International Journal of Mathematical, Engineering and Management Sciences
Vol. 7, No. 5, 613-623, 2022
https://doi.org/10.33889/IJMEMS.2022.7.5.040

Exact Solutions of (2+1) Dimensional Cubic Klein-Gordon (cKG) Equation

Lakhveer Kaur
Department of Mathematics,
Jaypee Institute of Information Technology, Noida, U.P., India.
E-mail: lakhveerscholor@gmail.com

Kuldeep
Department of Mathematics,
Jaypee Institute of Information Technology, Noida, U.P., India.
Corresponding author: kuldeepkr09@gmail.com

(Received on February 20, 2022; Accepted on June 22, 2022)

Abstract
In current study, (2+1)-dimensional cubic Klein-Gordon (cKG) equation illustrating dislocation propagation in crystals as well as the behaviour of elementary particles is investigated to establish a variety of new analytic exact solitary wave solutions. Modified exponential expansion method has been implemented to unfold certain wave solutions of considered model. As a result, three sorts of solutions emerge in a fairly systematic manner in the shape of hyperbolic, trigonometric, and rational functions. The kink and periodic wave solitons are acquired and presented geometrically, some 3D plots are simulated and displayed to respond the dynamic behavior of these obtained solutions. In this work we have used symbolic package maxima to obtained our solutions. Our acquired solutions might be most helpful to analyze physical issues that arise from nonlinear complicated dynamical systems.

Keywords- (2+1)-dimensional cubic Klein-Gordon (cKG) equation, Modified exponential expansion method, Exact solutions, Trigonometric functions, Hyperbolic functions.

1. Introduction

Nonlinear evolution equations (NLEEs) appear in many scientific applications such as beam propagation, particle physics, fluid mechanics, marine engineering, nonlinear optics, elasticity theory, solid-state physics, and quantum field theory, provide a crucial part for the modeling of problems which occur in the real world. Analytical solutions are becoming progressively more significant in various applied science and engineering fields. The exact solution plays a vital role in analyzing the nonlinear phenomena that have occurred in many areas of nonlinear science, especially in particle physics. One of the main objectives in nonlinear science and engineering is to find exact solutions to NLEEs by using analytical methods. Since finding analytical solutions to NLEEs is not possible every time, such types of solutions have a pretty important role in applied sciences, analytical solutions could be used to analyse the wave motions. An exact solution can also test the accuracy, consistency, and error of many numerical and approximate methods. In recent years, various leading approaches for obtaining the exact solution of NLEEs, such as the homogeneous balancing method (Fan et al., 1998), the extended F-expansion method (Abdou, 2007), tanh method (Wazwaz, 2004a), sine- cosine method (Wazwaz, 2004b), the hirota's direct method (Wazwaz, 2008), homogeneous balance method (Fan et al., 1998), $\left(\frac{\omega}{g}\right)$-expansion method (An et al., 2008), tan $\left(\frac{\phi(\xi)}{2}\right)$-expansion method (Kadkhoda and Feckan, 2018), generalized $\left(\frac{G'}{G}\right)$-expansion methods (Bekir, 2008; Kaur, 2014; Zhang et al., 2008, 2009; Wang et al., 2014; Kaur and Gupta, 2013; Alam et al., 2014), $\left(\left(\frac{G'}{G}\right), \left(\frac{1}{G}\right)\right)$-expansion method (Zayed et al., 2016; Duran, 2020), exp $(-\phi(\xi))$-

expansion method (Khan and Akbar, 2014a; Khan and Akbar, 2014b; Hafez et al., 2014; Abdelrahman et al., 2015) have been devised. Recently many NLEEs has been solved using the methods, first integral method (Zhang et al., 2020; Ghosh et al., 2021), Hirota's bilinear method (Ma et al., 2021), the functional variable method (Rezazadeh et al., 2020), extended Sinh-Gordon equation expansion method (Bezgabadi et al., 2021), collocation method (Kadkhoda et al., 2021), and a generalized ($\frac{G'}{G}$)- expansion method (Urazboev et al., 2021).

This article aims to examine a new form of the exact solutions of (2+1)-dimensional cubic Klein-Gordon (cKG) equation (Khan and Akbar, 2014a), which play a significant role in nonlinear science. The Cubic Klein-Gordon equation is employed to mold several nonlinear phenomena, including the propagation of fluxions in Josephson junctions, propagation of dislocation in crystals, and elementary particles' behavior. We have used a reliable significant mathematical technique modified exp $(-\phi(\xi))$-expansion method to obtained exact solutions to the cKG equation. In literature, exp $(-\phi(\xi))$-expansion method has been applied for acquiring exact solutions of NLEEs (Khan and Akbar, 2014a; Roshid et al., 2015; Zahran and Khater, 2015; Khater, 2016).

It assumes that travelling wave solutions can be written as a polynomial in exp $(-\phi(\xi))$, which satisfies the ordinary differential equation (ODE):

$$\phi'(\xi) = \exp(-\phi(\xi)) + \mu \exp(\phi(\xi)) + \lambda, \tag{1}$$

where, $\xi = a_1 x + a_2 y + a_3 t, a_1, a_2$ and a_3 are constants.

A (homogeneous) balance between the nonlinear term and the highest order derivative, present in the nonlinear evolution equation (NLEE), is maintained to find the polynomial exponent. To acquire a set of algebraic equations which has been solved to obtain the unknowns of a polynomial. We use the modified exp $(-\phi(\xi))$-expansion method to get many travelling wave solutions of nonlinear evolution equations.

On the other hand, partial differential equations of hyperbolic types have occurred in many areas of engineering and science. Hyperbolic type of the partial differential equations are used to model the vibrations of structures (e.g., buildings and machines), which are the integral part of the atomic physics. Many notable hyperbolic equations are such as Klein–Gordon, and sine-Gordon equations, telegraph etc.

In recent years, in the literature, special attention has been given to the establishment of numerical schemes for hyperbolic equations of the second order, such as the differential quadrature method (Verma et al., 2014; Jiwari, 2015). A numerical procedure for computational modeling of hyperbolic type wave equations, which is based on modified cubic trigonometric B-spline functions has been developed (Alshomrani et al., 2017). For computational modeling of nonlinear hyperbolic type wave equations, a numerical method based on the Harr wavelets operational matrix has been established (Jiwari et al., 2012; Pandit et al., 2017). The present work attempts to solve the cKG equation through a modified exp $(-\phi(\xi))$-expansion method. A critical equation in mathematical physics is $(2+1)$-dimensional cKG equation (Khan and Akbar, 2014a). It is a nonlinear, completely integrable NLEE and is as follows:

$$v_{xx} + v_{yy} - v_{tt} + \alpha v + \beta v^3 = 0, \tag{2}$$

where, $v = v(x, y, t)$ and the partial derivatives are represented by the subscripts and $\alpha,\ \beta$ are nonzero arbitrary constants.

Organization of this paper is as follows. The working procedure of modified expansion method exp $(-\phi(\xi))$ has been explained in section 2. This method has been imposed to Eq. (2) in section 3. The dynamic nature of the obtained exact solutions in the surface has been explained in section 4. In section 5 conclusion of the paper has been given.

2. Elucidation of the Modified exp $(-\phi(\xi))$-expansion Scheme

In this section, we discuss the approach of the modified exp $(-\phi(\xi))$-expansion method (Khan and Akbar, 2014a; Kaur, 2017) for finding the solution of NLEEs, suppose that the dependent variable is a function of independent variables present in NLEEs is $v = v(x, y, t)$. Let the general form of a solution to the NLEEs:

$$F\left(v, v_x, v_y, v_t \dots, v_{xy}, v_{yt}, v_{xt} \dots, v_{xx}, v_{yy}, v_{tt} \dots\right) = 0, \tag{3}$$

where, $v = v(x, y, t)$ is a dependent function, subscripts of above equations are its partial derivatives.

Step 1: In this step, we will convert our NLEEs into linear ODE, for which we have merged all independent variables present in the Eq. (3) by a new variable ξ

$$v(x, y, t) = v(\xi), \xi = a_1 x + a_2 y + a_3 t, \tag{4}$$

Now Eq.(4) allow us to convert Eq.(3) into linear ODE of the form:

$$F(v, a_1 v', a_2 v', a_1^2 v'' \dots \dots \dots \dots) = 0, \tag{5}$$

Step 2: We seek the general solution of the Eq.(3) as follows:

$$v(\xi) = \sum_{k=1}^{n_1} a_k (\exp(-\phi(\xi)))^k + \sum_{l=0}^{n_1} b_l (\exp(\phi(\xi)))^l, \tag{6}$$

where $\phi(\xi)$ satisfies the following Linear ODE:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi) = \mathrm{e}^{-\phi(\xi)} + \mu \mathrm{e}^{\phi(x)} + \lambda, \tag{7}$$

The general solution of Eq. (7) are illustrated as follows:

When $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$\phi(\xi) = \ln\left(\frac{1}{2}\frac{-\tanh\left(\frac{1}{2}\_C1\sqrt{\lambda^2 - 4\mu} + \frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu}\right)\sqrt{\lambda^2 - 4\mu} - \lambda}{\mu}\right),$$

$$\phi(\xi) = \ln\left(\frac{1}{2}\frac{-\coth\left(\frac{1}{2}\_C2\sqrt{\lambda^2 - 4\mu} + \frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu}\right)\sqrt{\lambda^2 - 4\mu} - \lambda}{\mu}\right),$$

When $\lambda^2 - 4\mu < 0, \mu \neq 0$,

$$\phi(\xi) = \ln\left(\frac{1}{2}\frac{\tan\left(\frac{1}{2}\_C3\sqrt{-\lambda^2+4\mu}+\frac{1}{2}\xi\sqrt{-\lambda^2+4\mu}\right)\sqrt{\lambda^2-4\mu}-\lambda}{\mu}\right),$$

$$\phi(\xi) = \ln\left(-\frac{1}{2}\frac{\cot\left(\frac{1}{2}\_C4\sqrt{-\lambda^2+4\mu}+\frac{1}{2}\xi\sqrt{-\lambda^2+4\mu}\right)\sqrt{\lambda^2-4\mu}-\lambda}{\mu}\right),$$

When $-4\mu+\lambda^2>0, \mu=0, \lambda\neq 0$,

$$\phi(\xi) = -\ln\left(\frac{\lambda}{exp(-C5\lambda+\xi\lambda)-1}\right),$$

When $4\mu-\lambda^2=0, \lambda\neq 0, \mu\neq 0$,

$$\phi(\xi) = \ln\left(-\frac{2(\_C6+\xi)\lambda+2}{\lambda^2(-C6+\xi)}\right),$$

When $4\mu-\lambda^2=0, \lambda=0, \mu=0$,

$$\phi(\xi) = \ln(\_C\gamma+\xi).$$

where _ Cm's $(m=1,2,\ldots..7)$ are arbitrary constants and λ,μ and $a_k,(k=1,2,3)$ are constants to be determined later.

Step 3: In this step we discover the unknown n_1 after solving Eq. (5) with the help of Eq. (4).

Step 4: In Eq. (3), put Eq. (4) and required derivatives then take all terms of exact degree of $\exp(-\phi(\xi))$together, collecting the coefficients of $\exp(\phi(\xi))^p, (p=0,\pm1,\pm2\ldots\ldots)$, then we equate the coefficients from both side of Eq. (3), a set of equations has been obtained.

Step 5: Solve the set of equations which were obtained in step 4, use the results obtained in step 5 and general solution of Eq. (1) in Eq. (5), we acquired exact solutions of the nonlinear equation (3).

3. Application of (2+1)-Dimensional Cubic Klein- Gordon (cKG) Equation

In the present part of the article, we will give the elucidate to find the explicit solution of the cKG equation, by balancing the highest derivative and nonlinear term present in Eq. (2), we attain $n_1=1$. The travelling wave transformation $v(x,y,t)=v(\xi),\ \xi=a_1x+a_2y+a_3t$, transform the Eq. (2) in the following linear ODE:

$$(a_1^2+a_2^2+a_3^2)v''+\alpha v+\beta v^3=0, \tag{8}$$

Suppose the solution of Eq. (2) in the following explicit form:

$$v(\xi)=b_1\exp(-\phi(\xi))+b_0+b_2\exp(\phi(\xi)), \tag{9}$$

where, $G=G(\xi)$ satisfies Eq. (1), $\xi=a_1x+a_2y+a_3t$.

By using Eq. (3.2) and (1.1) it is derived that

$$v_{xx} = \left(\left(b_1 \mathrm{e}^{-\phi(\xi)} + b_2 \mathrm{e}^{\phi(\xi)}\right)\left(\mathrm{e}^{-\phi(\xi)} + \mu \mathrm{e}^{\phi(\xi)} + \lambda\right) + \left(-b_1 \mathrm{e}^{-\phi(\xi)} + b_2 \mathrm{e}^{\phi(\xi)}\right)\left(\mu \mathrm{e}^{\phi(\xi)} - \mathrm{e}^{-\phi(\xi)}\right), \quad (10)$$

Substituting Eq. (10) into Eq. (8), equating the coefficients of exp $(-3\phi(\xi))$ to be zero, yields a set of algebraic equations for unknowns a_i, for $(i = 1,2,3), b_j$ for $(j = 1, 2, 3), \alpha, \beta, \mu$ and λ as follows:

$$\begin{aligned}
&\beta b_1^3 + 2a_1^2 b_1 + 2a_2^2 b_1 - 2a_3^2 b_1 = 0,\\
&3\beta b_0 b_1^2 + 3\lambda a_1^2 b_1 + 3\lambda a_2^2 b_1 - 3\lambda a_3^2 b_1 = 0,\\
&3\lambda\mu a_1^2 b_2 + 3\lambda\mu a_2^2 b_2 - 3\lambda\mu a_3^2 b_2 + 3\beta b_0 b_2^2 = 0,\\
&2\mu^2 a_1^2 b_2 + 2\mu^2 a_2^2 b_2 - 2\mu^2 a_3^2 b_2 + \beta b_2^3 = 0,\\
&\lambda^2 a_1^2 b_2 + \lambda^2 a_2^2 b_2 - \lambda^2 a_3^2 b_2 + 3\beta b_0^2 b_2 + 3\beta b_1 b_2^2\\
&+2\mu a_1^2 b_2 + 2\mu a_2^2 b_2 - 2\mu a_3^2 + \alpha b_2 = 0,\\
&\lambda^2 a_1^2 b_1 + \lambda^2 a_2^2 b_1 - \lambda^2 a_3^2 b_1 + 3\beta b_0^2 b_1 + 3\beta b_1^2 b_2\\
&+2\mu a_1^2 b_1 + 2\mu a_2^2 b_1 - 2\mu a_3^2 b_1 + \alpha b_1 = 0,\\
&\lambda\mu a_1^2 b_1 + \lambda\mu a_2^2 b_1 - \lambda\mu a_3^2 b_1 + \beta b_0^3 + 6\beta b_0 b_1 b_2\\
&+\lambda a_1^2 b_2 + \lambda a_2^2 b_2 - \lambda a_3^2 b_2 + \alpha b_0 = 0,
\end{aligned} \quad (11)$$

Solving above Eq. (11) we have received variety of exact solutions in term of a_i, for $(i = 1,2,3), b_j$ for $(j = 1,2,3), \alpha, \beta, \mu$ and λ are established and enumerated in the following two set of solutions:

Set 1:

$$\alpha = \frac{2\mu\left(\mu {a_1}^2 {b_0}^2 + \mu {a_2}^2 {b_0}^2 - \mu {a_3}^2 {b_0}^2 - {a_1}^2 {b_2}^2 - {a_2}^2 {b_2}^2 + {a_3}^2 {b_2}^2\right)}{{b_2}^2}, \quad \beta = -\frac{2\mu^2\left({a_1}^2 + {a_2}^2 - {a_3}^2\right)}{{b_2}^2}, \lambda = \frac{2\mu b_0}{b_2}, b_1 = 0, \quad (12)$$

Set 2:

$$\alpha = \frac{\mu {a_1}^2 {b_1}^2 + \mu {a_2}^2 {b_1}^2 - \mu {a_3}^2 {b_1}^2 - {a_1}^2 {b_0}^2 - {a_2}^2 {b_0}^2 + {a_3}^2 {b_0}^2}{{b_1}^2}, \frac{{a_1}^2 + {a_2}^2 - {a_3}^2}{{b_1}^2}, \quad \lambda = \frac{2b_0}{b_1},\ b_2 = 0, \quad (13)$$

Case 1: Substituting the values of unknown found in Eq. (12) into Eq. (9), we have obtained the solution:

$$v_1(\xi) = b_0 + b_2 \mathrm{e}^{\phi(\xi)}, \quad (14)$$

where $\xi = a_1 x + a_2 y + a_3 t$,

Substituting the solution of Eq. (1) into Eq. (13), acquired the solutions of the cKG Eq. (2) as follows:

Sub Case 1:
When $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$v_{11}(\xi) = b_0 + \frac{1}{2}\frac{b_2}{\mu}\left(\tanh\left(\frac{1}{2}(\_C1 + \xi)\sqrt{\frac{4\mu^2 {b_0}^2}{{b_2}^2} - 4\mu}\right)\sqrt{\frac{4\mu^2 {b_0}^2}{{b_2}^2} - 4\mu} - \frac{2\mu b_0}{b_2}\right), \quad (15)$$

$$v_{12}(\xi) = b_0 + \frac{1}{2}\frac{b_2}{\mu}\left(-\coth\left(\frac{1}{2}(\_C2+\xi)\sqrt{\frac{4\mu^2 {b_0}^2}{{b_2}^2} - 4\mu}\right)\sqrt{\frac{4\mu^2 {b_0}^2}{{b_2}^2} - 4\mu - \frac{2\mu b_0}{b_2}}\right), \quad (16)$$

where b_0, b_2,_ C1,_ C2 and μ are arbitrary parameters.

Sub Case 2:
When $\lambda^2 - 4\mu < 0, \mu \neq 0$,

$$v_{13}(\xi) = b_0 - \frac{1}{2}\frac{b_2}{\mu}\left(\tan\left(\frac{1}{2}(\_C3+\xi)\sqrt{-\frac{4\mu^2 {b_0}^2}{{b_2}^2} + 4\mu}\right)\sqrt{-\frac{4\mu^2 {b_0}^2}{{b_2}^2} + 4\mu + \frac{2\mu b_0}{b_2}}\right), \quad (17)$$

$$v_{14}(\xi) = b_0 + \frac{1}{2}\frac{b_2}{\mu}\left(-\cot\left(\frac{1}{2}(\_C4+\xi)\sqrt{-\frac{4\mu^2 {b_0}^2}{{b_2}^2} + 4\mu}\right)\sqrt{-\frac{4\mu^2 {b_0}^2}{{b_2}^2} + 4\mu + \frac{2\mu b_0}{b_2}}\right), \quad (18)$$

where b_0, b_2, $\_C3$, $\_C4$ and μ are arbitrary parameters.

Sub Case 3:
When $\lambda^2 - 4\mu = 0, \lambda \neq 0, \mu \neq 0$,

$$v_{15}(\xi) = b_0 - \frac{1}{2}\frac{{b_2}^3}{{b_0}^2\mu^2(\_C5+\xi)}\left(\frac{2C1\mu b_0}{b_2} + \frac{2\mu b_0 \xi}{b_2} + 2\right), \quad (19)$$

where, b_0, b_2 , _C 5 and μ are arbitrary parameters.

Sub Case 4:
When $\lambda^2 - 4\mu = 0, \lambda = 0, \mu = 0$,

$$v_{16}(\xi) = b_0 + b_2(\_C6 + \xi), \quad (20)$$

where, b_0, b_2 and _C6 are arbitrary parameters.

Case 2:
Substituting the values of unknown found in Eq. (13) into Eq. (9), we have acquired the solution:

$$v_2(\xi) = b_0 + b_1 e^{-\phi(\xi)}, \quad (21)$$

where, $\xi = a_1 x + a_2 y + a_3 t$.

Substituting the solution of Eq. (1) into (21), acquired the solutions of the cKG Eq. (2) as follows:

Sub Case 1:
When $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$v_{21}(\xi) = b_0 + 2\mu b_1\left(-\tanh\left(\frac{1}{2}(\_C7+\xi)\sqrt{-\frac{4{b_0}^2}{{b_1}^2} + 4\mu}\right)\sqrt{-\frac{4{b_0}^2}{{b_1}^2} + 4\mu - \frac{2b_0}{b_1}}\right)^{-1}, \quad (22)$$

$$v_{22}(\xi) = b_0 + 2\mu b_1\left(-\coth\left(\frac{1}{2}(\_C8+\xi)\sqrt{-\frac{4{b_0}^2}{{b_1}^2} + 4\mu}\right)\sqrt{-\frac{4{b_0}^2}{{b_1}^2} + 4\mu - \frac{2b_0}{b_1}}\right)^{-1}, \quad (23)$$

where b_0, b_1, _C7,_ C8 and μ are arbitrary parameters.

Sub Case 2:
When $\lambda^2 - 4\mu < 0, \mu \neq 0,$

$$v_{23}(\xi) = b_0 - 2\mu b_1 \left(\tan\left(\frac{1}{2}(\_C9 + \xi)\sqrt{-\frac{4b_0^2}{b_1^2} + 4\mu} \right) \sqrt{-\frac{4b_0^2}{b_1^2} + 4\mu + \frac{2b_0}{b_1}} \right)^{-1}, \quad (24)$$

$$v_{24}(\xi) = b_0 + 2\mu b_1 \left(-\tan\left(\frac{1}{2}(\_C10 + \xi)\sqrt{-\frac{4{b_0}^2}{{b_1}^2} + 4\mu} \right) \sqrt{-\frac{4{b_0}^2}{{b_1}^2} + 4\mu - \frac{2b_0}{b_1}} \right)^{-1}, \quad (25)$$

where $b_0, b_1, \_C9, \_C10$ and μ are arbitrary parameters.

Sub Case 3:
When $\lambda^2 - 4\mu > 0, \mu = 0, \lambda \neq 0,$

$$v_{25}(\xi) = b_0 + 2b_0 \left(e^{\frac{2\_C11 b_0}{b_1} + 2\frac{b_0 \xi}{b_1}} - 1 \right)^{-1}, \quad (26)$$

where b_0, b_1 and $\_C11$ are arbitrary parameters.

Sub Case 4:
When $\lambda^2 - 4\mu = 0, \lambda \neq 0, \mu \neq 0,$

$$v_{26}(\xi) = b_0 - \frac{b_0^2(\_C12 + \xi)}{\_C12 b_0 + \xi b_0 + b_1}, \quad (27)$$

where, b_0, b_1 and $\_C12$ are arbitrary parameters.

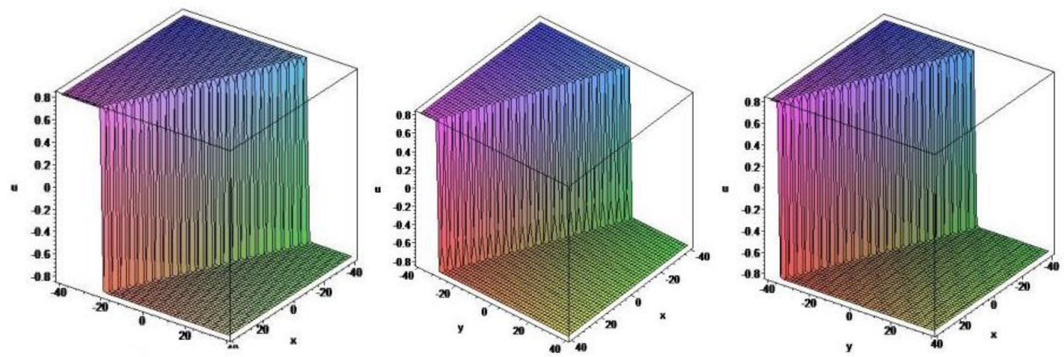
Sub Case 5:
When $\lambda^2 - 4\mu = 0, \lambda = 0, \mu = 0,$

$$v_{27}(\xi) = \frac{b_1}{\xi + \_C13}, \quad (28)$$

where, b_0, b_1 and $\_C13$ are arbitrary parameters.

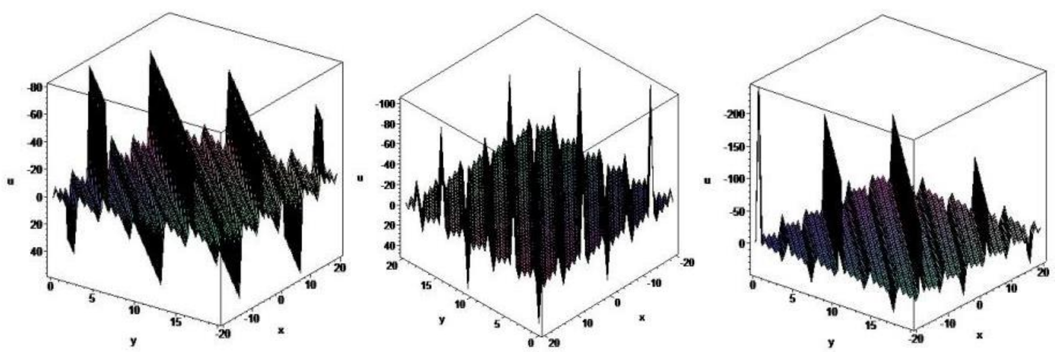
4. Geometrical Representation of few Obtained Solutions

This part of the article shows some obtained solutions 3-dimensional (3-D) profiles. We have analyzed the nature of the obtained solutions taking distinct values of unknowns and then visualizing the reduced form of exact solutions geometrically.



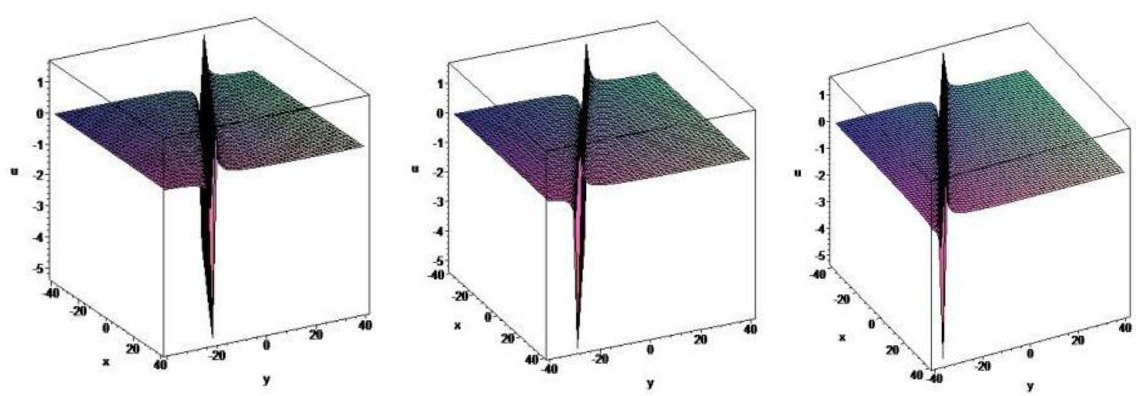
(a) (b) (c)

Figure 1. The 3-dimensional profile of solutions of Eq. (2) given in Eq. (15) when $\lambda = 6, \mu = 3, a_1 = 1,$ $a_2 = 2, a_3 = 3, b_0 = 1, b_1 = 0$ and $b_2 = 1$, _C1 = 1, in the interval $-40 \leq x \leq 40$ and $-40 \leq y \leq 40$, specifically describing depicting kink waves when (a) $t = 0$, (b) $t = 5$, (c) $t = 10$.



(d) (e) (f)

Figure 2. The 3-dimensional profile of solutions of Eq. (2) given in (17) when $\lambda = 3, \mu = 5, a_1 = 1, a_2 = 2,$ $a_3 = 3, b_0 = 3, b_1 = 0$ and $b_2 = 10$, _C3 = 1, within the interval $-20 \leq x \leq 20$ and $0 \leq y \leq 20$, specifically describing depicting periodic breathers when (d) $t = 0$, (e) $t = 5$, (f) $t = 10$.



(g) (h) (i)

Figure 3. The 3-dimensional profile of solutions of Eq. (2) given in Eq. (27) when $\lambda = 4, \mu = 4, a_1 = 1,$ $a_2 = 2, a_3 = 3, b_0 = 4, b_1 = 2$ and $b_2 = 0$, _C12 = 1 within the interval $-40 \leq x \leq 40$ and $-40 \leq y \leq 40$, specifically describing depicting dark solitons when (g) $t = 0$, (h) $t = 5$, (i) $t = 10$.

5. Conclusion

In this study, the analytical soliton solutions to the (2+1)-dimensional cubic Klein-Gordon (cKG) equation were obtained efficiently with little processing cost by employing the modified exp $(-\phi(\xi))$-expansion method. We created the standard soliton solutions, such as kink, periodic soliton, and breather, by picking specific free parameter values for the equations mentioned earlier. The answers discovered in this study are well-suited and practical. The presence of arbitrary parameters in the formulated exact solutions makes them rich physically. Our solutions are validated with the Maple by direct substituting the solutions into the analyzed equation. Our results with existing solutions have been a confrontation, and we realized that the results explored in this research work are entirely new. Moreover, this research demonstrated that the modified exp $(-\phi(\xi))$-expansion approach is efficient and powerful technique for finding exact solutions to wide range of questions. In future direction, other complex nonlinear problems can also be examined and can be attempted to solve.

Conflict of Interest

In bringing out this publication, there is no conflict of interest from the author's side.

Acknowledgments

Authors are thankful to all the reviewer's for their valuable comments to make this publication more appropriate.

References

Abdelrahman, M., Zahran, E. & Khater, M. (2015). The exp(-φ(ξ))-expansion method and its application for solving nonlinear evolution equations. *International Journal of Modern Nonlinear Theory and Application*, *4*, 37-47.

Abdou, M.A. (2007). The extended F-expansion method and its application for a class of nonlinear evolution equations. *Chaos Soliton Fractals, 31* (1), 95-104. https://doi.org/10.1016/j.chaos.2005.09.030.

Alam, M.N., & Akbar, M.A. (2014). Traveling wave solutions of the Boussinesq equation via the new approach of generalized $\left(\frac{G'}{G}\right)$-expansion method. *Springer Plus*, *3*(1), 1-9.

Alshomrani, A.S., Pandit S., Alzahrani, A.K., Alghamdi, M.S., & Jiwari, R. (2017). A numerical algorithm based on modified cubic trigonometric B-spline functions for computational modelling of hyperbolic-type wave equations. *Engineering Computations*, *34* (4), 1257-1276. https://doi.org/10.1108/EC-05-2016-0179.

An, L.W., Hao, C., & Guo-Cai, Z. (2008). The $\left(\frac{\omega}{g}\right)$-expansion method and its application to Vakhnenko equation. *Chinese Physical Society and IOP Publishing Ltd*, *18*, 1674-1056.

Bekir, A. (2008). Application of the $\left(\frac{G'}{G}\right)$-expansion method for nonlinear evolution equations. *Physics Letters A*, *372*, 3400-3406. http://dx.doi.org/10.1016/j.physleta.2008.01.057.

Bezgabadi, A.S., & Bolorizadeh, M. (2021). Analytic combined bright-dark, bright and dark solitons solutions of generalized nonlinear Schrodinger equation using extended Sinh-Gordon equation expansion method. *Results in Physics*, *30*, 104852. https://doi.org/10.1016/j.rinp.2021.104852.

Duran, S. (2020). Solitary wave solutions of the coupled Konno-Oono equation by using the functional variable method and the two variables $\left(\left(\frac{G'}{G}\right), \left(\frac{1}{G}\right)\right)$-expansion method. *Adıyaman University Journal of Science*, *10*(2), 585-594. https://doi.org/10.37094/adyujsci.827964.

Fan, E., & Zhang, H. (1998). A note on the homogeneous balance method. *Physics Letters A*, *246* (5), 403-406. https://doi.org/10.1016/S0375-9601(98)00547-7.

Ghosh, A., & Maitra, S. (2021). The first integral method and some nonlinear models. *Computational and Applied Mathematics*, *40*(3), 1-16. https://doi.org/10.1007/s40314-021-01470-1.

Hafez, M.G., Alam, N.M., & Akbar, M. A. (2014). Application of the exp $(-\phi(\xi))$-expansion method to find exact solutions for the solitary wave equation in an unmagnatized dusty plasma. *World Applied Sciences Journal*, *32* (10), 2150-2155.

Jiwari, R. (2015). Lagrange interpolation and modified cubic B-spline differential quadrature methods for solving hyperbolic partial differential equations with Dirichlet and Neumann boundary conditions. *Computer Physics Communications*, 193, 55-65. https://doi.org/10.1016/j.cpc.2015.03.021

Jiwari, R., Mittal, R.C., & Pandit, S. (2012). Numerical simulation of two-dimensional sine-Gordon solitons by differential quadrature method. *Computer Physics Communication*, *183*(3), 600-616. https://doi.org/10.1016/ j.cpc.2011.12.004.

Kadkhoda, N., & Feckan, M. (2018). Application of tan $\left(\frac{\phi(\xi)}{2}\right)$-expansion method to burgers and foam drainage equations. *Mathematica Slovaca*, *68*, 1057-1064.

Kadkhoda, N., Jafari, H., & Ganji, R.M. (2021). A numerical solution of variable order diffusion and wave eqautions. *International Journal of Nonlinear Analysis and Applications*, *12*(1), 27-36.

Kaur, L. (2017). Generalized exp $(-\phi(\xi))$ -expansion method for Camassa-Holm equation with variable coefficients. *International Journal of Nonlinear Science*, *23* (3), 131-136.

Kaur, L., & Gupta, R.K. (2013). Kawahara equation and modified Kawahara equation with time dependent coefficients: symmetry analysis and generalized $\left(\frac{G'}{G}\right)$-expansion method. *Mathematical Methods in the Applied Sciences*, *36*, 584-600.

Khan, K., & Akbar, M.A. (2014a). The exp $(-\phi(\xi))$-expansion method for finding travelling wave solution of Vakhnenko-Parkes equation. *International Journal of Dynamical Systems and Differential Equations*, *5*(1), 72-83. https://doi.org/10.1504/IJDSDE.2014.067119.

Khan, K., & Akbar, M.A. (2014b). Exact travelling wave solutions to the $(2+1)$-dimensional cubic Klein-Gordon equation. *Journal of the Association of Arab Universities for Basic and Applied Sciences*, *15*, 74-81. https://doi.org/10.1016/j.jaubas.2013.05.001.

Khater, M.A. (2016). Exact traveling wave solutions for the generalized Hirota-Satsuma couple KdV system using exp $(-\phi(\xi))$-expansion method. *Congent Mathematics*, *3* (1), 1172397.

Ma, Y.L., Wazwaz, A.M., & Li, B.Q. (2021). A new (3+1)-dimensional Kadomtsev-Petviashvili equation and its integrability, multiple-solitons, breathers and lump waves. *Mathematics and Computers in Simulation*, *187*, 505-519. https://doi.org/10.1016/j.matcom.2021.03.012.

Pandit, S., Jiwari, R., Bedi, K., & Koksal, M.E. (2017). Haar wavelets operational matrix based algorithm for computational modelling of hyperbolic type wave equations. *Engineering Computations*, *34*(8), 2793-2814. https://doi.org/10.1108/EC-10-2016-0364.

Rezazadeh, H., Vahidi, J., Zafar, A., & Bekir, A. (2020). The functional variable method to find new exact solutions of the nonlinear evolution equations with dual-power-law nonlinearity. *International Journal of Nonlinear Science.and Numerical. Simulation*, *21*(3-4), 249-257.

Roshid, H.O., Alam, M.N., & Akbar, M.A., (2015). Traveling wave solutions for fifth order (1+1)- dimensional Kaup-Kupershmidt equation with the help of exp $(-\phi(\xi))$- expansion method. *Walailak Journal of Science and Technology*, *12*, 1063-1073. https://doi.org/10.14456/vol12iss7pp.

Urazboev, G.U., Baltaeva, I.I., & Rakhimov, I.D. (2021). A generalized $\left(\frac{G'}{G}\right)$- expansion method for the loaded Korteweg –de Vries equation. *Sibirskii Zhurnal Industrial'noi Matematiki*, *24* (4), 139-147.

Verma, A., Jiwari, R., & Kumar, S. (2014). A numerical scheme based on differential quadrature method for numerical simulation of nonlinear Klein-Gordon equation. *International Journal of Numerical Methods for Heat & Fluid F*, *24*(7), 1390-1404. http://dx.doi.org/10.1108/HFF-01-2013-0014.

Wang, M.L., Li, X., & Zhang, J. (2014). The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*, *372*, 417-423.

Wazwaz, A.M. (2004a). A sine-cosine method for handling nonlinear wave equations. *Mathematical and Computer Modeling*, *40*, 499-508.

Wazwaz, A.M. (2004b). The tanh method for travelling wave solutions of nonlinear equations. *Applied Mathematics and Computation*, *154*, 713-723.

Wazwaz, A.M. (2008). The Hirota's direct method for multiple soliton solutions for three model equations of shallow water waves. *Applied Mathematics and Computation*, *201*, 489-503.

Zahran, E.H.M., & Khater, M.A. (2015). Exact traveling wave solutions for some nonlinear evolution equations by using the exp $(-\phi(\xi))$-expansion method. *Asian Journal of Mathematics and Computer Research*, *4*(4), 195-207.

Zayed, E.M.E., & Alurr, K.A.E. (2016). The $\left(\left(\frac{G'}{G}\right), \left(\frac{1}{G}\right)\right)$-expansion method and its application to two nonlinear Schrodinger equations describing the propagation of femtosecond pulses in non linear optical fibers. *Optik*, *127*, 1581-1589.

Zhang, Q., Xiong, M., & Chen, L. (2020). Exact solutions of two nonlinear partial differential equations by the first integral method. *Advances in Pure Mathematics*, *10*, 12-20.

Zhang, S., Dong, L., Ba, J.M., & Sun, Y.N. (2009). The ($\frac{G'}{G}$)-expansion method for nonlinear differential difference equations. *Physics Letters A*, *373*, 905910.

Zhang, S., Tong, J.L., & Wang, W. (2008). A generalized ($\frac{G'}{G}$)-expansion method for the mKdV equation with variable coefficients. *Physics Letters A*, *372*, 2254-2257.