f(x)$  with  $\widetilde{k}_{n,q} f(x)$  for  $n = 20$  and  $q = 5$ . Similarly, **Figure 10** provides illustrative comparison of the same sequences for  $n = 20$  and  $q = 10$ . Finally **Table 3** provides the absolute errors,  $E_{20}(f) = |f(x) - \widetilde{k}_{20,q} f(x)|$  and  $E_{20}^*(f) = |f(x) - \widetilde{k}_{20,q}^* f(x)|$  for  $q = 2, 5, 10$ , and fixed  $n = 20$ . It follows that King type modification provides a better degree of approximation than the ordinary sequence of Barbosu type operators  $\widetilde{k}_{20,q} f(x)$ .



**Figure 9.**  $f(x)$  versus  $\widetilde{k}_{n,q} f(x)$  versus  $\widetilde{k}_{n,q}^* f(x)$  for  $n = 20$  and  $q = 5$ .



**Figure 10.**  $f(x)$  versus  $\widetilde{k}_{n,q} f(x)$  versus  $\widetilde{k}_{n,q}^* f(x)$  for  $n = 20$  and  $q = 10$ .

**Table 3.** Comparison of  $E_{20}(f) = |f(x) - \tilde{k}_{20,q}f(x)|$  and  $E_{20}^*(f) = |f(x) - \tilde{k}_{20,q}^*f(x)|$  for  $q = 2, 5, 10$ .

x	n = 20					
	q = 2		q = 5		q = 10	
	$E_{20}(f)$	$E_{20}^*(f)$	$E_{20}(f)$	$E_{20}^*(f)$	$E_{20}(f)$	$E_{20}^*(f)$
0	0.0582061	0.0254206	0.0545039	0.0300277	0.0488749	0.0326045
0.1	0.133203	0.00697708	0.119927	0.00874408	0.102849	0.00989334
0.2	0.0523565	0.0728121	0.0467783	0.0654278	0.0396892	0.0560953
0.3	0.0499855	0.114673	0.045345	0.103076	0.0392849	0.0882274
0.4	0.091117	0.107715	0.0811736	0.0958724	0.0686593	0.0809844
0.5	0.0975783	0.0975783	0.0863079	0.0863079	0.072306	0.072306
0.6	0.091117	0.107715	0.0811736	0.0958724	0.0686593	0.0809846
0.7	0.0499855	0.114673	0.045345	0.103076	0.0392849	0.0882278
0.8	0.0523565	0.0728121	0.0467783	0.0654273	0.0396892	0.0560821
0.9	0.133203	0.00697699	0.119927	0.00874837	0.102849	0.0101305
1.0	0.0582061	0.02542	0.0545039	0.0300583	0.0488749	0.0352889

### 5. Conclusion

This study shows that the absolute error  $E_n(f) = |f(x) - \tilde{k}_{n,q}f(x)|$  corresponding to a sufficiently smooth function  $f$  diminishes with increasing values of  $q$  for a fixed  $n$  and behaves like  $n^{-1}$ . Furthermore, it is observed that to achieve a higher rate of approximation, it is more effective to employ the suitably modified operator  $\tilde{k}_{n,q}^*f(x)$  rather than increasing the degree of usual operator  $\tilde{k}_{n,q}f(x)$ . The error bound  $E_n^*(f) = |f(x) - \tilde{k}_{n,q}^*f(x)|$  for the function  $f$  decreases with increasing  $n$  and  $q$ .

### Conflicts of Interest

The authors states that there is no competing interest related to this work.

### Acknowledgments

The authors would like to express their sincere gratitude to Doon University, Dehradun for providing the necessary support and research facilities that made this work possible.

### AI Disclosure

During the preparation of this work the author(s) used generative AI in order to improve the language of the article. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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