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Advertising and Pricing Policies for a Diffusion Model Incorporating Price Sensitive Potential Market in Segment Specific Environment

Pradeep Kumar
Department of Mathematics, AIAS,
Amity University, Noida, U.P., India.
E-mail: pradeepvipro@gmail.com

Kuldeep Chaudhary
Department of Mathematics, AIAS,
Amity University, Noida, U.P., India.
Corresponding author: chaudharyiitr33@gmail.com

Vijay Kumar
Department of Mathematics, AIAS,
Amity University, Noida, U.P., India.
E-mail: vijay_parashar@yahoo.com

V. B. Singh
School of Computer and System Sciences,
Jawaharlal Nehru University, Delhi, India.
E-mail: vbsingh@mail.jnu.ac.in

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Abstract
This paper suggests an optimal control formulation of a diffusion model to determine the optimal advertising and pricing policy for a new product in segmented market involving price dependent market potential where total potential market is a decreasing function of price. We assume that sales rate is proportional to the number of adopters and evolved under the joint effect of a single advertising channel and segment specific advertising process that affect sales rate in unsaturated market. Single channel advertising with a fixed spectrum is available for each segment in whole market while segment specific advertising process can reach each target segment individually. The optimal dynamic price and advertising policies are obtained by applying Pontryagin's maximum principle. Finally, numerical examples of two cases for a discrete version of the proposed control problem are provided to illustrate the efficacy of the proposed method

Keywords- Optimal control theory, Segmented market, Price-dependent potential market, Advertising and price policy.

1. Introduction

The whole world is now easily accessible owing to globalisation and industrialization. As a result, competition is fierce between product to product, service to service, and industry to industry. Because of industrialization, technology has advanced at a rapid pace, and advertising mediums have evolved as well. In order to enhance the sales of a new product and compete for a considerable market, businesses must develop policies to reach out to consumers and spread awareness about the product's quality and innovative features through advertising and promotions. Firms always try to invest a large amount in advertising activities in order to increase product sales and improve their advertising activities in order to gain a targeted market share. Aside from the advertising process, firms try to differentiate themselves through their pricing and other marketing strategies.

In the past few years, we have seen a rapid increase in the number of new products entering in the market, which opens up new opportunities for different innovative and inclusive pricing approaches, allowing consumers to choose the price they want to pay. Firms are also considering various types of combined pricing and advertising opportunities. In this paper, we focus on optimal pricing and advertising in segmented market with price-dependent total potential market. The optimal price of a new product in targeted market means that customers can easily concentrate on exploring for the lowest price and if the price decreases, then the potential market means that the product's sales will increase accordingly. If the company focuses solely on lowering the price of the product in order to increase sales, the company will have to compromise on product quality, which can have a negative impact on the company's sales and revenue. As a result, while keeping the product quality in mind, the company should always try to keep its price as low as possible and accordingly advertise its products everywhere.

Diffusion theory is commonly used in marketing to describe the dynamics of a product's life cycle. Bass (1969) presented the simplest diffusion model related to marketing and this diffusion model has been widely used in the field of marketing since then. Many researchers, including Dockner and Jorgensen (1988), Simon and Sebastian (1987) and Horsky and Simmon (1983) have studied diffusion models that examine the impact of advertising policies on diffusion of the product. Several researchers contributed towards investigating the influence of pricing policies on diffusion of the product (Sethi and Bass, 2003; Kamakura and Balasubramanium, 1988; Kalish, 1985; Kalish, 1983; Robinson and Lakhani, 1975). Feichtinger (1982) proposed a diffusion model to investigate the optimal pricing policy of a new product, assuming that the entire market potential is a concave decreasing function of price. Jorgensen (1983) developed a diffusion model for a new product acceptance in order to investigate the optimal pricing policy with price-dependent market potential. Bass et al. (1994) developed a generalized form of Bass model (1969) that included both price strategies and advertising strategies. Thompson and Teng (1996) used the concept of the production learning curve and analyzed advertising and pricing in an oligopolistic market. In marketing, optimal control theoretic models have been used to analyze the dynamic behaviour of sales, advertising and pricing. Some researchers (Stummer et al., 2015; Thompson and Teng, 1984; Xiaojun and Li, 2016; Chutani and Sethi, 2012; Helmes et al., 2013; Huang et al., 2012) used an optimal control theoretic approach to study advertising and pricing policies for newly launched products in a market. Najafi-Ghobadi et al. (2021) developed a new diffusion model by considering pricing and advertising as key marketing-mix variables in presence of heterogeneous environment and analyzed the proposed model using maximum principle approach. Mesak et al. (2020) used an analytical approach to identify the optimal dynamic policies for pricing and advertising in both duopolistic and monopolistic markets.

It is obvious that the market is not homogeneous, so it is critical to divide purchasers into different segments based on their requirements. Advertising policies and design strategies are tailored to potential customers with similar requirements and preferences who belong to the same segment. These potential customers are categorized based on demographic, geographic characteristics, and mindset. Companies finalize their strategies for capturing market share and increasing sales of the product in the market.

Several sales-advertising models for market segmentation in dynamic advertising have been proposed in the literature using an optimal control theoretic approach (Grosset and Viscolani, 2005; Seidmann et al., 1987; Mariusz and Dominika, 2017; Mehta et al., 2020). Seidmann et al. (1987) proposed a general sales-advertising model. The potential population is distributed across various factors in this distributed diffusion model, such as family size, geographic location, income levels and so on. Buratto et al. (2006) developed a model for a segmented market based on Nerlove-Arrow's goodwill dynamics (Nerlove and Arrow, 1962). Later, Jha et al. (2009) introduced some market segmentation concepts in a diffusion model in which sales evolve as a result of segment-specific and single channel advertising. They derived advertising strategies in order to maximize total profit. Chaudhary et al. (2022) proposed a model for a new product that incorporates brand goodwill in a segment specific market. They discussed local and global stability of a dynamic system. This paper considers the advertising and pricing policies for a diffusion model incorporating price dependent total market potential in a segmented market. We have considered a firm that wants to plan advertising strategies that use both single channel and segment-specific advertising process at the same time in order to maximize total profit.

The reminder of this paper is organized as follows: Section 2 introduces a mathematical model development involving a price dependent potential market and formulation of its optimal control model. In section 3, the optimal pricing and advertising policies of the proposed model are determined and analyzed using the Pontryagin's Maximum principle. Numerical examples for discretized version of the optimal control model are provided in section 4. Finally, the paper is concluded, and the scope for future research is discussed in Section 5.

2. Model Development

Consider a monopolistic firm that needs to decide about its advertising and pricing policies for a single product in segmented market. Here, we assume that the firm wants to plan the advertising process in each segment simultaneously by using segment specific advertising and single channel advertising to capture the potential market for the product with the objective of maximizing its profit. Single channel advertising with fixed segment spectrum reaches each segment with relative segment-effectiveness. Segment specific advertising process reaches each segment with desired advertising effort intensity.

Assuming t is continuous and have a finite planning horizon $[0, T]$. Let the total consumer market be divided in N segments, each one specified by some segmentation attribute i and the function $\overline{Y_i} \geq 0$ denotes the number of potential consumers who belongs to the i^{th} segment recognized by segmentation attribute. Let $y_i(t)$ and $p_i(t)$ be sales rate and price per unit for i^{th} segment at time t with the assumption that sales are proportional to the number of adopters. Here, we assume that total market potential $\overline{Y}_i(p_i)$ for each segment is a concave decreasing function of the price i.e., $\overline{Y_i}'(p_i) < 0, \overline{Y_i}''(p_i) < 0, \overline{p_i} \leq p_i(t) \leq \overline{P_i}$. This means that the increase in $\overline{Y_i}$ due to a decrease in price has 'decreasing returns to scale'. To incorporate the single channel and segment specific advertising process in segmented market, we extend the well-known diffusion model to the following dynamic equation of sales evolution:

$$\frac{dy_i(t)}{dt} = \left(a_i + b_i\, y_i(t)\right)\left(u_i(t) + \alpha_i\, u(t)\right)\left(\overline{Y_i}(p_i) - y_i(t)\right),\ i = 1,2, \dots, N \tag{1}$$

with the initial condition $y_i(0) = y_{i0},\ \alpha_i > 0$ & $\sum_{i=1}^{N} \alpha_i = 1$. Where, $\alpha_i > 0$ is the channel medium spectrum, and to provide the different effectiveness of the advertising medium on the market segments; a_i and b_i are coefficients of external and internal influence in i^{th} segment; $u_i(t) \geq 0$ and $u(t) \geq 0$ denote the segment specific advertising effort rate and single channel advertising effort rate for i^{th} segment at time t that influence the sales in untapped market in each segment. Without loss of generality and technical simplicity, we assume that total market potential decreases with the price and $\overline{Y_i}(p_i) = \overline{Y_i} - g_i\, p_i(t), g_i > 0$.

On the assumption that the firm now is cherished more highly than profit later. Using optimal values of single channel and segment specific advertising effort rates over a planning horizon T, the present value of net discounted profit with a fixed rate r can be written as:

$$Max\,J = \int_0^T e^{-rt} \left(\sum_{i=1}^{N} \left[(p_i(t) - c_i)\, y_i(t) - \phi_i\big(u_i(t)\big) \right] - \psi\big(u(t)\big) \right) dt \tag{2}$$

subjected to the dynamics in (1) and the constraint $c_i < p_i(t) \leq \frac{\bar{Y}_i - y_{i0}}{g_i}$ for all $t \in [0, T]$. Where, $c_i > 0$ denotes marginal costs and for simplicity constant; $\phi_i\,(u_i(t))$ and $\psi(u(t))$ are the segment specific advertising and single channel advertising effort costs and $\phi_i\,(u_i(t))$ and $\psi(u(t))$ twice continuously differentiable functions such that $\phi_{i_{u_i}}\big(u_i(t)\big) > 0$; $\psi_u\big(u(t)\big) > 0$; $\phi_{i_{u_i u_i}}\big(u_i(t)\big) > 0$; $\psi_{uu}\big(u(t)\big) > 0$.

Now, objective function Eq. (2) and state equation Eq. (1) and constraint, the problem of optimization can be defined as:

$$\left.\begin{aligned} Max\,J = \int_0^T e^{-rt} \left(\sum_{i=1}^{N} \left[(p_i(t) - c_i)\, y_i(t) - \phi_i\big(u_i(t)\big) \right] - \psi\big(u(t)\big) \right) dt \\ \frac{dy_i(t)}{dt} = \big(a_i + b_i\, y_i(t)\big) \big(u_i(t) + \alpha_i\, u(t)\big)\big(\bar{Y}_i(p_i) - y_i(t)\big),\; y_i(0) = y_{i0}\;\; \forall\, i = 1,2,\dots,N \end{aligned}\right\} \tag{3}$$

In the above formulated optimal control problem framework, $p_i(t), u_i(t)$ and $u(t)$ are control variables and $y_i(t)$ is a state variable.

3. The Optimal Dynamic Pricing and Advertising Strategy

We use Pontryagin's Maximum Principle to obtain the optimal price and advertising effort policies for optimal control theory problem expressed in Eq. (3). We formulate the current-value Hamiltonian function as

$$H = \sum_{i=1}^{N} \left[(p_i(t) - c_i) y_i(t) - \phi_i\big(u_i(t)\big) + \lambda_i(t)\, \dot{y}_i(t) \right] - \psi\big(u(t)\big) \tag{4}$$

The Hamiltonian function is composed of the current profit $\left(\sum_{i=1}^{N} \left[(p_i(t) - c_i)\, y_i(t) - \phi_i\big(u_i(t)\big) \right] - \psi\big(u(t)\big) \right)$ and future profit $\sum_{i=1}^{N} \lambda_i(t)\, \dot{y}_i(t)$. As such, H measures the surrogate profit to be maximized. Where, $\lambda_i(t)$ represent the marginal valuation of sales variables $y_i(t)$ at time *t* on the intertemporal profit and known as adjoint variables. The necessary conditions of Pontryagin's Maximum Principle (Sunita et al., 2020; Sethi and Thompson, 2000; Seierstad and Sydsaeter, 1987) for controls $p_i^*(t), u_i^*(t)$ and $u^*(t)$ to be optimal controls are:

$$H(t, y_i^*, p_i^*, u_i^*, u^*, \lambda) = H(t, y_i^*, p_i, u_i, u, \lambda) \tag{5}$$

$$\frac{\partial H^*}{\partial u_i} = 0 \tag{6}$$

$$\frac{\partial H^*}{\partial u} = 0 \tag{7}$$

$$\frac{\partial H^*}{\partial p_i} = 0 \tag{8}$$

$$\frac{d\lambda_i(t)}{dt} = r\,\lambda_i(t) - \frac{\partial H^*}{\partial y_i(t)}, \lambda_i(T) = 0 \tag{9}$$

for all $t \in [0, T]$. From Mangasarian Sufficiency Theorem specified in Seierstad and Sydsaeter, 1987, the optimal value of the control variables $p_i^*(t), u_i^*(t)$ and $u^*(t)$ for each segment from Eq. (6), (7) and Eq. (8) are:

$$u_i^*(t) = \phi_{u_i}^{-1}\left(\lambda_i(t)(a_i + b_i\, y_i(t)\,)\left(\overline{Y_i}(p_i) - y_i(t)\right)\right)\ , \qquad i = 1, 2, \ldots, N \tag{10}$$

$$u^*(t) = \psi_u^{-1}\left(\textstyle\sum_{i=1}^{N}\left[\lambda_i(t)(a_i + b_i\, y_i(t)\,)\alpha_i\left(\overline{Y_i}(p_i) - y_i(t)\right)\right]\right), \quad i = 1, 2, \ldots, N \tag{11}$$

$$p_i^*(t) = \bar{Y}_i^{-1}\left(-\frac{y_i(t)}{\lambda_i(t)(a_i + b_i\, y_i(t)\,)\left(u_i(t) + \alpha_i\, u(t)\right)}\right), \qquad i = 1, 2, \ldots, N \tag{12}$$

where, $\bar{Y}_i^{-1}$ is inverse function of $\frac{\partial \bar{Y}_i}{\partial p_i}$; $\phi_{u_i}^{-1}$ is inverse function of segment specific marginal advertising effort at i^{th} segment; ψ_u^{-1} is inverse function of single channel marginal advertising effort. For particular case, where the market potential is linear decreasing function of price i.e., $\overline{Y_i}(p_i) = \overline{Y_i} - g_i\, p_i(t), g_i > 0$. then

$$p_i^*(t) = \begin{cases} \overline{p_i}\ ,\ \lambda_i(t) > \dfrac{y_i(t)}{g_i\,(a_i + b_i\, y_i(t)\,)\left(u_i(t) + \alpha_i\, u(t)\right)} \\ \overline{P_i}\ ,\ \lambda_i(t) < \dfrac{y_i(t)}{g_i\,(a_i + b_i\, y_i(t)\,)\left(u_i(t) + \alpha_i\, u(t)\right)} \end{cases}, \qquad i = 1, 2, \ldots, N \tag{13}$$

where, $\overline{p_i}$ and $\overline{P_i}$ are minimum and maximum admissible prices, and moreover $\overline{p_i} < c_i, \overline{P_i} < \frac{\overline{Y_i} - y_{i0}}{g_i}$. From expressions Eq. (10) and Eq. (11), we found that the optimal advertising effort rate policy indicates that when market is nearly saturated, then the segment specific advertising effort rate and single channel advertising effort rate respectively will diminish over time. With that interpretation, it is easy to understand that there is no necessity of any type of advertising effort rate in the market in the case of market saturation. Eq. (13) shows the optimal price policy over the planning period. For adjoint variable $\lambda_i(t)$, we have adjoint equation as

$$\frac{d\lambda_i(t)}{dt} = r\lambda_i(t) - \frac{\partial H}{\partial y_i}, \lambda_i(T) = 0 \tag{14}$$

Solving Eq. (14), we get

$$\lambda_i(t) = \int_t^T e^{-r(\tau - t)}\left(p_i(t) - c_i + \frac{\partial \phi_i}{\partial u_i}\,\frac{\frac{\partial \dot{y_i}}{\partial y_i}}{\frac{\partial \dot{y_i}}{\partial u_i}}\right) d\tau \tag{15}$$

Preceding expression Eq. (15) correspond to the value of future profit of getting one more unit of sale or marginal value per unit of sale at time *t*.

3.1 Special Cases of General Formulation

We consider two scenarios of advertising cost function to depict the behavior of proposed optimal control problem. We have assumed the different advertising effort cost functional forms to analyze the related optimal control policies:

3.1.1 Case (1)

Generally, advertising process are costly and define as an increasing convex function of advertising effort rate. Without loss of generality and technical simplicity, we assume advertising effort costs take quadratic forms $\phi_i\left(u_i(t)\right) = \frac{\kappa_i}{2}\, u_i^2(t)$ and $\varphi\left(u(t)\right) = \frac{\kappa}{2}\, u^2(t)$ that is common in literature mentioned in (Teng and Thompson, 1983; Chaudhary et al. 2021), where $\kappa_i > 0$ and $\kappa > 0$ as representing the magnitude of advertising effort rates. Then the current-value Hamiltonian function is

$$H = \sum_{i=1}^{N}\left[(p_i(t) - c_i)y_i(t) - \frac{\kappa_i}{2}\, u_i^2(t) + \lambda_i(t)\, \dot{y_i}(t)\right] - \frac{\kappa}{2}\, u^2(t) \tag{16}$$

For adjoint variable $\lambda_i(t)$, we have adjoint equation as

$$\lambda_i(t) = \int_t^T e^{-r(\tau-t)} \left(p_i(t) - c_i + k_i u^*(t) \frac{\frac{\partial \dot{y}_i}{\partial y_i}}{\frac{\partial \dot{y}_i}{\partial u_i}} \right) d\tau \tag{17}$$

By maximum principle, the optimal value of the control variables $u_i^*(t)$ and $u^*(t)$ for each segment respectively from Eq. (6) and Eq. (7) are:

$$u_i^*(t) = \frac{1}{\kappa_i} \left(\lambda_i(t)(a_i + b_i\, y_i(t)\,)\left(\overline{Y_i}(p_i) - y_i(t)\right) \right) \;, i = 1, 2, \ldots, N \tag{18}$$

$$u^*(t) = \frac{1}{k} \left(\sum_{i=1}^{N} \left[\lambda_i(t)(a_i + b_i\, y_i(t)\,)\alpha_i \left(\overline{Y_i}(p_i) - y_i(t)\right) \right] \right), i = 1, 2, \ldots, N \tag{19}$$

From above, the optimal segment specific and single channel advertising effort rates policy indicates that when the market is nearly saturated, then both the type of advertising effort rates decrease over time. Furthermore, it is optimal to advertise with minimum effort near the end of planning period.

3.1.2 Case (2)

In contrast with quadratic forms of advertising cost functions discussed in special case 3.1.1, we have considered a case when the advertising cost functions are linear function of advertising effort rate and given in Grosset and Viscolani, (2005). Many researchers used this advertising cost function form to demonstrate the linear relationship between advertising cost and advertising effort rate in their pioneer work in the field of marketing and economics. Therefore, the advertising cost functions are: $\phi_i\left(u_i(t)\right) = k_i\, u_i(t)$ and $\varphi\left(u(t)\right) = \kappa\, u(t)$, $u_i(t) \in [0, \overline{A_i}], u(t) \in [0, \bar{A}]$, where $\overline{A_i}$ and $\bar{A}$ are positive constants and denotes the maximum permissible segment-specific and single channel effort rate which are determined by availability of advertising budget, limitations of advertising media etc.; $\kappa_i > 0$ denotes the degree of advertising effort rates towards the i^{th} segment for segment-specific advertising process and $\kappa > 0$ represents the degree of effort rate in all segments for single advertising channel. Now, Hamiltonian function becomes

$$H = \sum_{i=1}^{N} [(p_i(t) - c_i) y_i(t) - \kappa_i\, u_i(t) + \lambda_i(t)\, \dot{y}_i(t)] - \kappa\, u(t) \tag{20}$$

The above Hamiltonian function is linear in $u_i(t)$ and $u(t)$. So, we attain optimal value of advertising effort rates and given by the following expression:

$$u_i^*(t) = \begin{cases} 0 & if\ W_i(t) < 0 \\ \overline{A_i} & if\ W_i(t) > 0 \end{cases} \tag{21}$$

$$u(t) = \begin{cases} 0 & if\ W(t) < 0 \\ \bar{A} & if\ W(t) > 0 \end{cases} \tag{22}$$

where, $W_i(t) = -\kappa_i + \lambda_i(t)\left(a_i + b_i y_i(t)\right)\left(\overline{Y_i}(p_i) - y_i(t)\right)$ and $W(t) = -\kappa + \sum_{i=1}^{N}\left[\alpha_i\, \lambda_i(t)(a_i + b_i\, y_i(t)\,)\left(\overline{Y_i}(p_i) - y_i(t)\right)\right]$ are advertising effort switching function. In terminology of optimal control theory, this type of optimal control described by Eq. (21) and (22) are known as "Bang-Bang" control. Nevertheless, interior controls are possible on an arc along $u_i(t)$ and $u(t)$ and these arcs are called as "Singular arcs" (Seierstad and Sydsaeter, 1987; Sethi and Thompson, 2000). By using optimal policy $(u^*(t), u_i^*(t), Y_i(p^*))$, the sales trajectory is given as

$$y_i^*(t) = \frac{\bar{Y}_i(p^*) - a_i\left(\frac{\bar{Y}_i(p^*) - y_{i0}}{a_i + b_i y_{i0}}\right) exp - \{a_i + b_i \bar{Y}_i(p^*)\} \int_0^t (u_i^* + \alpha_i u^*) d\tau}{1 + b_i\left(\frac{\bar{Y}_i(p^*) - y_{i0}}{a_i + b_i y_{i0}}\right) exp - \{a_i + b_i \bar{Y}_i(p^*)\} \int_0^t (u_i^* + \alpha_i u^*) d\tau} \ \forall i \in M \quad (23)$$

If we define $U^*(t) = \int_0^t (u_i^* + \alpha_i u^*) d\tau$ is the cumulative advertising effort rate, then above equation becomes

$$y_i^*(t) = \frac{\bar{Y}_i(p^*) - a_i\left(\frac{\bar{Y}_i(p^*) - y_{i0}}{a_i + b_i y_{i0}}\right) exp - \{a_i + b_i \bar{Y}_i(p^*)\} U^*(t)}{1 + b_i\left(\frac{\bar{Y}_i(p^*) - y_{i0}}{a_i + b_i y_{i0}}\right) exp - \{a_i + b_i \bar{Y}_i(p^*)\} U^*(t)} \ \forall i \in M \quad (24)$$

If $y_i(0) = y_{i0} = 0$, then we obtained

$$y_i^*(t) = \bar{Y}_i(p^*) \left[\frac{\left(1 - a_i e^{-\left(a_i + b_i \bar{Y}_i(p^*)\right) U^*(t)}\right)}{1 + \frac{b_i}{a_i} \bar{Y}_i(p^*) e^{-\left(a_i + b_i \bar{Y}_i(p^*)\right) U^*(t)}} \right] \ \forall i \in M \quad (25)$$

The sales trajectory resembles the generalized Bass- Model' (Bass et al., 1994) sales trajectories. In situations given above, the proposed optimal control model is solved by Maximum Principle theoretically. The model is nonlinear in nature, and it leads to complex analytical expression, so next step is to determine the optimal pricing and advertising policies that maximize the total profit per unit time with the help of numerical examples.

4. Numerical Illustration

In this section, we discuss the optimal advertising and price dynamics of the optimal control problem numerically. Using the numerical example, dynamic optimization model established here can be applied to illustrate the applicability for single product in segmented market. In multi-cultural countries, firms encourage their product in regional and national language to influence customers. As data available in discrete form, the equivalent discrete problem is attained from the continuous optimal control problem (3) as stated in Rosen (1968). The discrete version of optimal control problem (3) is given as

$$\left.\begin{aligned} &Max\ J = \sum_{k=1}^{T} \left(\left(\left[\sum_{i=1}^{N} (p_i(k) y_i(k) - c_i) - \frac{k_i}{2} u_i^2(k) \right] - \frac{k}{2} u^2(k) \right) \left(\frac{1}{(1+k)^{k-1}} \right) \right) \\ &subjected\ to \\ &y_i(k+1) = y_i(k) + \left(a_i + b_i y_i(k)\right)\left(u_i(k) + \alpha_i u(k)\right)\left(\left(\bar{Y}_i - g_i\, p(k)\right) - y_i(k)\right), i = 1,2, \dots, N \\ &c_i \le p_i(t) \le \frac{\bar{Y}_i - y_{i0}}{g_i}, i = 1,2, \dots, N \end{aligned}\right\} \quad (16)$$

This discretized version of the proposed optimal control model is solve using Lingo11 (Thirez, 2000). We consider a case of three market segments ($N = 3$) and time horizon is assumed to be divide into 12 time periods with equal duration. The value of parameters used for the purpose of illustration are given in Table 1.

Table 1. The values of parameters.

Segments	$\bar{Y}_i$	a_i	b_i	α_i	g_i	c_i	k_i	Initial Sales	r
S1	1000	0.001161	0.4805750	0.19	0.1	3000	2000	100	0.09
S2	1000	0.00138	0.5403950	0.18	0.1	3200	2000	110	
S3	1000	0.00549	0.3136200	0.18	0.1	3300	2000	110	

Total advertising budget is assumed to be allocated for single channel advertising and segment specific advertising process in all the three segments. The discrete problem is coded and solved by using Lingo11.

4.1 Case (1)

Here, we assume that single channel and segment specific advertising cost are quadratic function of effort rate. According to our results, the optimal approximate value of total profit is 24403180 units. The optimal allocation of segment specific and single channel advertising efforts rate for each segment are given in Table 2 and the optimal value of sales and price for each segment are tabulated in Table 3.

Table 2. Optimal value of segment specific and single channel advertising effort rate.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$u_1(t)$	1	0.1	0.5161	0.1	0.1	0.1	0.1	0.5095	0.1	0.5095	0.1	1
$u_2(t)$	1	0.1	0.1	0.4724	0.1	0.1	0.1	0.4659	0.1	0.4659	0.1	1
$u_3(t)$	1	0.1	0.7102	0.1	0.1	0.7102	0.1	0.7038	0.1	0.7038	0.1	1
$u(t)$	0.64	0.1	0.1	0.1	0.1	0.1	0.1	0.1561	0.1	0.1561	0.1	0

Table 3. Optimal value of sales and price (In Units).

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
y_1	100	357	342	356	349	349	349	342	356	342	356	0
y_2	110	346	339	333	345	339	339	333	345	333	345	0
y_3	110	346	322	346	333	322	346	322	346	322	346	0
p_1	8952	6430	6574	6437	6507	6507	6511	6574	6440	6574	6608	9000
p_2	8864	6534	6609	6664	6545	6607	6609	6664	6548	6664	6698	8900
p_3	8838	6549	6772	6548	6671	6772	6557	6772	6557	6772	6808	8900

4.2 Case (2)

In this subsection, we assume that single channel and segment specific advertising cost are linear function of effort rate. According to our results, the optimal approximate value of total profit is 24389330 units. The optimal allocation of segment specific and single channel advertising efforts rate for each segment are given in Table 4 and the optimal value of sales and price for each segment are tabulated in Table 5.

Table 4. Optimal value of segment specific and single channel advertising effort rate.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$u_1(t)$	1	0.1	0.1	0.1	0.1	0.1	0.1	0.3366	0.1	0.1	0.1	1
$u_2(t)$	1	0.1	0.1	0.2831	0.1	0.1	0.1	0.2831	0.1	0.2831	0.1	1
$u_3(t)$	1	0.1	0.5830	0.1	0.1	0.5830	0.1	0.5830	0.1	0.5830	0.1	1
$u(t)$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0

Table 5. Optimal value of sales and price (In Units).

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
y_1	100	358	349	349	349	349	349	343	356	349	349	0
y_2	110	347	339	334	344	339	339	335	344	335	344	0
y_3	110	347	323	346	334	323	323	322	346	323	346	0
p_1	8947	6428	6508	6508	6508	6508	6511	6564	6448	6508	6683	9000
p_2	8861	6535	6609	6652	6658	6607	6609	6652	6560	6652	6711	8900
p_3	8833	6551	6767	6553	6671	6767	6562	6769	6562	6767	6814	8900

In both the cases, it shows clearly when price is increasing then sales are decreased and when price is decreased then sales are increased.

5. Managerial Implication and Conclusion

5.1 Managerial Implications

Pricing and advertising are integral parts of the market planning of all the business activities. With the arduous growth in the new products and rising need of the customers, the competition among the firms has grown. Consequently, a suitable marketing plan is required to avoid failure of the product. To understand the growth and strategies of a market system, mathematical models have proven to be perfect tools for pricing and advertising policies for durable or new product sales.

The research aim of this study is to develop a more general optimal control theoretic mathematical model for determining the advertising and pricing policies for a product in segmented market with price dependent demand. Such model is relevant for those products that have price sensitive potential market. The model incorporates price dependent potential market in segment specific environment.

5.2 Conclusion and Future Directions

When a new product introduces in any market, then several important factors like advertising, product awareness and the purchasing power of the customers etc. influencing the total size of the market potential. The purpose for market segmentation in advertising process is to enable the advertising programs to focus on target market of customers that are almost certainly to purchase the product. In this paper, we formulated an optimal control problem for a diffusion model incorporating price dependent total market potential in segmented envirnoment under the assumption that the potential market is a concave decreasing function of price. This research is attempting to determine the optimal advertising and price policies for a product in segmented market using simultaneously both single channel and segment specific advertising process.

The sales of the product are assumed to be evolved through a single advertising channel as well as segment specific advertising process. Single channel advertising is available with a fixed spectrum for each segment in whole market while segment specific advertising process can reach each target segment independently. To analyses the proposed model, the corresponding optimal advertising effort policies and price policies for a new product is derived by using Pontryagin's maximum principle. The potential market size in each segment increases or decrease over time due to a decrease or increase in price. After discretizing, the proposed optimal control problems have been solved by using Lingo11. Numerical examples for both quadratic and linear advertising cost are delivered to demonstrate the efficacy of the proposed optimal control model. For further research, the proposed optimal control model may be extended in a competitive environment. Another direction for future research is to incorporate quality together with segment specific and a single channel advertising.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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