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Study of Dynamical Behavior of a Delayed Stage-Structured Predator-Prey Model with Disease in Prey

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Abstract
The present paper deals with the dynamics of a stage-structured predator-prey model, with a ratio-dependent functional response including gestational delay in the predator. The prey is carrying an infection which affects the predator adversely. The boundedness of solutions and the stability of equilibrium points have been investigated. There is a Hopf-bifurcation arising out of the variation in the time-delay parameter. Numerical simulations of phase-plane diagrams, and bifurcation diagrams illustrate the dependence of the system on the delay -time. The effect of the disease transmission from prey to predator has also been illustrated through simulations.

Keywords- Stage structured prey-predator model with delay, Boundedness and permanence, Extinction criterion, Stability analysis, Hopf bifurcation, Global stability, Numerical simulation.

1. Introduction

Prey-predator interactions have always been of interest to ecologists in the study of sustainability of eco-systems and persistence of populations. The first mathematical representation of the simultaneous dynamics of such interactions were proposed separately by Alfred James Lotka (Lotka, 1920) and Vito Volterra (Volterra, 1926). It was seen that under the simple assumptions given, the system showed periodic fluctuations. The Lotka-Volterra model has subsequently been modified to impart structural stability to the system. The mathematical analysis of prey-predation and other forms of interaction between species has been studied extensively (May, 2001 and Freedman, 1980).

In ecological modelling of interacting species, it is important to know the rate of consumption of prey by a predator. The term functional response (trophic function) of the predator to the prey refers to the change in density of the prey per unit time per predator. This response can be either prey-density dependent, or ratio-dependent. Three types of density-dependent trophic functions were proposed by Holling, but all of these are seen to lead to some contradictions between theoretical calculations and natural observations, for example, if the environment is enriched resulting in increase of carrying capacity, it is seen theoretically that the predator equilibrium is increased but not the prey-equilibrium. This leads to instability in the system. However, in nature the opposite is observed, that is, there will be increase in prey-density and no resulting instability is seen. This is known as the paradox of enrichment. Another paradox that arises with the density-dependent trophic functions is that of biological control, where we see that stable equilibrium does not exist for lower values of prey, although, in nature, prey-populations can exist well below their carrying capacity. These suggest that a trophic function dependent solely on prey-density might not represent the ecological situation, and some further improvement of the theory is required. This led to the idea of inclusion of the predator-density in the trophic function (Abrams et al., 2003; Arditi et al., 2001). Various trophic functions depending on both predator and prey-densities have been suggested (Murray, 1993 and Freedman, 1980). A model with a trophic-function depending on the ratio of prey to predator density was suggested by Adriti and Ginzburg (1989) in the following form;

$$f\left(\frac{s}{t}\right)=\frac{\left(\frac{s}{t}\right)}{1+m\left(\frac{s}{t}\right)}=\frac{cs}{ms+t},$$

where, s and t are density of prey and predator respectively.

The ratio-dependent prey-predator model, where response function is also known as a Michaelis-Menten-Holling type response, has been extensively studied by researchers and work is still being done on this model. The dynamics of such models have been discussed by Baretta and Kuang (1998), Hsu et al. (2001), Xaio and Ruan (2001), Xu et al. (2004), Freedman (1993) and others. Davidson et al. (2004) proposed a ratio-dependent predator-prey model with harvesting effort and solved through an algorithm on Adomian's decomposition method and have shown that the convergence of the decomposition series is enhanced using Padé approximation technique. The effect of environmental fluctuation and stochastic stability of a ratio-dependent predator-prey model was studied by Bandyopadhyay and Chattopadhyay (2005). A review of ratio-dependence in predator-prey system was given by Tyutyunov and Titova (2021). Panja et al. (2021) analyzed the dynamical behavior of a stage-structured prey-predator model with ratio-dependent functional response and anti-predator behavior of prey. Khajanchi and Banerjee (2017) studied the role of prey-refuge in a stage structured predator-prey model. Stage-structures are usually found in many species in nature; usually these have two distinct stage in life such as immature and mature. In a predator-prey model with stage-structure, it is a logical assumption that immature predators lack the ability to catch prey. Dynamics of predator-prey systems with stage-structuring in the predator have been studied by Wang et al. (2008), Xu et al. (2004), Devi (2013), Song et al. (2014), Gourley and Gorgescue, (2010) and Kuang et al. (2004), etc. The introduction of time-delay into the governing differential equations brings more complexity to the system, and the dynamics of the system is usually affected by the delay times. A time delay can cause a stable equilibrium to become unstable equilibrium or cause a population to fluctuate. Existence of time delay is frequently a cause of some sort of instability. Many researchers (Freedman et al., 1993; Zhao et al., 1997; Zeng et al., 2008; Xu et al., 2009; Xu, 2001, and others) have discussed ecological models with time-delay to make their models more realistic. An impulsive delay differential predator–prey model was studied by Zeng et al. (2008) who analyzed the stability of the trivial equilibrium by impulsive Floquet theory providing a sufficient condition for extinction. Xu (2001) formulated a predator-prey model with gestation delay of the predator and discussed the local stability of positive equilibrium and a semi trivial boundary equilibrium and showed Hopf-bifurcation. Ali et al. (2017) have studied the effect of time delay on the global dynamics of HIV-1 model.

The present paper considers the dynamics of a predator-prey system where the predator is divided into two stages, immature and mature. The mature predator interacts with the prey and the trophic function is taken in the Michelis-Menten Ratio-dependent form. Further, it is assumed that the prey is carrying a pathogen of some sort, which does not directly affect the prey, but can affect predator mortality. It can happen that in a prey-predator system, the prey is the carrier of some pathogen/disease/toxin which does not harm the prey directly; however, vertical transmission of the disease from prey to predator is possible when the predator comes into contact with the prey. In this paper such a model is considered, where the effect of the toxin in the prey is negligible. For simplicity, it is assumed that the pathogen is not capable of horizontal transmission in the predator. The effect of the pathogen or toxin is seen only in the decline of the predators in contact with prey. This model would be useful to understand the dynamics of such a situation from the point of permanence and sustainability of populations.

This article is organized in the following steps. The mathematical model is established in section-2. The positivity of the solutions, boundedness and permanence of all positive solutions are shown in section-3. Extinction criterion of the system are established in section-4. Study of the existence and stability of possible equilibria with both zero delay and non-zero delay are discussed and conditions derived in section-5. Hopf bifurcation of interior equilibrium point with respect to delay parameter is done in section-6. Global stability and numerical analysis with respect to parameters in section-7 and 8 respectively.

2. Model Formulation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{\alpha x(t) y_2(t)}{mx(t) + y_2(t)},$$
$$\frac{dy_1(t)}{dt} = ay_2(t) - by_1(t) - uy_1^2(t), \tag{1}$$
$$\frac{dy_2(t)}{dt} = py_2(t)\big(1 - y_2(t)\big) + \frac{\beta x(t-T) y_2(t-T)}{mx(t-T)+y_2(t-T)} + by_1(t) - \gamma x(t) y_2(t),$$
$$x(t) = \emptyset_1(t), y_1(t) = \emptyset_2(t), y_2(t) = \emptyset_3(t)\ where, \emptyset_i(t) \ge 0,$$
$$for\ -T \le t \le 0\ and\ \emptyset_i(t) > 0\ for\ i = 1,2,3.$$

(i) Prey: $x(t)$ denotes density of prey. r is the growth rate, k is the maximum carrying capacity, α is the predatation rate of prey by the mature predator and ’m’ is the handling time. Prey individuals is infected by some disease.

(ii) Immature Predator: $y_1(t)$ denotes density of immature predator. ’a’ is growth rate of immature predator, ’b’ is the transform rate from immature to mature and ’u’ is the intra-specific competition rate.

(iii) Mature Predator: $y_2(t)$ denotes density of mature predator. ’p’ is the growth rate, β is the growth rate and ’γ’ is decline rate of mature predator for the disease in prey. $T(>0)$ is the constant delay due to the gestation of mature predator.

3. Boundedness and Permanence of the Solutions

The positivity, boundedness and persistence are established in the following way (Xu et al., 2004; Ali et al. 2017)

Lemma: All the solution of the system with initial conditions are positive for all $t \ge 0$.

Proof: We consider from 2nd equation $\frac{dy_1(t)}{dt} = ay_2(t) - by_1(t) - uy_1^2(t) > -by_1(t) - uy_1^2(t)$
Therefore, by standard comparison shows that $y_1(t) > \frac{b_1 y_1(0)}{b_1 + uy_1(0)(e^{b_1 t} - 1)} > 0$.
Since, $y_1(0)$ is positive. So, $y_1(t)$ is positive for $t \ge 0$.

Again from 3rd equation

$$\frac{dy_2(t)}{dt} = py_2(t)\big(1-y_2(t)\big) + \frac{\beta x(t-T)y_2(t-T)}{mx(t-T)+y_2(t-T)} + by_1(t) - \gamma x(t)y_2(t) > -\gamma x(t)y_2(t).$$

Therefore, from the above in-equation we get $y_2(t) > y_2(0)e^{-\int_0^t x(t)dt}$.

Since, $y_2(0)$ is positive. So, $y_2(t)$ is positive for $t \geq 0$.

At last, from 1st equation $\frac{dx(t)}{dt} = rx(t)\left(1-\frac{x(t)}{k}\right) - \frac{\alpha x(t)y_2(t)}{mx(t)+y_2(t)} > -\frac{\alpha x(t)y_2(t)}{mx(t)+y_2(t)} > -\alpha x(t)$.

The above inequality implies that $x(t) > x(0)e^{-\alpha t}$.

Since, $x(0)$ is positive. So, $x(t)$ is positive for $t \geq 0$.

Hence, all the solution of the system with initial conditions are positive for all $t \geq 0$.

Theorem-3.1: All the positive solution of the system with the initial condition are uniformly bounded for $T = 0$.

Proof: Let $(x(t), y_1(t), y_2(t)\,)$be the positive solution of the system satisfying the initial condition. Let us consider the function $A(t) = x(t) + y_1(t) + \; y_2(t)$ and differentiating with respect of 't', we get

$$\frac{dA(t)}{dt} = \frac{dx(t)}{dt} + \frac{dy_1(t)}{dt} + \frac{dy_2(t)}{dt}$$
$$= rx\left(1-\frac{x}{k}\right) - (\alpha-\beta)\frac{xy_2}{m+y_2} + ay_2 - \; uy_1^2 + py_2(1-y_2) - \gamma xy_2$$
$$\leq rx\left(1-\frac{x}{k}\right) + ay_2 - \; uy_1^2 + py_2(1-y_2)$$
$$= rx\left(1-\frac{x}{k}\right) + y_2(a+p-py_2) - \; uy_1^2$$
$$= rx\left(1-\frac{x}{k}\right) + y_2(a+2p-py_2) + by_1 - \; uy_1^2 - rx - by_1 - py_2$$
$$\leq rk + \frac{(a+2p)^2}{4p} + \frac{b^2}{4u} - k_1A(t) \; where, k_1 = \min\{r, b, k\}$$
$$\frac{dA(t)}{dt} + k_1A(t) \leq rk + \frac{(a+2p)^2}{4p} + \frac{b^2}{4u}$$
$$A(t) \leq \frac{1}{k_1}\left(rk + \frac{(a+2p)^2}{4p} + \frac{b^2}{4u}\right) + c_1e^{-k_1t}$$

Therefore, $\limsup_{t\to\infty} A(t) \leq \frac{1}{k_1}\left(rk + \frac{(a+2p)^2}{4p} + \frac{b^2}{4u}\right)$.

Hence, all the positive solution of the system with the initial condition are uniformly bounded.

Theorem-3.2: The system (1) is permanent if $r > \alpha$ and $p > k\gamma$.

Proof: From 1st equation of (1) $\frac{dx(t)}{dt} \geq \left[r\left(1-\frac{x(t)}{k}\right)-\alpha\right]x(t)$.

From 2nd equation of (1) $\frac{dy_1(t)}{dt} = ay_2(t) - by_1(t) - uy_1^2(t)$.

From 3rd equation of (1) $\frac{dy_2(t)}{dt} \geq \left[p\left(1-y_2(t)\right)-\gamma x(t)\right]y_2(t)$.

Using standard comparison theorem [Xiao and Chen, 2001], we notice that $\lim_{t\to\infty} x(t) = s_1$,

$\lim_{t\to\infty} y_1(t) = s_2$ and $\lim_{t\to\infty} y_2(t) = s_3$ where, $s_1 = \frac{k}{r}(r-\alpha)$, $s_2 = \frac{-b\pm\sqrt{b^2+4au\left(\frac{r(p-k\gamma)+k\alpha\gamma}{rp}\right)}}{2u}$
and $s_3 = \frac{r(p-k\gamma)+k\alpha\gamma}{rp}$.

s_1, s_2, s_3 are positive if $r > \alpha$ and $p > k\gamma$.

4. Extinction Criterion

Theorem-4.1: If $r < 1$ and $\frac{x}{y_2} < \frac{\alpha-1}{m}$ then the prey population of the given system (1) extinct as $t \to \infty$.

Proof: From the 1st equation of (1)

$$\frac{dx(t)}{dt} = rx(t)\left(1-\frac{x(t)}{k}\right) - \frac{\alpha x(t)y_2(t)}{mx(t)+y_2(t)}$$
$$< (r-1)x(t)\ if\ \frac{x}{y_2} < \frac{\alpha-1}{m}.$$

Therefore $x(t) < x(0)e^{(r-1)t} \to 0\ as\ t \to \infty$.

Hence, if $r < 1$ and $\frac{x}{y_2} < \frac{\alpha-1}{m}$ then the prey population of the given system (1) extinct as $t \to \infty$.

Theorem-4.2: If $p + rk\beta + \epsilon\beta < \gamma k_2$ where, $k_2 = \inf x(t)$ then mature predator population of the given system (1) extinct as $t \to \infty$ for $T = 0$.

Proof: From 3rd equation of (1)

$$\frac{dy_2(t)}{dt} = py_2(t)\left(1-y_2(t)\right) + \frac{\beta x(t-T)y_2(t-T)}{mx(t-T)+y_2(t-T)} + by_1(t) - \gamma x(t)y_2(t)$$
$$\leq py_2(t) + \beta x(t)y_2(t) - \gamma x(t)y_2(t)$$
$$\leq (p + rk\beta + \epsilon\beta - \gamma k_2)y_2(t)\ \ (k_2 = \inf x(t)).$$

Therefore, $y_2(t) < y_2(0)e^{(p+rk\beta+\epsilon\beta-\gamma k_2)t} \to 0\ as\ t \to \infty\ for\ p + rk\beta + \epsilon\beta < \gamma k_2$.

Hence, if $p + rk\beta + \epsilon\beta < \gamma k_2$ then mature predator population of the given system (1) extinct as $t \to \infty$ for $T = 0$.

5. Equilibrium Points and their Stability Analysis

(i) The trivial equilibrium point $E_0(0,0,0)$.
(ii) The axial equilibrium point $E_1(k,0,0)$.
(iii) The positive equilibrium point $E_2(x^*,y_1^*,y_2^*)$.

5.1 Stability of Trivial Equilibrium

Theorem-5.1: The trivial equilibrium point is unstable $E_0(0,0,0)$.

Proof: Clearly, we see that $E_0(0,0,0)$ is the trivial equilibrium. The eigen values corresponding the trivial equilibrium are $\left(m-\frac{1}{k}\right), -b, p$. At least one of them is positive. So trivial equilibrium is unstable.

5.2 Stability of Axial Equilibrium

Theorem-5.2: The axial equilibrium point is $E_1(k,0,0)$ unstable for $p(1-k\gamma)+\frac{b\beta ke^{-\lambda T}}{mk}>b$ and $ab+pb(1-k\gamma)+\frac{b\beta ke^{-\lambda T}}{mk}>0$.

Proof: We see that, one eigen value corresponding to the axial equilibrium point $E_1(k,0,0)$ is $-\frac{r}{m}\left(m+\frac{1}{k}\right)$, which is negative and at least one eigen value is positive between other two eigen value if $p(1-k\gamma)+\frac{b\beta ke^{-\lambda T}}{mk}>b$ and $ab+pb(1-k\gamma)+\frac{b\beta ke^{-\lambda T}}{mk}>0$. This shows that the axial equilibrium point is $E_1(k,0,0)$ unstable for $p(1-k\gamma)+\frac{b\beta ke^{-\lambda T}}{mk}>b$ and $ab+pb(1-k\gamma)+\frac{b\beta ke^{-\lambda T}}{mk}>0$.

5.3 Stability of Positive Equilibrium

Theorem-5.3: The positive equilibrium $E_2(x^*,y_1^*,y_2^*)$ is exist for $r^2(1-mk)^2>4rkm(by_1{}^*+uy_1{}^*-r)$

Proof: For finding the positive equilibrium point of the system (1), we conclude three equations

$$y_2^*=\frac{r}{\alpha}\left(1-\frac{x^*}{k}\right)(y_2^*+\mathrm{m}x^*) \tag{2}$$
$$y_2^*=\frac{1}{\alpha}[by_1^*+u(y_1^*)^2] \tag{3}$$
$$\frac{ab}{b+uy_1^*}=\left(\gamma-\frac{\beta}{y_2^*+mx^*}\right)x^*+p\left[\frac{r}{\alpha}\left(1-\frac{x^*}{k}\right)(y_2^*+\mathrm{m}x^*)-1\right] \tag{4}$$

From (2) and (3) we conclude that

$$\frac{r}{\alpha}\left(1-\frac{x^*}{k}\right)(y_2^*+\mathrm{m}x^*)=\frac{1}{\alpha}[by_1^*+u(y_1^*)^2] \tag{5}$$

From equation (4) and (5) we find at least one positive value of y_1^*. Then from equation (5), we calculate the positive value of x^* under the condition $r^2(1-mk)^2>4rkm(by_1{}^*+uy_1{}^*-r)$. Again from equation (2), we get the positive value of y_2^*.

The characteristic equation corresponding equilibrium point $E_2(x^*,y_1^*,y_2^*)$ is

$$\lambda^3 + G_2\lambda^2 + G_1\lambda + G_0 + e^{-\lambda T}[H_2\lambda^2 + H_1\lambda + H_0] = 0 \tag{6}$$

where, $G_2 = B - C - L$, $G_1 = LC - LB - BC - \alpha\gamma L_1 y_2^*$, $G_0 = B(LC - \alpha\gamma L_1 y_2^*)$, $H_2 = -\beta L_1$, $H_1 = -\beta L_1(L - B) + \alpha\beta L_1 L_2$, $H_0 = B\beta L_1(L + \alpha L_2)$ and $L = r\left(1 - \frac{2x^*}{k}\right) - \frac{\alpha(y_2^*)^2}{(mm^* + y_2^*)^2}$, $B = b + 2uy_1^*$, $C = p(1 - 2y_2^*) - \gamma x^*$, $L_1 = \frac{mx^*}{(mm^* + y_2^*)^2}$, $L_2 = \frac{(y_2^*)^2}{(mm^* + y_2^*)^2}$.

5.3.1 Stability of Positive Equilibrium without Delay ($T = 0$)

Theorem-5.4: The positive equilibrium $E_2(x^*, y_1^*, y_2^*)$ is locally asymptotically stable for $T = 0$ if s_0, s_2 and $s_1 s_2 - s_0$ all are positive. Where, $s_1 = LC - LB - BC - \alpha\gamma L_1 y_2^* + \beta L_1(L - B) + \alpha\beta L_1 L_2$, $s_0 = B(LC - \alpha\gamma L_1 y_2^*) + LL_1 B\beta + \alpha\beta B L_1 L_2$, $s_2 = B - C - L - \beta L_1$ (B, C, L, L_1, L_2 are already define in the last part of Theorem 5.3).

Proof: For $T = 0$, the characteristic equation becomes, $\lambda^3 + s_2\lambda^2 + s_1\lambda + s_0 = 0$, where, $s_1 = LC - LB - BC - \alpha\gamma L_1 y_2^* + \beta L_1(L - B) + \alpha\beta L_1 L_2$, $s_0 = B(LC - \alpha\gamma L_1 y_2^*) + LL_1 B\beta + \alpha\beta B L_1 L_2$, $s_2 = B - C - L - \beta L_1$

By the Routh-Hurwitz criteria, all the roots of this equation are negative real part if s_0, s_2 and $s_1 s_2 - s_0$ all are positive. Hence, the positive equilibrium $E_2(x^*, y_1^*, y_2^*)$ is locally asymptotically stable for $T = 0$ if s_0, s_2 and $s_1 s_2 - s_0$all are positive.

5.3.2 Stability of Positive Equilibrium for non-zero Delay ($T \neq 0$)

Theorem-5.5: When $T \neq 0$, the positive equilibrium point $E_2(x^*, y_1^*, y_2^*)$ is locally asymptotically stable for (A_1): $N_2 > 0$, (A_2): $N_1 N_2 - N_0 > 0$ and (A_3): $N_0(N_1 N_2 - N_0) > 0$, where $N_0 = (B - C - L)^2 - \beta^2 L_1^2$

$N_1 = (LC - \alpha\gamma L_1 y_2^*)^2 + B^2(L + C)^2 + (\beta L_1 B)^2 - \beta^2 L_1^2(L + \alpha L_1)^2 - 2B^2(LC - \alpha\gamma L_1 y_2^*)^2$,

$N_2 = B^2(LC - \alpha\gamma L_1 y_2^*)^2 - L_1 B\beta(L + \alpha L_1)^2 - 2LC - LB - BC - \alpha\gamma L_1 y_2^*$ otherwise, at least one positivity condition fails as T increases, a finite number of stability switches and ultimately $E_2(x^*, y_1^*, y_2^*)$ becomes unstable (B, C, L, L_1, L_2 are already define in the last part of Theorem 5.3).

Proof: When $T \neq 0$, the characteristic equation is

$$\lambda^3 + G_2\lambda^2 + G_1\lambda + G_0 + e^{-\lambda T}[H_2\lambda^2 + H_1\lambda + H_0] = 0, \tag{7}$$

where $G_2 = B - C - L$, $G_1 = \mathrm{LC} - \mathrm{LB} - \mathrm{BC} - \alpha\gamma L_1 y_2^*$, $G_0 = B(LC - \alpha\gamma L_1 y_2^*)$, $H_2 = -\beta L_1$, $H_1 = -\beta L_1(L - B) + \alpha\beta L_1 L_2$, $H_0 = B\beta L_1(L + \alpha L_2)$.

When $T = 0$ then the eigen values of (7) is either negative or negative real part. Thus positive equilibrium point is stable under some condition, as shown in previous theorem.

When $T \neq 0$, that is T increases, then real part of complex roots are changes that way from negative to positive. Let the eigen values are $\mu(T) + i\omega(T)$. So, for $T = 0$ all $(T) < 0$. For $T > 0$, it is seen whether there can exist a purely imaginary eigen value $\omega(T)$, where $\omega(T)$ is real. We are looking for a purely imaginary solution of this characteristic equation. So, we substitute $\lambda = i\omega$ (ω is real number) in (7) and separate real part and imaginary part, then the characteristic equation becomes,

$$\omega^6 + N_2\omega^4 + N_1\omega^2 + N_0 = 0, \quad (8)$$

where,

$$N_0 = (B - C - L)^2 - \beta^2 L_1^2,$$
$$N_1 = (LC - \alpha\gamma L_1 y_2^*)^2 + B^2(L + C)^2 + (\beta L_1 B)^2 - \beta^2 L_1^2 (L + \alpha L_1)^2 - 2B^2(LC - \alpha\gamma L_1 y_2^*)^2,$$
$$N_2 = B^2(LC - \alpha\gamma L_1 y_2^*)^2 - L_1 B\beta(L + \alpha L_1)^2 - 2LC - LB - BC - \alpha\gamma L_1 y_2^*.$$

Which is the quadratic equation of ω^2. Let $z = \omega^2$, then equation (8) becomes

$$z^3 + N_2 z^2 + N_1 z + N_0 = 0. \quad (9)$$

According to the Routh-Hurwitz criteria, all roots of this equation are either negative real number or negative real part if N_0, N_2 and $N_1N_2 - N_0$ all are positive. Therefore the positive equilibrium $E_2(x^*, y_1^*, y_2^*)$ is asymptotically stable.

If there exist at least one of the coefficients of N_0, 1, N_2 is negative then equation (9) has at least one positive root ω_0^2 (say). Then equation (8) has a pair of imaginary roots $\pm\omega_0$. Hence, T increases there will be switch the stability and undergoes to Hopf-bifurcation.

6. Hopf Bifurcation

Theorem-6.1: If at least one of the positivity condition of (A1), (A2), (A3) define in the previous theorem does not hold and $N_1 > 0$, $N_2 > 0$ but $N_0 < 0$ then,

a) The positive equilibrium $E_2(x^*, y_1^*, y_2^*)$ is stable for $< T^*$.
b) The positive equilibrium $E_2(x^*, y_1^*, y_2^*)$ is unstable for $T > T^*$.
c) At $T = T^*$, the Hopf Bifurcation occurs as T passes through the point T^*,
where, $T^* = \frac{1}{\omega_0} cos^{-1}\left\{\frac{(H_1 - G_2H_2)\omega_0^4 + (G_2H_0 + G_0H_2 - G_1H_1)\omega_0^2 - G_0H_0}{\left(H_0 - H_2\omega_0^2\right)^2 + {H_1}^2\omega_0^2}\right\}$.

Proof: If at least one of the positivity condition of (A1), (A2), (A3) does not hold and $N_1 > 0$, $N_2 > 0$ but $N_0 < 0$ then, equation (9) has at least one positive roots ω_0^2 (say). Then the equation (8) has a pair of roots $\pm\omega_0$ and the characteristic equation has pair of imaginary roots $\pm\omega_0$. for $\mu(T^*) = 0$ and $\omega(T^*) = \omega_0$.

Now we calculete T^* from real and imaginary part those are comes from by putting $\lambda = i\omega$ in (7).

Eleminating sin (ωT) from real and imaginary part, then we get,

$$T^* = \frac{1}{\omega} cos^{-1}\left\{\frac{(H_1 - G_2H_2)\omega^4 + (G_2H_0 + G_0H_2 - G_1H_1)\omega^2 - G_0H_0}{(H_0 - H_2\omega^2)^2 + {H_1}^2\omega^2}\right\} + \frac{2n\pi}{\omega} \; where, n = 0,1,2 \ldots.$$

We denote $T_{0n} = \frac{1}{\omega_0} cos^{-1}\left\{\frac{(H_1-G_2H_2)\omega_0^4+(G_2H_0+G_0H_2-G_1H_1)\omega_0^2-G_0H_0}{\left(H_0-H_2\omega_0^2\right)^2+{H_1}^2\omega_0^2}\right\} + \frac{2n\pi}{\omega_0}\ where, n = 0,1,2$ ….

When at least one positivity condition fails, then positive equilibrium stable for $T < T^*$. (Here $T_{00} = T^*$) where, $T^* = \frac{1}{\omega_0} cos^{-1}\left\{\frac{(H_1-G_2H_2)\omega_0^4+(G_2H_0+G_0H_2-G_1H_1)\omega_0^2-G_0H_0}{\left(H_0-H_2\omega_0^2\right)^2+{H_1}^2\omega_0^2}\right\}$.

There are countably many bifurcations point in which T^* $(= T_{00})$ is least. Differentiating the characteristic equation (7) with respect to $'T'$ and after calculating we conclude the result,

$$\left(\frac{d\lambda}{dT}\right)^{-1} = -\frac{1}{\lambda}\left[\frac{3\lambda^2+2G_2\lambda+G_1}{\lambda^3+G_2\lambda^2+G_1\lambda+G_0} - \frac{2H_2\lambda+H_1}{H_2\lambda^2+H_1\lambda+H_0} + T\right]. \quad (10)$$

Now at $\lambda = i\omega_0$,

$$\lambda^3 + G_2\lambda^2 + G_1\lambda + G_0 = (G_0 - G_2\omega_0^2) + i(G_1\omega_0 - \omega_0^3),$$
$$H_2\lambda^2 + H_1\lambda + H_0 = (H_0 - H_2\omega_0^2)^2 + iH_1\omega_0,$$
$$3\lambda^2 + 2G_2\lambda + G_1 = (G_1 - 3\omega_0^2) + 2iG_2\omega_0,$$
$$2H_2\lambda + H_1 = H_1 + 2H_2 i\omega_0.$$

Using above relation in (10) we find the real part of $\left(\frac{d\lambda}{dT}\right)^{-1}$ at $T = T^*$.

$$Re\left(\frac{d\lambda}{dT}\right)^{-1} = \frac{(G_1 - 3\omega_0^2)(G_1 - \omega_0^2) - 2G_2(G_0 - G_2\omega_0^2)}{(G_0 - G_2\omega_0^2)^2 + (G_1\omega_0 - \omega_0^3)^2} + \frac{2H_2(H_0 - H_2\omega_0^2) - H_1^2}{(H_0 - H_2\omega_0^2)^2 + H_1^2\omega_0^2}.$$

Therefore,

$$Re\left(\frac{d\lambda}{dT}\right)^{-1} = \frac{3\omega_0^3 + 2N_2 + N_1}{(H_0 - H_2\omega_0^2)^2 + H_1^2\omega_0^2}.$$

Hence, $Re\left[\left(\frac{d\lambda}{dT}\right)^{-1}\right]_{T=T^*} > 0.$

Therefore $sgn\left\{\frac{d(Re\lambda)}{dT}\right\}_{\lambda=i\omega_0} = sgn\left\{Re\left(\frac{d\lambda}{dT}\right)^{-1}\right\}_{\lambda=i\omega_0} > 0.$

The real part of the characteristic root is positive for $T > T^*$ and $E_2(x^*, y_1^*, y_2^*)$ becomes unstable. Therefore, Hopf-Bifurcation occur at $T = T^*$.

7. Global Stability

Theorem-7.1: In this case, $r\beta(x - x^*) - u\alpha y_1(y_1 - y_1^*) + p\alpha(y_2 - y_2 + y_2y_2^*) + \alpha(a + \beta x^*)y_2^* \leq 0$ then the positive equilibrium $E_2(x^*, y_1^*, y_2^*)$ of the given system is globally asymptotically stable.

Proof: Let us consider the following equation,

$$V(t) = c_1\left(x - x^* - x^* log\frac{x}{x^*}\right) + c_2\left(y_1 - y_1^* - y_1^* log\frac{y_1}{y_1^*}\right) + c_3\left(y_2 - y_2^* - y_2^* log\frac{y_2}{y_2^*}\right).$$

Differentiating with respect to ‘t’ we get

$$\frac{dV}{dt} = c_1\left(1 - \frac{x^*}{x}\right)\frac{dx}{dt} + c_2\left(1 - \frac{y_1^*}{y_1}\right)\frac{dy_1}{dt} + c_3\left(1 - \frac{y_2^*}{y_2}\right)\frac{dy_2}{dt},$$

$$= c_1(x - x^*)\left[r\left(1 - \frac{x}{k}\right) - \frac{\alpha y_2}{mx + y_2}\right] + c_2(y_1 - y_1^*)\left(\frac{\alpha y_2}{y_1} - b - uy_1\right)$$

$$+c_3(y_2 - y_2^*)\left[p(1 - y_2) + \frac{\beta y_2(t-T)}{y_2\big(mx(t-T) + y_2(t-T)\big)} + \frac{by_1}{y_2} - \gamma x\right],$$

$$\leq c_1(x - x^*)\left[r\left(1 - \frac{x}{k}\right) - \frac{\alpha y_2}{mx+y_2}\right] + c_2(y_1 - y_1^*)\left(\frac{\alpha y_2}{y_1} - b\right) - uc_2y_1(y_1 - y_1^*)$$

$$+c_3(y_2 - y_2^*)\left[p(1 - y_2) + \frac{\beta y_2(t-T)}{y_2\big(mx(t-T) + y_2(t-T)\big)} + \frac{by_1}{y_2}\right],$$

$$\leq rc_1(x - x^*)\left(1 - \frac{x}{k}\right) - \frac{\alpha c_1 y_2}{mx + y_2}(x - x^*) + c_2y_1\left(\frac{\alpha y_2}{y_1} - b\right) - uc_2y_1(y_1 - y_1^*)$$

$$+pc_3(y_2 - y_2^*)(1 - y_2) + c_3(y_2 - y_2^*)\left[\frac{\beta y_2(t-T)}{y_2\big(mx(t-T) + y_2(t-T)\big)} + \frac{by_1}{y_2}\right],$$

$$\leq rc_1(x - x^*)\left(1 - \frac{x}{k}\right) - \frac{\alpha c_1 x y_2}{mx+y_2} + \frac{\alpha c_1 x^* y_2}{mx+y_2} + \alpha c_2 y_2 - bc_2y_1 - uc_2y_1(y_1 - y_1^*)$$

$$+pc_3(y_2 - y_2^*)(1 - y_2) + c_3y_2\left[\frac{\beta y_2(t-T)}{y_2(mx(t-T)+y_2(t-T))} + \frac{by_1}{y_2}\right],$$

$$\leq rc_1(x - x^*) - \frac{\alpha c_1 x y_2}{mx+y_2} + \alpha c_1 x^* y_2 + \alpha c_2 y_2 - bc_2y_1 - uc_2y_1(y_1 - y_1^*)$$

$$+pc_3(y_2 - y_2^*)(1 - y_2) + c_3\left[\frac{\beta y_2(t-T)}{mx(t-T) + y_2(t-T)} + by_1\right],$$

$$\leq rc_1(x - x^*) - \frac{\alpha c_1 x y_2}{mx+y_2} + c_1\alpha x^* y_2 + \alpha c_2 y_2 - bc_2y_1 - uc_2y_1(y_1 - y_1^*)$$

$$+pc_3(y_2 - y_2^*)(1 - y_2) + \frac{c_3\beta y_2(t-T)}{mx(t-T) + y_2(t-T)} + bc_3y_1.$$

Now, we define a function

$$V_1(t) = V(t) + \int_{t-T}^{t}\left(\frac{x(s)y_2(s)}{mx(s)+y_2(s)} - \frac{x^*y_2^*}{mx^*+y_2^*}\right)ds$$
$$+\int_0^t [b(c_2-c_3)(y_1(s)-y_1^*) - (ac_2+\alpha c_1 x^*)(y_2(s)-y_2^*)]\,ds,$$

$$\frac{dV_1(t)}{dt} \le rc_1(x-x^*) - uc_2y_1(y_1-y_1^*) + pc_3(y_2-y_2^*)(1-y_2)$$
$$+(ac_2+\alpha c_1x^*)y_2^* - b(c_2-c_3)y_1^*.$$

Set $\alpha c_1 = c_3\beta$ and $c_2 = c_3$, then we get

$$\frac{dV_1(t)}{dt} \le \frac{rc_3\beta}{\alpha}(x-x^*) - uc_3y_1(y_1-y_1^*) + pc_3(y_2-y_2^*)(1-y_2) + (ac_3+c_3\beta x^*)y_2^*,$$

$$\le \frac{rc_3\beta}{\alpha}(x-x^*) - uc_3y_1(y_1-y_1^*) + pc_3(y_2-y_2^*+y_2y_2^*) + c_3(a+\beta x^*)y_2^*,$$

$$\le 0 \;\; if \;\; r\beta(x-x^*) - u\alpha y_1(y_1-y_1^*) + p\alpha(y_2-y_2^*+y_2y_2^*) + \alpha(a+\beta x^*)y_2^* \le 0.$$

Equality holds if and only if $x = x^*$, $y_1 = y_1^*$ and $p(y_2-y_2^*+y_2y_2^*) + (a+\beta x^*)y_2^* = 0$.

Now we find an invariant subset E within the following set

$Q = \{(x,y_1,y_2):\; x = x^*, y_1 = y_1^*$ and $p(y_2-y_2^*+y_2y_2^*) + (a+\beta x^*)y_2^* = 0\}$.

Since $x = x^*$, $y_1 = y_1^*$ on E and consequently $0 = \frac{dy_1}{dt} = ay_2 - by_1^* - u(y_1^*)^2$. That implies $y_2 = y_2^*$. Hence, the only invariant set in Q is $E = \{(x^*, y_1^*, y_2^*)\}$. According to LaSalle's invariant principle, globally asymptotical stability of $E_2(x^*, y_1^*, y_2^*)$ follows (Xu, 2011).

8. Numerical Simulation

Now we discuss the numerical simulation of this problem in this paragraph. We take into account an initial solution is $(x,y_1,y_2) = (25, 8, 10)$ and all the parameters value using in this problem are in the given bellow the following table-1. Different phase portrait and different character of the population of this model can be seen for different values of parameters. From theorem-5.3, the positive equilibrium $E_2(x^*,y_1^*,y_2^*) = (4.063, 2.189, 2.945)$ exist, because $r^2(1-mk)^2 - 4rkm(\mathrm{b}y_1^* + uy_1^* - r) = 3.897 > 0$ and which is locally asymptotically stable since $s_0 = 2.469$, $s_2 = 1.294$ and $s_1s_2 - s_0 = 1.402$ without delay ($T = 0$) according to theorem-5.4.

When T increases from 2.2 to 2.7, the characters of the positive equilibrium changes. Sometimes the phase portrait is stable which means all trajectories of the phase portrait moves to the positive equilibrium point (see Figure 2), corresponding population status is shown beside the stable phase portrait (see Figure 1). Sometimes a limit cycle arises which means the positive equilibrium point is unstable and undergoes a Hopf bifurcation. According to theorem 6.1 the positive equilibrium $E_2(x^*, y_1^*, y_2^*)$ is stable for $T < T^*$ where $T^* = 2.364$ and if we increase the value of T further, Hopf-bifurcation will happen at the point $T = T^*$. After crossing the value of T^*, the system still gives stable limit cycle (Figure 4, Figure 6 and Figure 8) and the character of the population (Figure 3, Figure 5 and Figure 7) are given beside. The positive equilibrium point $E_2(x^*, y_1^*, y_2^*)$is unstable for $T > T^*$. The status of the system due to change in the value of T are shown in following figures:

Table 1. Value of all parameters.

Parameter	Value	Parameter	Value
r	1.6	P	0.4
k	35	A	0.2
α	0.5	B	0.05
m	0.01	U	0.1
β	0.2	γ	0.01
T	As follows in the graph		

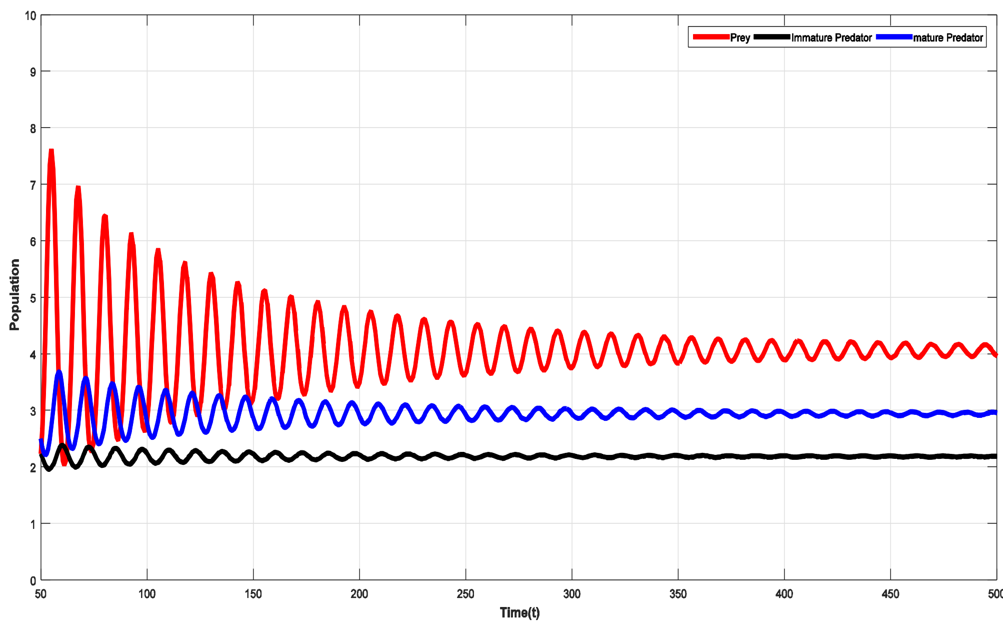

Figure 1. Population graph for T=2.2.

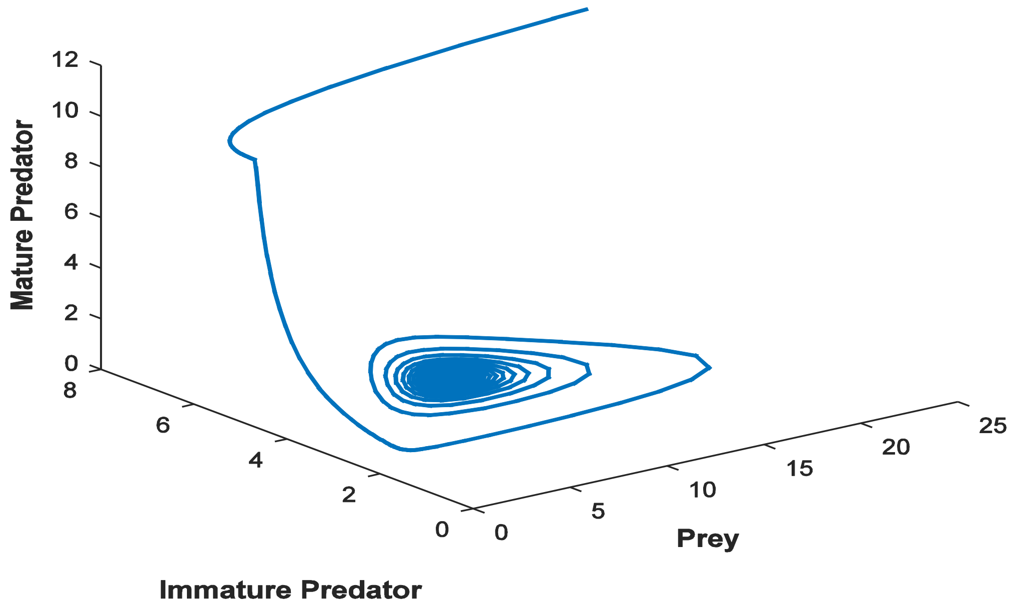

Figure 2. Phase portrait for T=2.2.

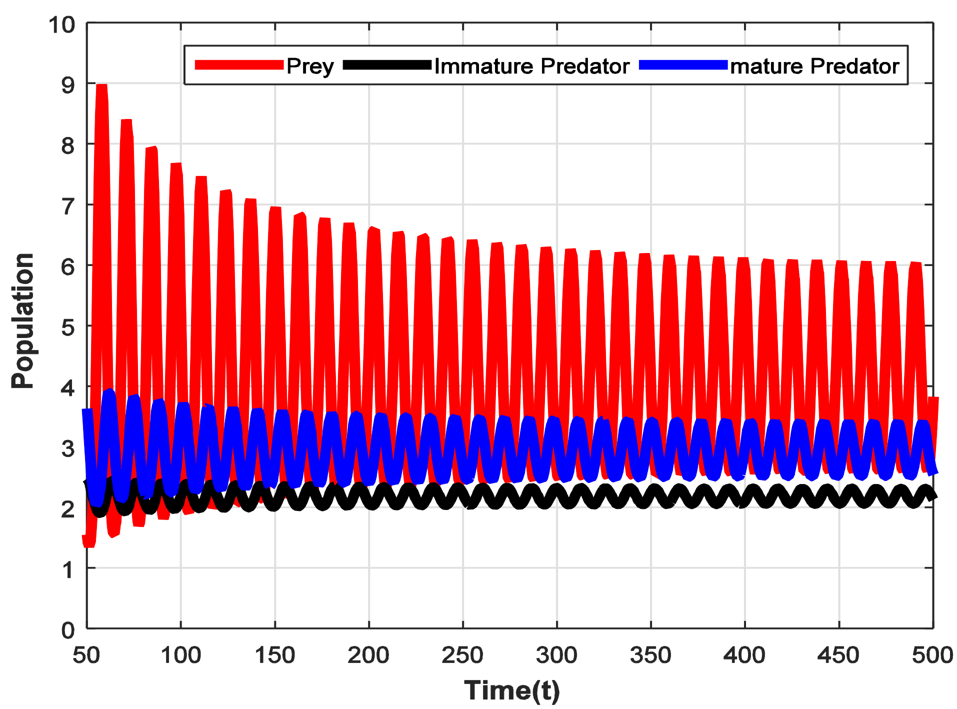

Figure 3. Population graph for T=2.4.

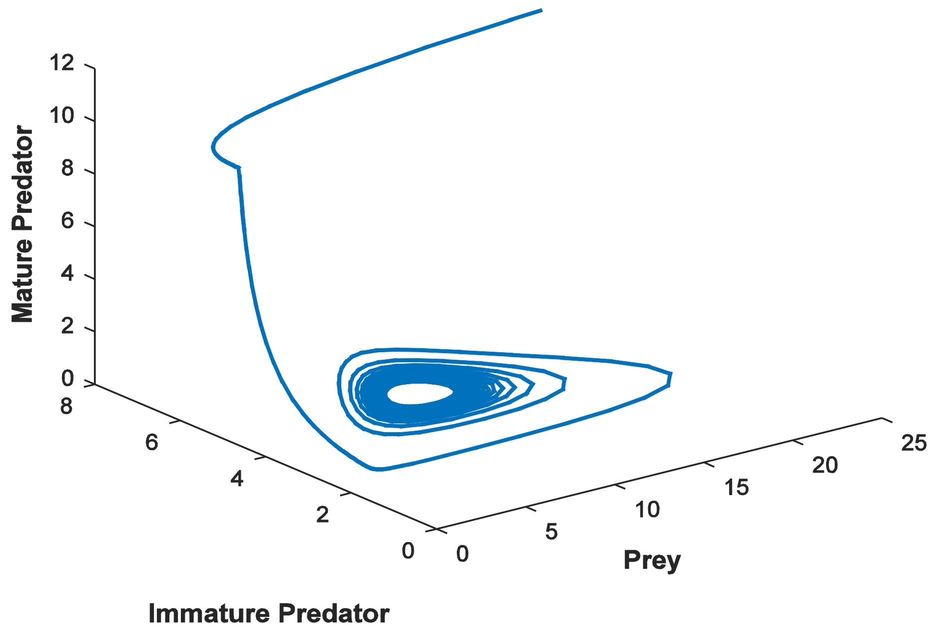

Figure 4. Phase portrait for T=2.4.

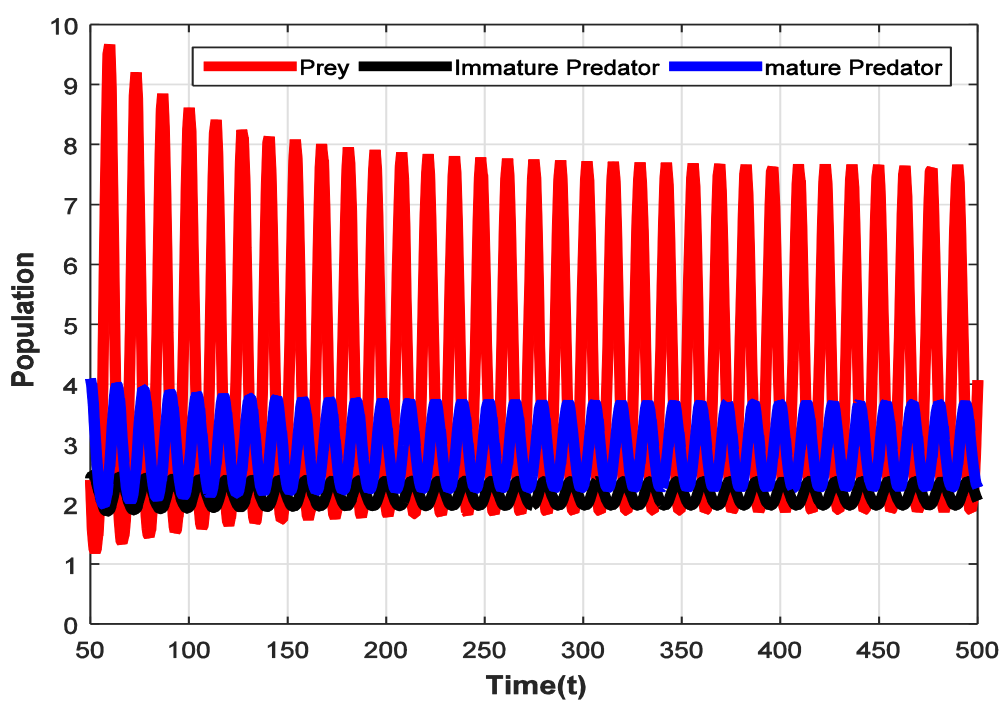

Figure 5. Population graph for T=2.5.

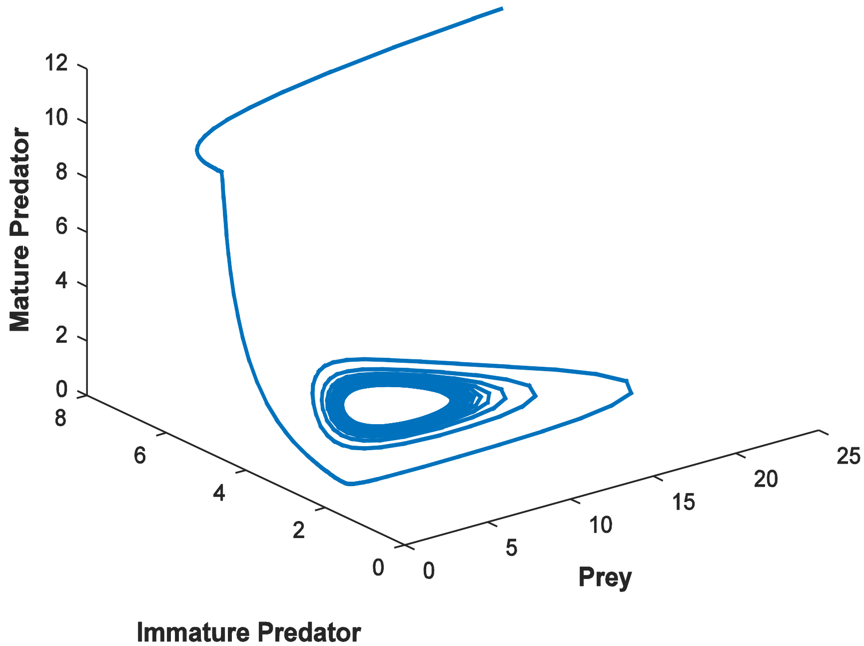

Figure 6. Phase portrait for T=2.5.

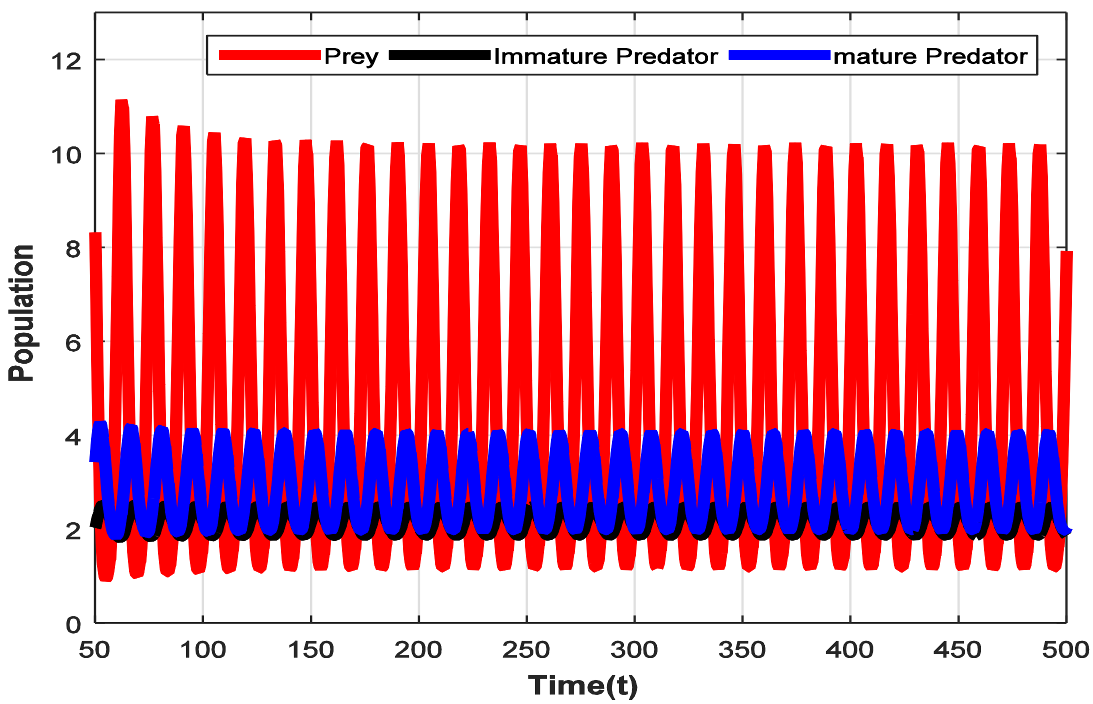

Figure 7. Population graph for T=2.7.

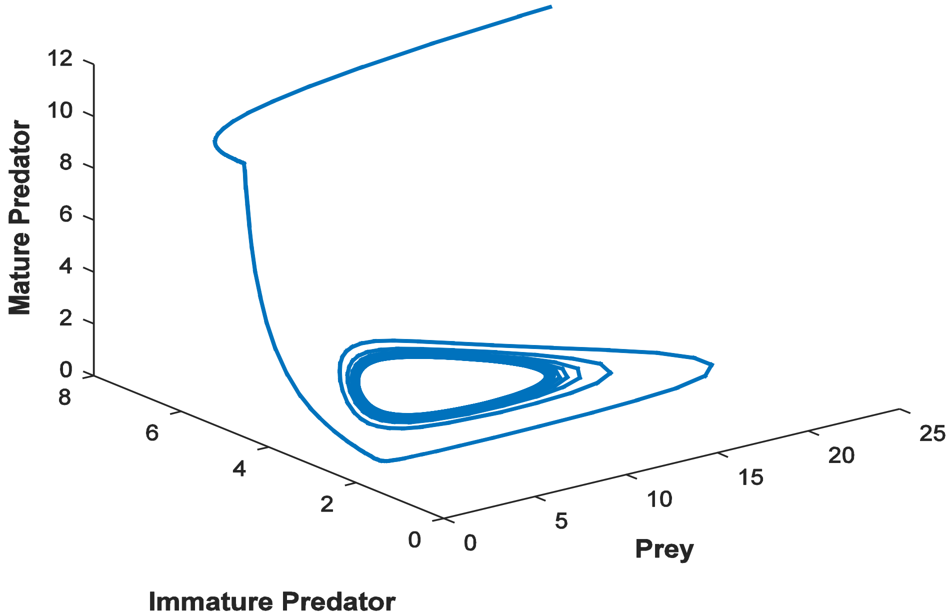

Figure 8. Phase portrait for T=2.7.

To understand the decline rate γ affects the stability of the system, numerical simulation is done by keeping all parameters of Table 1 except γ unchanged with $T = 2.1$. The phase portrait are shown in Figure 10, 12 and 14 and Figure 9, 11, 13 show the variation of population with time. We see that as gamma (γ) changes from 0.01 to 0.05, the positive equilibrium point loses its stability and a limit cycle generated.

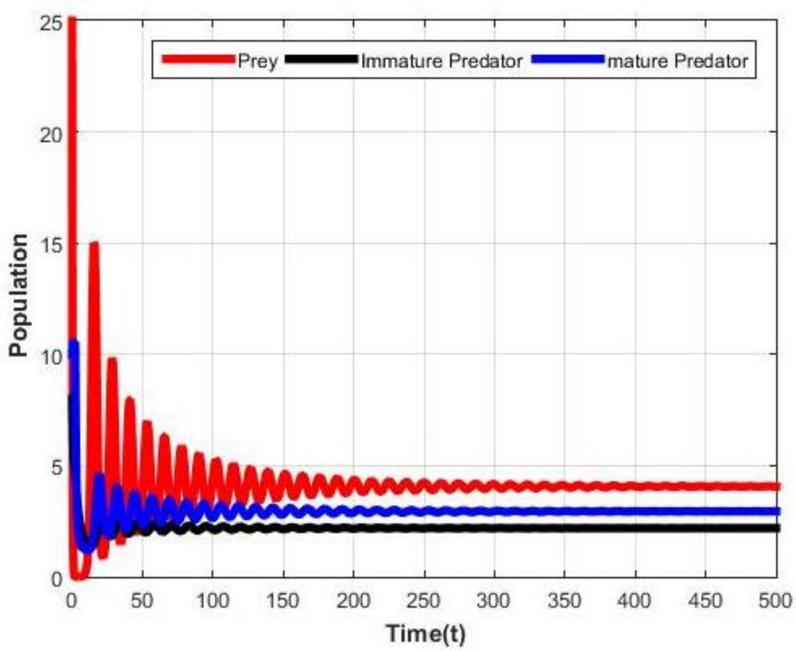

Figure 9. Population graph for $\gamma = 0.01$.

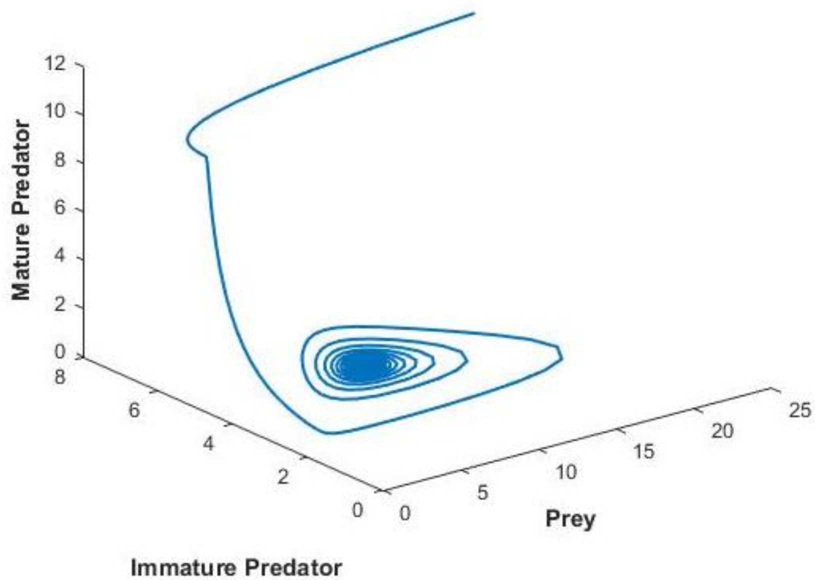

Figure 10. Phase portrait for $\gamma = 0.01$.

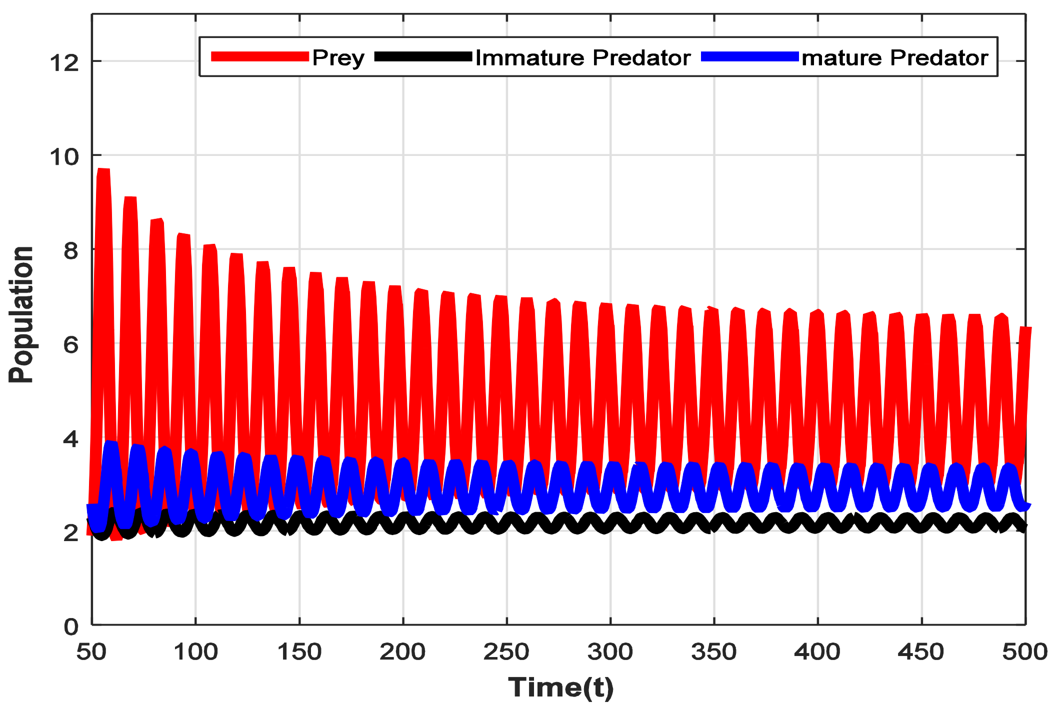

Figure 11. Population graph for $\gamma = 0.03$.

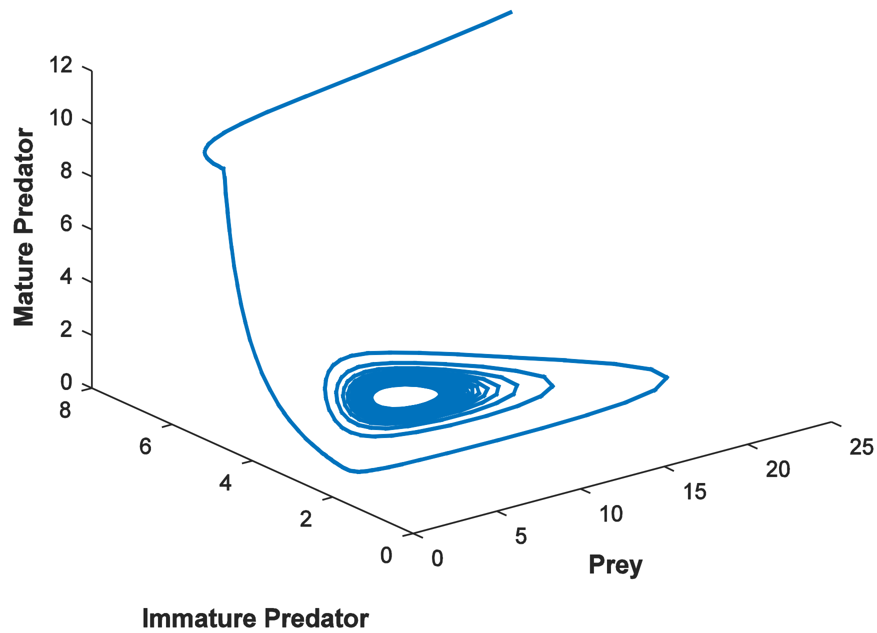

Figure 12. Phase portrait for $\gamma = 0.03$.

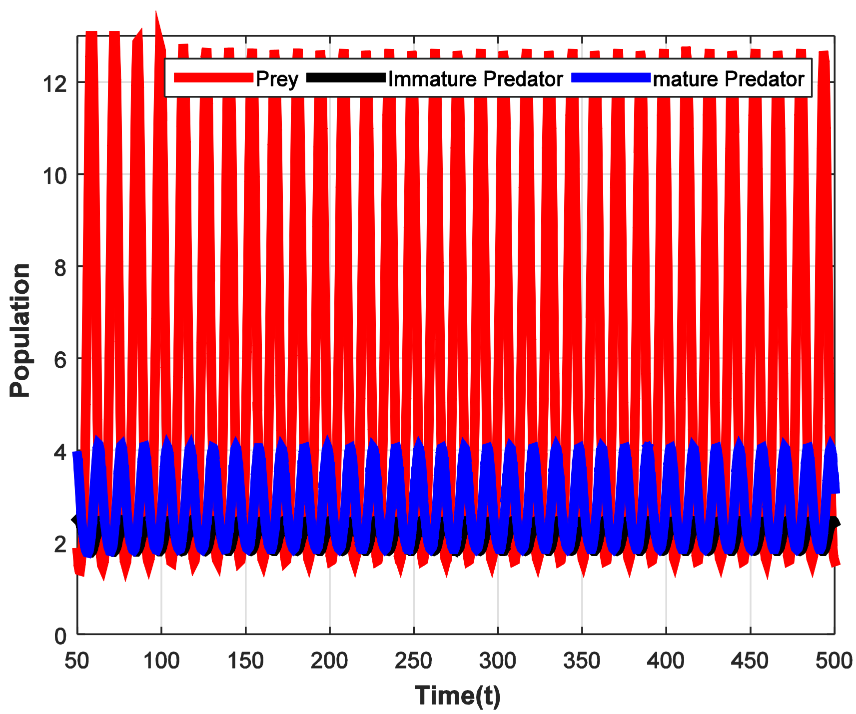

Figure 13. Population graph for $\gamma = 0.05$.

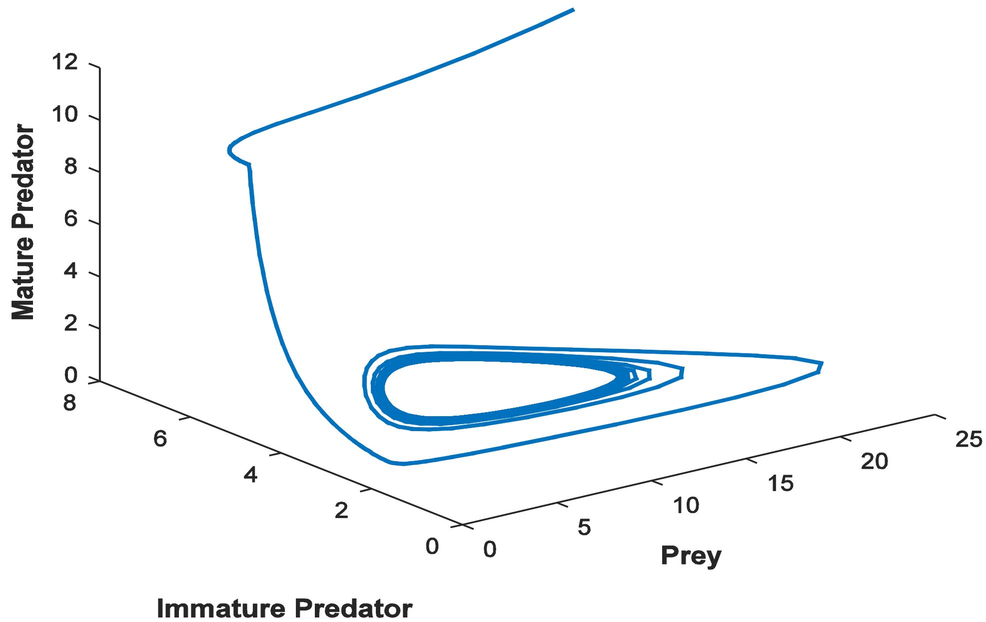

Figure 14. Phase portrait for $\gamma = 0.05$.

9. Conclusion

In this paper a prey-predator model with stage-structured in predator has been analyzed by including the effect of gestation delay, ratio-dependent functional response and including the presence of transferable disease from prey to predator. It was shown that subject to certain conditions the system will be bounded and persistent. The extinction criterion has been derived. The stability conditions of positive equilibrium point have been derived in presence and absence of delay. The change in stability leading to Hopf-bifurcation with respect to delay parameter has been established theoretically and numerically. The effect of the decline rate of predator on the system has been investigated numerically. It is seen that the system moves from stable to unstable with increase in the decline rate. Numerical show the appearance of a limit cycle and stable oscillation of the system.

In future, the work may be extended to study 1) the non-local effects arising out of prey and predator dispersion. 2) The effect of SI disease in the predator. 3) The direction and stability of Hopf bifurcation.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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