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A Two-Echelon Supply Chain with a Two-Warehouse Retailer and a Green Manufacturer under Advertising and Product Expiry-Dependent Demand

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Abstract
This study aims to develop a sustainable and coordinated supply chain model that addresses increasing environmental concerns and growing consumer demand for green products. The model considers a two-echelon system comprising a green manufacturer and a retailer managing both their own warehouse (OW) and a rented warehouse (RW), with customer demand segmented according to price sensitivity, advertising effectiveness, product greenness, and time-based accessibility. Specifically, demand in the OW segment is modeled as a function of the effective selling price, the retailer's advertising expenditure, and the greenness level of the product, capturing the preferences of consumers who are more responsive to promotional and environmental attributes. In contrast, the RW demand function incorporates a time-dependent decay factor, reflecting reduced customer interest in products stored for longer durations or in less accessible locations. The supply chain is analyzed using a Stackelberg game-theoretic framework, where the green manufacturer leads by setting the wholesale price, investing in preservation technology and emission control, and determining advance payment terms. The retailer follows by optimizing advertising expenditure and allocating inventory across warehouse segments to serve distinct customer groups. Numerical simulations and sensitivity analyses show that preservation investment, carbon costs, and green subsidies significantly affect profitability and environmental performance. The findings emphasize the importance of segment-specific demand modeling and integrated decision-making in enhancing economic outcomes and sustainability, providing valuable insights for green supply chain coordination and policy development.

Keywords- Green product, Advertisement, Carbon emission, Advance payment, Demand segmentation.

1. Introduction

Environmental sustainability has emerged as a cornerstone of modern supply chain management, driven by intensifying global concerns over carbon emissions, regulatory pressures, and rising consumer environmental awareness (Wu et al., 2020; Haleem et al., 2023). In this context, green supply chains aim to balance profitability with ecological responsibility, particularly in industries dealing with perishable or deteriorating products, where shelf-life, freshness, and greenness are critical to both customer satisfaction and sustainable outcomes (Tan et al., 2020; Yadav & Khanna, 2021).

A significant body of research has examined deteriorating inventory models under green considerations. For example, Das et al. (2024), Sharma et al. (2024), and Yadav et al. (2025) analyze pricing, deterioration, and carbon emissions, while Choudhury et al. (2023) focus on expiration-sensitive multistage production-inventory systems. However, many of these studies do not fully account for how eco-conscious consumer preferences, demand elasticity, and behavioral drivers interact with green operational practices (Chang et al., 2019; Qiao et al., 2021). Similarly, several works emphasize pricing and replenishment decisions (Giri et al., 2018; Xin et al., 2022) yet often overlook sustainability investments such as preservation technology, carbon reduction initiatives, and green marketing efforts (Ruidas et al., 2022; Ahmed et al., 2025).

Dual-warehouse systems—typically involving a combination of owned and rented warehouses—are widely studied for improving storage flexibility and cost efficiency. Models by Mondal et al. (2024), Sharma et al. (2024), Yadav et al. (2024), and Shekhar et al. (2025) explore such structures under varying holding costs, backlogging, and green technology considerations. Still, most analyses remain logistics-oriented, without sufficiently examining how warehouse configurations interact with product greenness, preservation strategies, and consumer visibility to shape demand. Furthermore, research on integrating freshness-preserving technologies and advertising policies has largely progressed in isolation, despite evidence that these levers jointly influence both product quality and consumer awareness (Noh et al., 2019; Yang et al., 2019; Devi et al., 2025).

Another important gap lies in the behavioral and financial dimensions of sustainable supply chains. Factors such as consumer goodwill, trust, and green visibility are emerging as key market drivers (Ghosh et al., 2021; Ghosh & Goswami, 2024) yet are rarely linked to operational levers like hybrid payment mechanisms, subsidies, and carbon cap-and-tax regulations. For instance, Mashud et al. (2021) and Ruidas et al. (2022) examine the effects of emission controls and subsidies, while Giri & Dash (2022) and Shekhar et al. (2025) study hybrid payment and advertisement-driven demand. Nevertheless, these models often fail to integrate consumer behavior with long-term sustainability investments in a coordinated way (Soni & Suthar, 2019; Yadav et al., 2024).

Recent works highlight the importance of game-theoretic perspectives in capturing the strategic interactions among supply chain actors. Studies such as Noh et al. (2019), Ghosh et al. (2021) and Choudhury et al. (2023) adopt Stackelberg or negotiation-based frameworks to address coordination issues, while Xin et al. (2022) and Bhavani et al. (2023) incorporate carbon pricing and dynamic pricing into multi-echelon settings. Yet, there is still limited research that simultaneously integrates deterioration control, advertising, dual-warehouse management, carbon reduction policies, and green technology investments within a unified decision framework (Devi et al., 2025; Yadav et al., 2025).

Building on these developments, recent advances by Yadav et al. (2024), Ahmed et al. (2025) and Shekhar et al. (2025) attempt to bridge multiple dimensions—dual-warehouse design, green technology investment, payment strategies, and carbon emissions. However, there remains a need to develop more comprehensive models that explicitly capture segmented and time-dependent demand, preservation-sensitive deterioration, and advertising-driven market awareness, while embedding these within sustainability-oriented game-theoretic frameworks (Chang et al., 2019; Qiao et al., 2021; Giri & Dash, 2022).

1.1 Research Gap and Objective

A significant gap exists in the literature: the absence of a unified, sustainability-integrated, and behaviorally responsive dual-warehouse supply chain model that incorporates segmented demand, green investments, and real-world policy tools. Specifically, no existing model simultaneously addresses:

- segmented consumer demand across warehouse types,
- coordinated investment in preservation and green efforts,
- emission penalties and subsidy structures, and
- hierarchical decision-making under a Stackelberg framework.

This study aims to fill this gap by developing a two-echelon sustainable supply chain model involving a green manufacturer and a retailer operating a dual-warehouse system. The model captures segmented demand based on product greenness, price, advertising, and warehouse accessibility, while integrating deterioration, preservation investment, carbon emissions, advertising strategy, and advance payments. A Stackelberg game-theoretic approach is employed, wherein the manufacturer (leader) makes strategic decisions on pricing, green technology investment, preservation level, and payment policies, while the retailer (follower) optimizes advertising and warehouse-level inventory allocation. Government subsidies are incorporated to promote eco-friendly investments.

1.2 Main Contributions and Novelty

This research offers the following key contributions:

(i) Development of a novel dual-warehouse coordination model that integrates deterioration control, preservation investment, and warehouse-specific cost and accessibility factors—an area scarcely explored in sustainable inventory literature.

(ii) Introduction of a segmented demand structure that reflects consumer behavior differentiated by warehouse origin, product greenness, pricing sensitivity, and advertising responsiveness, adding realism to inventory demand modeling.

(iii) Integration of multiple sustainability drivers including advertising, preservation, and green investment within a Stackelberg game-theoretic framework, enabling strategic coordination between the manufacturer and retailer.

(iv) Incorporation of real-world financial and regulatory mechanisms, such as advance payment strategies and carbon emission penalties/subsidies, to guide environmentally aligned decision-making.

(v) Proposal of a comprehensive and practical decision-making framework that links operational strategies with behavioral demand segmentation and sustainability policies—bridging theoretical modeling with implementable green practices.

The novelty of this study lies in its simultaneous consideration of segmented green demand, dual-warehouse dynamics, coordinated preservation and advertising investment, and incentive-based environmental policies under a Stackelberg game structure. Unlike existing models, this research provides an integrated and policy-responsive framework that captures the multi-faceted realities of sustainable supply chain coordination, offering new insights for both academia and practice.

2. Assumptions and Notations

2.1 Assumptions

The following assumptions are considered in developing the integrated two-echelon supply chain model:

(i) The supply chain features a green manufacturer and retailer, each using owned and rented warehouses for flexible inventory management. The manufacturer produces eco-friendly products, while the retailer optimizes storage and distribution to enhance efficiency and sustainability.

(ii) The manufacturer produces imperfect-quality items, with a set portion reworked to meet quality standards, reducing defects and ensuring product reliability.

(iii) The manufacturer faces costs from carbon emissions and pollution control efforts, reflecting their environmental impact and commitment to regulatory compliance and sustainability.

(iv) The demand faced by the manufacturer depends on price, advertising, and product greenness. Higher advertising and product greenness increase demand, while higher prices reduce it. This relationship is given by

$$D_p = \kappa p^{-\alpha} A^{\beta} G^{\gamma}.$$

where, κ is a scaling constant, p is the price, A is advertising, and G represents the product's eco-friendliness (Giri & Dash, 2022). The exponents α, β, and γ indicate how sensitive demand is to price, advertising, and greenness.

(v) The retailer manages perishable green products through a dual-warehouse system comprising an Owned Warehouse (OW) and a Rented Warehouse (RW)**.** This setup enhances storage flexibility and introduces operational and strategic complexity due to differentiated product deterioration, cost structures, and customer demand sensitivities.

(vi) The Own Warehouse is fully controlled by the retailer, allowing implementation of highly efficient preservation technologies**.** Although it has limited capacity, it offers superior environmental control (e.g., temperature, humidity), resulting in lower deterioration rates and better-quality retention of stored products.

(vii) Demand Function for Own Warehouse (OW): The demand arising from the own warehouse is characterized by a warehouse-specific functional form that captures the influence of price, advertising, and product greenness. Specifically, the demand is modeled as: (motivated from Giri & Dash, 2022)

$$D_1 = \kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1},$$

where, λp_r is the effective selling price, A is the retailer's advertising expenditure, and G_r denotes the greenness level of the product. The parameters $\kappa_1, \alpha_1, \beta_1, \gamma_1$ are positive constants, reflecting the sensitivity of the own-warehouse market segment to these variables.

(viii) The RW provides flexible overflow storage with higher capacity, often leased from third parties. However, due to less control over storage conditions and standardized infrastructure, the RW incurs higher deterioration rates and weaker preservation capability**.** As a result, the quality of products stored here declines faster over time.

(ix) Demand Function for Rented Warehouse (RW): The demand corresponding to the rented warehouse follows a similar structure but incorporates a time-dependent decay factor to represent diminishing demand over the rental period. The demand (motivated by Soni & Suthar, 2020) is expressed as:

$$D_2 = \kappa_2(\lambda p_r)^{-\alpha_2} A^{\beta_2} G_r^{\gamma_2} \left(\frac{L-t}{L}\right),$$

where, $t \in [0, L]$ denotes the time within the planning horizon of length L. The factor $\left(\frac{L-t}{L}\right)$ captures the declining attractiveness or accessibility of products stored in the rented warehouse over time. The parameters $\kappa_2, \alpha_2, \beta_2, \gamma_2$ are positive and specific to the RW segment, reflecting differing customer responsiveness compared to the OW.

(x) In this model, the retailer uses preservation technology (PT) to reduce the deterioration rate of perishable products. The deterioration rate after investment is given by $\delta(v_2) = e^{\{-u\, v_2\}}$, where v_2 is the preservation investment and $u > 0$ represents how sensitive the deterioration rate is to the investment. This means that as the retailer invests more, the deterioration rate decreases, but the benefit from each additional unit of investment becomes smaller. This assumption captures the realistic trade-off between spending on preservation and reducing product spoilage (Mashud et al., 2021).

(xi) The model includes an advance payment from the retailer to the manufacturer, providing upfront capital that reduces production risk and promotes timely, efficient production planning.

(xii) During stock-outs, some unmet demand is partially backlogged while the rest is lost due to customers' unwillingness to wait or service limitations. The backlogging rate decreases as waiting time increases, reflecting customers' declining tolerance for delays.

$$\frac{1}{1+\delta(T_r-t)},$$

where, T_r denotes the start of the replenishment cycle, t is the time at which the demand occurs, and $\delta > 0$ is the backlogging sensitivity parameter. This functional form implies that the likelihood of backlogging diminishes as the waiting time increases, acknowledging that longer delays reduce customer tolerance. This assumption introduces a realistic and dynamic approach to managing shortages by accounting for both service-level impacts and evolving customer behavior during inventory depletion (Mashud et al., 2021).

2.2 Notations

The notations and symbols used in the formulation of the model are presented below. The parameters and decision variables related to the manufacturer are summarized in **Table 1**, while those corresponding to the retailer are listed in **Table 2**. These notations are used consistently throughout the mathematical formulation and analysis of the supply chain model.

Table 1. List of notations for the manufacturer.

Symbol	Description
P	Production rate of the product (units/time unit)
D_p	Demand rate of the product (units/time unit)
X	Rate of imperfect production (units/time unit)
R	Repairing rate for the imperfect items (units/time unit)
Q	Produced a lot of volume in each cycle (units)
x	Proportion of defective items in regular production run ($x > 0$)
S_c	Setup cost per production cycle ($/cycle)
θ	Deterioration rate for finished products
c_p	Production cost per unit product ($/unit)
c_r	Reworking the cost per unit product ($/unit)
h_c	Holding cost per unit product per unit time ($/unit/time unit)
d_c	Deterioration cost per unit ($/unit)
p_c	Pollution cost per unit pollution index ($)
G	Green innovation investment parameter
m	Efficiency of ERT investment in reducing emissions ($m > 0$)
ξ	Fractional reduction in average carbon emission after ERT investment ($0 < \xi < 1$)
s	Government subsidy intensity per unit green product
e_s	Carbon emission rate during the setup phase (kg/setup)
e_p	Carbon emission rate per unit production (kg/unit)
e_r	Carbon emission rate per unit repaired item (kg/unit)
e_h	Carbon emission rate during inventory holding (kg/unit/time unit)
M	A cap on emission level per unit time set by the government (kg/unit time)
C_t	Carbon tax per unit carbon emission ($/unit)
A_d	Advertisement cost per advertisement
ζ	Preservation cost
T_m	The duration of the production cycle
ϕ	Pollution absorption fraction
Decision Variables	
t_{m1}	Production run time (unit time)
p	Selling price of the product ($/unit)

Table 2. List of notations for the retailer.

Parameter	Description
h_r	Holding cost per unit per unit time (rented warehouse)
h_o	Holding cost per unit per unit time (own warehouse)
p_r	Selling price of the retailer
C_a	Advertisement cost per advertisement
B	Shortage Backorder
C_b	Shortage cost per unit per unit time
ℓ	Lost sales cost per unit
Φ	Fraction of demand lost during shortage
c_d	Deterioration cost per unit
$\wp$	Fraction of stock in the rented warehouse
τ	Deterioration rate in the rented warehouse
ρ	Deterioration rate in the own warehouse
C_p	Preservation investment rate
G_r	Environmental improvement factor
C_{tr}	Transportation cost
T_r	Retailer's cycle time
t_{r1}	Time stock finishes in the rented warehouse
t_{r2}	Time markdown ends
i_e	Interest rate at which interest is earned from the bank
i_p	Interest rate at which interest is paid to the bank
q	The period after/before which the actual payment is made by the retailer
w_1	Wholesale price for advanced payment
L	expiry date of the product

3. Model Formulation

This study develops a two-echelon supply chain model featuring a manufacturer and a retailer collaborating to improve profitability and sustainability. The retailer manages inventory using a dual-warehouse system, owned and rented to meet increasing demand efficiently. Key retailer decisions include pricing, advertising, preservation investment, and advance payments. The manufacturer addresses carbon emissions, pollution control, and defective products while promoting green production. Additionally, the manufacturer employs pricing and advertising strategies to shape market demand.

3.1 Retailer's Perspective

The inventory level of the retailer, which decreases over time due to demand and deterioration, is shown in **Figure 1**. This figure depicts the retailer's inventory cycle under the effects of preservation investment and replenishment. In this study, the retailer employs a dual-warehouse structure, comprising an Owned Warehouse (OW) and a Rented Warehouse (RW)**,** to manage inventory effectively. This configuration is particularly relevant for perishable and green products, where product quality, preservation, and storage costs are critical to profitability and sustainability. The proposed two-warehouse inventory model captures the dynamics of a retailer managing inventories from both an Own Warehouse (OW) and a Rented Warehouse (RW) under time-dependent deterioration and segmented demand. The retailer faces separate demand streams from each warehouse, defined by sensitivity to selling price, advertisement frequency, and product greenness. Let us assume that a retailer places an order of S units of a product and prepays a fraction of the purchasing cost to the green manufacturer through equal installments at uniform intervals over the lead time, while the remaining fraction is paid at the time of delivery, i.e., at $t = 0$. The manufacturer, having received the advance, invests the funds in an interest-bearing account at a rate i_e, thereby generating additional financial returns during the lead time. Upon receipt, W_1 units are allocated to the Own Warehouse (OW), and the remaining $(S - W_1)$ units are placed in a Rented Warehouse (RW), which incurs a higher holding cost due to its enhanced facilities. Consequently, the inventory in RW is prioritized for consumption.

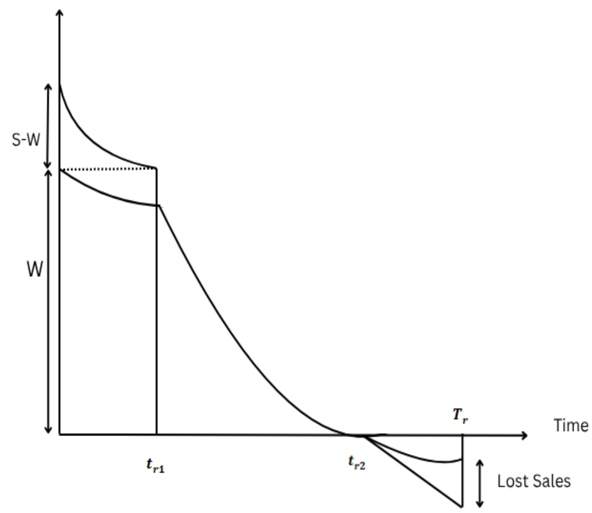

Figure 1. Inventory level for retailer.

The proposed two-warehouse inventory model captures the dynamics of a retailer managing stock under segmented market demand and time-dependent deterioration. The total demand is divided into two components, each based on the source warehouse. The demand from RW is denoted by

$$D_2 = \kappa_2(\lambda p_r)^{-\alpha_2} A^{\beta_2} G_r^{\gamma_2}\left(\frac{L-t}{L}\right),$$

While the demand from OW is

$$D_1 = \kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1},$$

where, p_r is the retail price, A is the advertisement frequency, G_r is the product's greenness index, and the exponents α_i, β_i, and γ_i $(i = 1,2)$ represent the sensitivities to price, advertising, and greenness, respectively. The time-dependent nature of RW demand captures the diminishing willingness of consumers to purchase from the rented stock as freshness declines.

Inventory in RW deteriorates at a constant rate τ and depletes over the interval $[0, t_{r1}]$ due to both demand and deterioration. The inventory level at RW follows a linear differential equation with initial condition $I_r(0) = S - W_1$, and becomes zero at $t = t_{r1}$. In parallel, inventory in OW undergoes three distinct phases. During $[0, t_{r1}]$, it is subject only to deterioration at rate ρ, modeled by an exponential decay. Between t_{r1} and t_{r2}, the OW inventory begins satisfying demand alongside deterioration, described by an inhomogeneous first-order differential equation. The inventory level reaches zero at $t = t_{r2}$, and a shortage emerges and accumulates until the end of the cycle. The total accumulated shortage at $t = T_r$ is evaluated using a logarithmic function.

3.1.1 Rented Warehouse Inventory Model Development

$$\frac{dI_r(t)}{dt} + \tau I_r(t) = -\kappa_2(\lambda p_r)^{-\alpha_2} A^{\beta_2} G_r^{\gamma_2}\left(\frac{L-t}{L}\right) \quad (1)$$

Boundary conditions: $I_r(t) = 0$ at $t = t_{r1}$, $I_r(0) = S - W_1$.

The solution to the differential equation:

$$I_r(t) = \frac{\kappa_2(\lambda p_r)^{-\alpha_2} A^{\beta_2} G_r^{\gamma_2}}{\tau^2 L}\left([\tau(L - t_{r1}) - 1]e^{\tau(t_{r1}-t)}[\tau(L - t) - 1]\right) \quad (2)$$

At $t = 0$:

$$S - W_1 = \frac{\kappa_2(\lambda p_r)^{-\alpha_2} A^{\beta_2} G_r^{\gamma_2}}{\tau^2 L}([\tau(L - t_{r1}) - 1]e^{\tau t_{r1}}[\tau L - 1]).$$

3.1.2 Own Warehouse Inventory Model Development

For $0 \leq t \leq t_{r1}$:

$$\frac{dI_1(t)}{dt} + \rho I_1(t) = 0 \quad (3)$$

$$I_1(t) = W_1 e^{-\rho t} \quad (4)$$

For $t_{r1} \leq t \leq t_{r2}$:

$$\frac{dI_2(t)}{dt} + \rho I_2(t) = -\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1} \quad (5)$$

$$I_2(t) = \frac{\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\rho}\left(e^{\rho(t_{r2}-t)} - 1\right) \quad (6)$$

Boundary at $t = t_{r1}$:

$$W_1 e^{-\rho t_{r1}} = \frac{\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\rho}\left(e^{\rho(t_{r2}-t_{r1})} - 1\right)$$

Thus,

$$t_{r2} = t_{r1} + \frac{1}{\rho}\ln\left(1 + \frac{\rho W_1 e^{-\rho t_{r1}}}{\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}\right) \quad (7)$$

For $t_{r2} \leq t \leq T_r$:

$$\frac{dI_3(t)}{dt} = -\frac{\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{1+\delta(T_r - t)} \quad (8)$$

Using boundary conditions, when $I_3(t) = -B$ at $t = T_r$

$$I_3(t) = \frac{\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\delta}\ln\left(1 + \delta(T_r - t_{r2})\right) - B$$

Accumulated shortage at $t = t_{r2}$, then $I_3(t) = 0$:

$$B = \frac{\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\delta}\ln\left(1 + \delta(T_r - t_{r2})\right) \quad (9)$$

3.1.3 Cost Components

- **Holding Cost for the Retailer**

The holding cost for inventory stored at the rented and owned warehouses per cycle is:

$$
\begin{aligned}
HC = \;& h_r \int_0^{t_{r1}} I_r\,(t)\,dt + h_o\left[\int_0^{t_{r1}} I_1\,(t)\,dt + \int_{t_{r1}}^{t_{r2}} I_2\,(t)\,dt\right] \\
= \;& h_r \frac{\kappa_{s2}(\lambda p_r)^{-\alpha_2} A^{\beta_2} G_r^{\gamma_2}}{\tau^2 L}\left[-\frac{[(-t_{r1}-2L)\tau+2]e^{\tau t_{r1}} + (2Lt_{r1}-t_{r1}^2) + (2L-4t_{r1})\tau - 2}{2\tau}\right] \\
& + h_o\left[\frac{W_1}{\rho}(1-e^{-\rho t_{r1}}) + \frac{\kappa_{s1}(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\rho}\left(\frac{1}{\rho}\left(e^{\rho(t_{r2}-t_{r1})}-1\right) + (t_{r2}-t_{r1})\right)\right].
\end{aligned}
$$

where, h_r is the holding cost per unit per unit time in a rented warehouse, and h_o is the holding cost per unit per unit time in the own warehouse.

- **Shortage Cost Function**

The shortage cost incurred during the markdown (clearance) period $[t_{r2}, T_r]$ is:

$$
\begin{aligned}
SC = \;& -C_b \int_{t_{r2}}^{T_r} I_3\,(t)\,dt \\
= \;& -C_b\left(\frac{\kappa_1(\lambda p)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\delta}\right)\left[\left(\frac{\ln\left(1+\delta(T_r-t_{r2})\right)\left(1+\delta(T_r-t_{r2})\right)}{\delta}\right) - (1+B)(T_r-t_{r2})\right].
\end{aligned}
$$

where, C_b is the shortage cost per unit per unit time.

- **Lost Sales Cost**

If shortages are not completely backlogged, a proportion Φ $(0 < \Phi < 1)$ of demand is lost. Then, the lost sales cost is:

$$
\begin{aligned}
LSC = \;& \ell\Phi \int_{t_{r2}}^{T_r} D_1\;dt \\
= \;& \ell\Phi\left[\frac{\kappa_1(\lambda p)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\delta}(T_r-t_{r2})\right].
\end{aligned}
$$

where, ℓ is the cost per unit of lost sales and $D(t)$ is the demand rate during the markdown period.

- **Transportation Cost Per Cycle for the Retailer**

Transportation cost per replenishment cycle considering supply to rented and own warehouse:

$$TC = C_{tr} \times (S).$$

where, C_{tr} is the transportation cost per unit.

- **Advertisement Expense for the Retailer**

Advertisement expenses per cycle are proportional to the frequency and intensity of advertising:

$$AC = C_a A.$$

where, C_a is the cost per unit advertisement effort, and A is advertisement frequency.

- **Deterioration Cost Per Cycle**

The deterioration cost due to spoilage at rented and owned warehouses:

$$
\begin{aligned}
DC = \; & c_d\left(\wp \int_0^{t_r} \tau\, I_r(t)\, dt + (1-\wp)\rho\left[\int_0^{t_{r1}} I_1\,(t)\, dt + \int_{t_{r1}}^{t_{r2}} I_2\,(t)\, dt\right]\right) \\
= \; & c_d\left(\wp\tau \frac{\kappa_2(\lambda p_r)^{-\alpha_2} A^{\beta_2} G_r^{\gamma_2}}{\tau^2 L}\left[-\frac{[(-t_{r1}-2L)\tau+2]e^{\tau t_{r1}} + (2Lt_{r1} - t_{r1}^2) + (2L - 4t_{r1})\tau - 2}{2\tau}\right]\right. \\
& \left. + (1-\wp)\rho\left[\frac{W_1}{\rho}(1-e^{-\rho t_{r1}}) + \frac{\kappa_{s1}(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}}{\rho}\left(\frac{1}{\rho}\left(e^{\rho(t_{r2}-t_{r1})} - 1\right) + (t_{r2} - t_{r1})\right)\right]\right)
\end{aligned}
$$

where, c_d is the deterioration cost per unit loss.

- **Preservation Cost**

If preservation technology is used to reduce deterioration, the preservation cost per cycle is:

$$PC = C_p\left(1 - e^{-\lambda T_r}\right),$$

where, C_p is the total preservation technology cost and λ measures effectiveness.

- **Sales Revenue**

Sales revenue collected during the full cycle is:

$$
\begin{aligned}
\text{Sales Revenue} = \; & p_r \times \left(\int_0^{T_r} D\,(t)\, dt - \Phi \int_{t_{r2}}^{T_r} D\,(t)\, dt\right) \\
= \; & p_r\left[\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1} T_r - \Phi\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}(T_r - t_{r2})\right] \\
= \; & p_r\kappa_1(\lambda p_r)^{-\alpha_1} A^{\beta_1} G_r^{\gamma_1}[T_r(1-\Phi) + \Phi t_{r2}]
\end{aligned}
$$

where, p is the selling price per unit.

- **Advance Payment**

Since the payment is made in advance, the corresponding amount can be invested in an interest-bearing financial instrument or account that yields a return at an effective interest rate denoted by i_e. This strategy allows the firm to utilize idle funds efficiently by generating additional income through interest accrual. Over the period represented by w_1DTq, where w_1 indicates the duration in weeks, D denotes the demand rate, T_m is the cycle time, w_1 is the whole price for advance payment, and q is a relevant scaling or adjustment factor, the total interest accumulated from this investment will be $i_e w_1 DTq$. This accrued interest contributes to the overall profitability of the system by offsetting some of the upfront financial commitments.

Interest paid:

$$IP = i_p w_1 D_1 T_r(T_r + q).$$

Interest earned:

$$IE = i_e p_r \int_0^{T_r} D_1\, t dt = \frac{i_e p_r D_1 T_r^2}{2}.$$

- **Total Profit of the Retailer**

$$
\begin{aligned}
\text{Total Profit} = \quad & \text{Sales Revenue} - HC - SC - LSC - TC - AC - DC - PC - IP + IE \\
= \quad & p_r\kappa_1(\lambda p_r)^{-\alpha_1}A^{\beta_1}G_r^{\gamma_1}[T_r(1-\Phi)+\Phi t_{r2}] - \ell\phi\left[\frac{\kappa_1(\lambda p)^{-\alpha_1}A^{\beta_1}G_r^{\gamma_1}}{\delta}(T_r - t_{r2})\right] \\
& -h_r\frac{\kappa_{s2}(\lambda p_r)^{-\alpha_2}A^{\beta_2}G_r^{\gamma_2}}{\tau^2 L}\left[-\frac{[(-t_{r1}-2L)\tau+2]e^{\tau t_{r1}}+\left(2Lt_{r1}-t_{r1}^2\right)+(2L-4t_{r1})\tau-2}{2\tau}\right] \\
& -h_o\left[\frac{W_1}{\rho}(1-e^{-\rho t_{r1}}) + \frac{\kappa_{s1}(\lambda p_r)^{-\alpha_1}A^{\beta_1}G_r^{\gamma_1}}{\rho}\left(\frac{1}{\rho}\left(e^{\rho(t_{r2}-t_{r1})}-1\right)+(t_{r2}-t_{r1})\right)\right] \\
& +C_b\left(\frac{\kappa_1(\lambda p)^{-\alpha_1}A^{\beta_1}G_r^{\gamma_1}}{\delta}\right)\left[\left(\frac{\ln\left(1+\delta(T_r-t_{r2})\right)\left(1+\delta(T_r-t_{r2})\right)}{\delta}\right)-(1+B)(T_r-t_{r2})\right] \qquad (10) \\
& -C_{tr}\times(S) - C_aA - C_p\left(1-e^{-\lambda T_r}\right) - i_p w_1 D_p T_m(T_m+q) + \frac{i_e p_r D_p T_m^2}{2} \\
& -c_d\left(\wp\tau\frac{\kappa_2(\lambda p_r)^{-\alpha_2}A^{\beta_2}G_r^{\gamma_2}}{\tau^2 L}\left[-\frac{[(-t_{r1}-2L)\tau+2]e^{\tau t_{r1}}+\left(2Lt_{r1}-t_{r1}^2\right)+(2L-4t_{r1})\tau-2}{2\tau}\right]\right. \\
& \left. +(1-\wp)\rho\left[\frac{W_1}{\rho}(1-e^{-\rho t_{r1}}) + \frac{\kappa_{s1}(\lambda p_r)^{-\alpha_1}A^{\beta_1}G_r^{\gamma_1}}{\rho}\left(\frac{1}{\rho}\left(e^{\rho(t_{r2}-t_{r1})}-1\right)+(t_{r2}-t_{r1})\right)\right]\right)
\end{aligned}
$$

3.2 Manufacturer's Perspective

The inventory variation of the manufacturer during the production and depletion periods is illustrated in **Figure 2**. This figure represents the manufacturer's inventory level over time, considering production rate, demand rate, and deterioration. In the proposed supply chain framework, the manufacturer plays a proactive role in stimulating market demand through strategic decisions on pricing and advertising investments. The manufacturer's demand function is modeled as

$$D_p = \kappa p^{-\alpha}A^{\beta}G^{\gamma},$$

where, p represents the selling price, A denotes the advertising expenditure, and G reflects the product's green quality or eco-friendliness. The manufacturer also contends with several operational and environmental challenges, including carbon emissions, pollution control, and the reprocessing of defective units through rework. The central objective of the manufacturer is to optimize profit while simultaneously minimizing environmental degradation, which involves carefully balancing production and rework quantities, associated costs, and penalties for emissions.

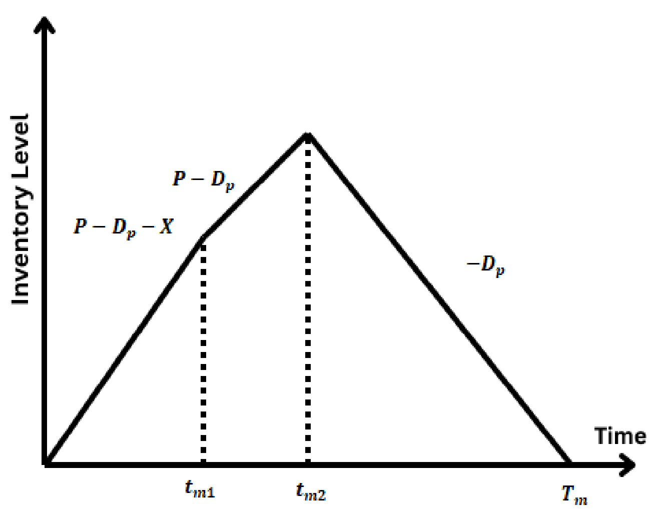

Figure 2. Inventory level for the manufacturer.

To capture the dynamics of the inventory system, a three-phase model is developed (motivated from Ruidas et al., 2022). In the first phase, $(0 \leq t \leq t_{m1})$, the inventory level $I_{m1}(t)$ evolves according to the differential equation

$$\frac{dI_{m1}(t)}{dt} + \theta I_{m1}(t) = P - D_p - X, \qquad 0 \leq t \leq t_{m1} \tag{11}$$

where, P is the production rate, D_p is the demand rate, X is the defective rate, and θ is the decay rate.

In the second phase $(t_{m1} \leq t \leq t_{m2})$, the defective items R are reworked, and the inventory level $I_{m2}(t)$ satisfies

$$\frac{dI_{m2}(t)}{dt} + \theta I_{m2}(t) = R - D_p \qquad t_{m1} \leq t \leq t_{m2} \tag{12}$$

Finally, during the third phase $(t_{m2} \leq t \leq T_m)$, no new units are produced or reworked, and the inventory depletes solely due to customer demand and deterioration, governed by

$$\frac{dI_{m3}(t)}{dt} + \theta I_{m3}(t) = -D_p \qquad t_{m1} \leq t \leq T_m \tag{13}$$

This integrated model enables the manufacturer to assess the impact of eco-conscious decisions on profitability while maintaining a responsive and sustainable inventory strategy.

Using the boundary condition, when $t = 0$, then $I_{m1}(t) = 0$, we get

$$I_{m1}(t) = \left[\frac{P(1-x) - D_p}{\theta}\right]\left[1 - e^{-\theta t}\right] \tag{14}$$

Using the boundary condition, when $t = t_{m2}$, then $I_{m1}(t_{m2}) = Q_2$, we get

$$I_{m2}(t) = \left(\frac{R - D_p}{\theta}\right)\left[1 - e^{\theta(t_{m2} - t)}\right] + Q_2 e^{\theta(t_{m2} - t)} \tag{15}$$

When $t = t_{m1}$, then $I_{m1}(t_{m1}) = I_{m2}(t_{m1})$ becomes

$$Q_2 = \frac{\left[\frac{P(1-x) - D_p}{\theta}\right]\left[1 - e^{-\theta t_{m1}}\right] - \left(\frac{R - D_p}{\theta}\right)\left[1 - e^{\theta(t_{m2} - t_{m1})}\right]}{e^{\theta(t_{m2} - t_{m1})}}.$$

Using the boundary condition, when $t = T_m$, then $I_{m3}(T_m) = 0$, we get

$$I_{m3}(t) = \frac{D_p}{\theta}\left[e^{\theta(T_m - t)} - 1\right] \tag{16}$$

$$t_{m2} = t_{m1} + \frac{1}{\theta} log\left[\frac{\frac{R - D_p}{\theta} + \left[\frac{P(1-x) - D_p}{\theta}\right]\left[1 - e^{-\theta t_{m1}}\right]}{Q_2 + \frac{R - D_p}{\theta}}\right] \tag{17}$$

3.2.1 Cost Components

- **Setup Cost**

The setup cost is incurred for initiating the production process.

$$SC = S_c.$$

- **Production Cost**

The production cost in the proposed model is influenced by the green level of the product, reflecting the sustainability efforts embedded in the manufacturing process. Specifically, the production cost is modeled as a function of both a fixed base cost and an additional component that increases with the square of the product's green level. Mathematically, it is expressed as

$$\begin{aligned} PC = & \ [P_{c0} + P_{c1}G]\int_0^{t_{m2}} P\, dt \\ = & \ [P_{c0} + P_{c1}G]Pt_{m2}. \end{aligned}$$

where, P_{c0} denotes the fixed production cost, P_{c1} is the coefficient associated with green-level-dependent cost, G is the greenness index of the product, P is the constant production rate, and t_{m2} is the time required for production. The quadratic dependence on G indicates that higher levels of environmental friendliness (such as use of biodegradable materials, clean energy, or recyclable packaging) result in increased production cost, thereby capturing the trade-off between ecological sustainability and economic expenditure. This formulation allows for a more realistic and environmentally conscious cost modeling in the green supply chain context.

- **Repairing Cost**

Rework the cost associated with repairing or modifying defective items produced.

$$RC = C_r P x t_{m1}.$$

- **Holding Cost**

Holding cost is incurred by the manufacturer for storing the finished goods in the inventory.

$$\begin{aligned} HC_m = & \ h_c\left[\int_0^{t_{m1}} I_{m1}(t)dt + \int_{t_{m1}}^{t_{m2}} I_{m2}(t)dt + \int_{t_{m2}}^{T_m} I_{m3}(t)dt\right] \\ = & \ h_c\left\{\left[\frac{P(1-x)-D_p}{\theta}\right]\left[t_{m1} + \frac{1}{\theta}\left(e^{-\theta t_{m1}} - 1\right)\right] + \frac{R-D_p}{\theta}\left[t_{m2} - t_{m1} + \frac{e^{\theta t_{m1}}}{\theta}\right] - \frac{Q_2 e^{\theta t_{m1}}}{\theta} \right. \\ & \left. + \frac{D_p}{\theta}\left[\frac{-e^{\theta t_{m2}}}{\theta} - (T_m - t_{m2})\right]\right\}. \end{aligned}$$

- **Deterioration Cost**

Deterioration cost represents the loss in value due to the item's perishability or quality degradation over time.

$$\begin{aligned} DC_m = & \ d_c\left[\int_0^{t_{m1}} I_{m1}(t)dt + \int_{t_{m1}}^{t_{m2}} I_{m1}(t)dt + \int_{t_{m2}}^{T_m} I_{m1}(t)dt\right] \\ = & \ d_c\left\{\left[\frac{P(1-x)-D_p}{\theta}\right]\left[t_{m1} + \frac{1}{\theta}\left(e^{-\theta t_{m1}} - 1\right)\right] + \frac{R-D_p}{\theta}\left[t_{m2} - t_{m1} + \frac{e^{\theta t_{m1}}}{\theta}\right] - \frac{Q_2 e^{\theta t_{m1}}}{\theta} \right. \\ & \left. + \frac{D_p}{\theta}\left[\frac{-e^{\theta t_{m2}}}{\theta} - (T_m - t_{m2})\right]\right\}. \end{aligned}$$

- **Advertisement Cost**

Advertisement costs are spent to influence market demand and promote the product.

$$AC = A_d A.$$

- **Preservation Cost**

The preservation technology cost (PTC) represents the expenditure associated with implementing technologies aimed at extending product shelf life or minimizing spoilage during storage and distribution.

In this model, the preservation cost is assumed to be directly proportional to the length of the production cycle. It is mathematically expressed as

$$PTC = \zeta T_m.$$

where, ζ denotes the per-unit time preservation cost incurred by the manufacturer.

- **Green Innovation Cost**

To develop environmentally friendly products, the manufacturer is required to allocate resources to research and development (R&D) activities. In the proposed model, it is assumed that the cost associated with achieving green innovation (GI) is a quadratic function of the product's degree of greenness, denoted by w. Specifically, the green innovation cost per cycle is given by $\frac{1}{2}g_i w^2$, where g_i represents the investment sensitivity parameter related to green innovation. Notably, this cost is independent of the production lot size in any given cycle, emphasizing that green innovation investment is driven solely by the targeted greenness level rather than the volume of production.

$$GI = \frac{g_i w^2}{2}.$$

- **Policy-driven Financial Support for Ecological Innovation from the Government**

Government subsidies or incentives received, typically in support of green manufacturing practices or emission reduction efforts.

$$Subs = Q_2 sw.$$

- **Carbon Emission Cost Analysis**

Carbon emission cost (CEC) is a critical component in evaluating the environmental impact of a green supply chain. In this model, the total carbon emission cost accounts for emissions generated during multiple stages of the supply chain, including setup, production, repairing, holding, and transportation.

- Setup Emission:

$$e_s$$

- *Production Emission*:

$$Q_2 e_p$$

- *Repairing Emission*:

$$Q_2 x e_r$$

- Transportation Emission: This component captures the emissions due to transportation activities across the supply chain. It considers the number of deliveries n, distance d, shipment quantity per delivery c_q, and the transportation emission coefficient c_t.

$$CET = 2ndc_q c_t.$$

- *Holding Emission*:

$$HC_m = e_h\left\{\left[\frac{P(1-x)-D_p}{\theta}\right]\left[t_{m1}+\frac{1}{\theta}\left(e^{-\theta t_{m1}}-1\right)\right]+\frac{R-D_p}{\theta}\left[t_{m2}-t_{m1}+\frac{e^{\theta t_{m1}}}{\theta}\right]-\frac{Q_2 e^{\theta t_{m1}}}{\theta} + \frac{D_p}{\theta}\left[\frac{-e^{\theta t_{m2}}}{\theta}-(T_m-t_{m2})\right]\right\}.$$

Combining these components, the total carbon emission cost (CEC) is expressed as:

$$CEC = e_s + Q_2 e_p + Q_2 x e_r + 2ndc_q c_t + e_h\left[\left(\frac{P(1-x)-D_p}{\theta}\right)\left(t_{m1}+\frac{1}{\theta}\left(e^{-\theta t_{m1}}-1\right)\right) + \left(\frac{R-D_p}{\theta}\right)\left(t_{m2}-t_{m1}+\frac{e^{\theta t_{m1}}}{\theta}\right)-\frac{Q_2 e^{\theta t_{m1}}}{\theta}+\frac{D_p}{\theta}\left(\frac{-e^{\theta t_{m2}}}{\theta}-(T_m-t_{m2})\right)\right].$$

- **Carbon Emissions after Emission Reduction Technology (ERT) Investment**

$$\begin{aligned} CE_{ERT} &= (1-\xi+\xi e^{-mk})CEC \\ &= (1-\xi+\xi e^{-mk})\left\{e_s + Q_2 e_p + Q_2 x e_r + 2ndc_q c_t + e_h\left[\left(\frac{P(1-x)-D_p}{\theta}\right)\left(t_{m1}+\frac{1}{\theta}\left(e^{-\theta t_{m1}}-1\right)\right) \right.\right. \\ &\quad \left.\left. + \left(\frac{R-D_p}{\theta}\right)\left(t_{m2}-t_{m1}+\frac{e^{\theta t_{m1}}}{\theta}\right)-\frac{Q_2 e^{\theta t_{m1}}}{\theta}+\frac{D_p}{\theta}\left(\frac{-e^{\theta t_{m2}}}{\theta}-(T_m-t_{m2})\right)\right]\right\}. \end{aligned}$$

Here, *m* represents the efficiency of the Emission Reduction Technology (ERT) in lowering emissions, and *k* denotes the maximum allowable investment in ERT. This formulation quantifies the environmental costs throughout the supply chain. By conducting a sensitivity analysis on each component, firms can pinpoint major emission sources and implement targeted strategies—like optimizing transportation or improving inventory turnover—to minimize their carbon footprint and meet sustainability objectives.

- **Carbon Tax**

The model is developed within a cap-and-trade carbon regulatory framework, where the government sets a maximum allowable limit (cap) on carbon emissions for firms over a given period. Emissions exceeding this cap are subject to a carbon tax, incentivizing firms to adopt greener technologies. Let *M* denote the permissible emission rate per unit time, and *T* the production cycle length, so the total allowed emissions per cycle is *M* multiplied by *T*. With Emission Reduction Technology (ERT), actual emissions per cycle reduce to a value called E_{aft}. If these emissions exceed the allowed limit ($M \times T$), the firm pays a carbon tax proportional to the excess emissions, calculated as the amount by which E_{aft} exceeds the cap, multiplied by the tax rate.

$$\begin{aligned} CTAX = \; & C_t(1-\xi+\xi e^{-mk})\left\{e_s + Q_2 e_p + Q_2 x e_r + 2ndc_q c_t + e_h\left[\left(\frac{P(1-x)-D_p}{\theta}\right)\left(t_{m1}+\frac{1}{\theta}\left(e^{-\theta t_{m1}}-1\right)\right) \right.\right. \\ & \left.\left. + \left(\frac{R-D_p}{\theta}\right)\left(t_{m2}-t_{m1}+\frac{e^{\theta t_{m1}}}{\theta}\right)-\frac{Q_2 e^{\theta t_{m1}}}{\theta}+\frac{D_p}{\theta}\left(\frac{-e^{\theta t_{m2}}}{\theta}-(T_m-t_{m2})\right)\right]\right\}. \end{aligned}$$

- **Estimation of Pollution Function**

In modern industrial production, each cycle generates pollution at a constant rate. However, natural processes like rainfall and photosynthesis, along with advanced manufacturing technologies, help reduce pollutant accumulation by removing or converting harmful substances into less damaging ones. Despite these efforts, some pollutants persist and contribute to long-term climate change. To capture the dynamics of pollution build-up and reduction, pollution concentration over time is modeled using a differential equation that balances ongoing pollution generation with its gradual absorption.

$$\begin{cases} \dfrac{dP(t)}{dt} + \phi P(t) = \eta r & 0 \le t \le \infty \\ \text{Subject to} & P(0) = \eta' \end{cases}$$

Here, ϕ represents the effective pollution absorption rate (accounting for both natural and engineered processes), ηr is the pollution generation rate per production cycle, and η' denotes the initial pollution concentration at the start of the observation period. This model helps to evaluate the long-term environmental impact of industrial operations and informs strategies for achieving sustainability by balancing production with ecological resilience. Solving the above equation, we get

$$P(t) = \eta' e^{-\phi t} + \eta r \frac{(1-e^{-\phi t})}{\phi}.$$

Now, applying the above expression after time $t_{m2,}$ the total amount of pollution is determined by

$$\begin{aligned} w &= \int_0^{t_{m2}} P\,(t)dt \\ &= \left[t_{m2} - \frac{1-e^{-\phi t_{m2}}}{\phi}\right]\frac{\eta r}{\phi} + \frac{\eta'}{\phi}\left(1-e^{-\phi t_{m2}}\right). \end{aligned}$$

- **Pollution Cost**

The total cost of pollution is represented by

$$PLC = p_c\left\{\left[t_{m2} - \frac{1-e^{-\phi t_{m2}}}{\phi}\right]\frac{\eta r}{\phi} + \frac{\eta'}{\phi}\left(1-e^{-\phi t_{m2}}\right)\right\}.$$

- **Sales Revenue**

The manufacturer from the sale of finished goods generates sales revenue.

$$SR_m = pQ_2.$$

- **Advance Payment**

Since the retailer pays for the products in advance, the manufacturer benefits from enhanced cash flow and can invest this upfront amount in interest-bearing instruments at a rate i_e during the lead time. This generates additional income, which can offset production or storage costs and improve overall supply chain profitability. Moreover, advance payments incentivize timely production planning and financial stability.

$$IE_m = i_e w_1 D_p T_m q.$$

This comprehensive profit formulation captures the interplay between economic, environmental, and operational factors, offering an integrated framework for evaluating the manufacturer's performance under a sustainable and policy-compliant supply chain model.

- **Total Profit of the Manufacturer**

In the proposed model, the total profit of the manufacturer, denoted as $TP_{m,}$ is formulated by systematically incorporating all relevant sources of revenue and cost components associated with the production and supply chain operations under a sustainability-oriented regulatory environment. The profit function is expressed as:

$$
\begin{aligned}
TP_m = {} & SR_m + Subs - SC_m - PC - RC - HC_m - DC_m - AC - PTC - GI - CTAX - PLC + IE_m \\
= {} & pQ_2 + Q_2 sw - S_c - [P_{c0} + P_{c1}G]Q_2 - C_r Pxt_{m1} - A_d A - \zeta T_m - \frac{g_i w^2}{2} \\
- {} & h_c\left\{\left[\frac{P(1-x)-D_p}{\theta}\right]\left[t_{m1} + \frac{1}{\theta}\left(e^{-\theta t_{m1}} - 1\right)\right] + \frac{R-D_p}{\theta}\left[t_{m2} - t_{m1} + \frac{e^{\theta t_{m1}}}{\theta}\right] - \frac{Q_2 e^{\theta t_{m1}}}{\theta}\right. \\
& \left. + \frac{D_p}{\theta}\left[\frac{-e^{\theta t_{m2}}}{\theta} - (T_m - t_{m2})\right]\right\} - p_c\left\{\left[t_{m2} - \frac{1-e^{-\phi t_{m2}}}{\phi}\right]\frac{\eta r}{\phi} + \frac{\eta'}{\phi}\left(1 - e^{-\phi t_{m2}}\right)\right\} \\
- {} & d_c\left\{\left[\frac{P(1-x)-D_p}{\theta}\right]\left[t_{m1} + \frac{1}{\theta}\left(e^{-\theta t_{m1}} - 1\right)\right] + \frac{R-D_p}{\theta}\left[t_{m2} - t_{m1} + \frac{e^{\theta t_{m1}}}{\theta}\right] - \frac{Q_2 e^{\theta t_{m1}}}{\theta}\right. \\
& \left. + \frac{D_p}{\theta}\left[\frac{-e^{\theta t_{m2}}}{\theta} - (T_m - t_{m2})\right]\right\} + i_e w_1 D_p T_m q \\
& C_t(1 - \xi + \xi e^{-mk})\left\{e_s + Q_2 e_p + Q_2 x e_r + 2ndc_q c_t + e_h\left[\left(\frac{P(1-x)-D_p}{\theta}\right)\left(t_{m1} + \frac{1}{\theta}\left(e^{-\theta t_{m1}} - 1\right)\right)\right.\right. \\
& \left.\left. + \left(\frac{R-D_p}{\theta}\right)\left(t_{m2} - t_{m1} + \frac{e^{\theta t_{m1}}}{\theta}\right) - \frac{Q_2 e^{\theta t_{m1}}}{\theta} + \frac{D_p}{\theta}\left(\frac{-e^{\theta t_{m2}}}{\theta} - (T_m - t_{m2})\right)\right]\right\}
\end{aligned} \tag{18}
$$

4. Concavity Analysis

4.1 Concavity Analysis of the Retailers Total Profit Function

In the Stackelberg game framework, confirming the concavity of the retailer's total profit function is crucial for ensuring the existence of a unique optimal solution at the follower level. The retailer's total profit, denoted by $TP_r(T_r, p_r, A_r, x_r, G_r)$, is a function of the retailer's decision variables: cycle time T_r, selling price p_r, advertising investment A_r, preservation investment x_r, and green investment G_r.

The retailer's profit function includes revenue from retail sales, minus costs such as product purchase from the manufacturer, holding, deterioration, preservation technology, advertising, green investment, carbon tax, and applicable tax credits.

To verify concavity, the second-order partial derivatives of TP_r with respect to its decision variables are derived, and the Hessian matrix is constructed:

$$
H_r = \begin{bmatrix}
\frac{\partial^2 TP_r}{\partial T_r^2} & \frac{\partial^2 TP_r}{\partial T_r\,\partial p_r} & \frac{\partial^2 TP_r}{\partial T_r\,\partial A_r} & \frac{\partial^2 TP_r}{\partial T_r\,\partial G_r} \\
\frac{\partial^2 TP_r}{\partial p_r\,\partial T_r} & \frac{\partial^2 TP_r}{\partial p_r^2} & \frac{\partial^2 TP_r}{\partial p_r\,\partial A_r} & \frac{\partial^2 TP_r}{\partial p_r\,\partial G_r} \\
\frac{\partial^2 TP_r}{\partial A_r\,\partial T_r} & \frac{\partial^2 TP_r}{\partial A_r\,\partial p_r} & \frac{\partial^2 TP_r}{\partial A_r^2} & \frac{\partial^2 TP_r}{\partial A_r\,\partial G_r} \\
\frac{\partial^2 TP_r}{\partial G_r\,\partial T_r} & \frac{\partial^2 TP_r}{\partial G_r\,\partial p_r} & \frac{\partial^2 TP_r}{\partial G_r\,\partial A_r} & \frac{\partial^2 TP_r}{\partial G_r^2}
\end{bmatrix}
$$

The total profit function TP_r is strictly concave if this Hessian matrix is negative semi-definite over the feasible region.

Based on analytical derivation and numerical validation, the following hold:

$\frac{\partial^2 TP_r}{\partial T_r^2} < 0$: longer replenishment cycles increase spoilage and holding costs.

$\frac{\partial^2 TP_r}{\partial p_r^2} < 0$: demand typically decreases at an increasing rate with price due to elasticity.

$\frac{\partial^2 TP_r}{\partial A_r^2} < 0$: advertising has diminishing marginal impact on demand stimulation.

$\frac{\partial^2 TP_r}{\partial G_r^2} < 0$: green investment also faces diminishing returns in profit impact.

All cross-partial derivatives like $\frac{\partial^2 TP_r}{\partial p_r\, \partial A_r}, \frac{\partial^2 TP_r}{\partial G_r\, \partial x_r}$, etc., are either negative or small in magnitude, and do not violate the requirement for negative semi-definiteness of the Hessian.

Therefore, the retailer's total profit function is strictly concave with respect to its decision variables. This ensures a unique optimal strategy exists for the retailer for any given set of manufacturer decisions, supporting the backward induction solution in the Stackelberg game.

4.2 Concavity Analysis of the Manufacturer's Total Profit Function

To ensure the existence of a unique optimal solution in the Stackelberg game framework, it is essential to verify the concavity of the manufacturer's total profit function concerning its decision variables. Let the manufacturer's decision variables be denoted by G (green investment), T_m (cycle time), A (advertising investment), p (wholesale price), and x (preservation investment). The total profit function of the manufacturer, $TP_m(G, T_m, A, p, x)$, incorporates revenue from wholesale transactions, costs associated with production, investment in green technologies and preservation, advertising, holding, and emissions penalties or credits.

To determine the concavity, the second-order partial derivatives of TP_m concerning each pair of decision variables are computed. The Hessian matrix H is then formed using these second derivatives:

$$H_m = \begin{bmatrix} \frac{\partial^2 TP_m}{\partial G_m^2} & \frac{\partial^2 TP_m}{\partial G_m\, \partial T_m} & \frac{\partial^2 TP_m}{\partial G_m\, \partial A_m} & \frac{\partial^2 TP_m}{\partial G_m\, \partial p_m} & \frac{\partial^2 TP_m}{\partial G_m\, \partial x_m} \\ \frac{\partial^2 TP_m}{\partial T_m\, \partial G_m} & \frac{\partial^2 TP_m}{\partial T_m^2} & \frac{\partial^2 TP_m}{\partial T_m\, \partial A_m} & \frac{\partial^2 TP_m}{\partial T_m\, \partial p_m} & \frac{\partial^2 TP_m}{\partial T_m\, \partial x_m} \\ \frac{\partial^2 TP_m}{\partial A_m\, \partial G_m} & \frac{\partial^2 TP_m}{\partial A_m\, \partial T_m} & \frac{\partial^2 TP_m}{\partial A_m^2} & \frac{\partial^2 TP_m}{\partial A_m\, \partial p_m} & \frac{\partial^2 TP_m}{\partial A_m\, \partial x_m} \\ \frac{\partial^2 TP_m}{\partial p_m\, \partial G_m} & \frac{\partial^2 TP_m}{\partial p_m\, \partial T_m} & \frac{\partial^2 TP_m}{\partial p_m\, \partial A_m} & \frac{\partial^2 TP_m}{\partial p_m^2} & \frac{\partial^2 TP_m}{\partial p_m\, \partial x_m} \\ \frac{\partial^2 TP_m}{\partial x_m\, \partial G_m} & \frac{\partial^2 TP_m}{\partial x_m\, \partial T_m} & \frac{\partial^2 TP_m}{\partial x_m\, \partial A_m} & \frac{\partial^2 TP_m}{\partial x_m\, \partial p_m} & \frac{\partial^2 TP_m}{\partial x_m^2} \end{bmatrix}$$

The function TP_m is concave in the decision variables if the Hessian matrix is negative semi-definite, i.e., all its leading principal minors alternate in sign starting from negative, or all eigenvalues are non-positive.

By symbolic derivation and numerical verification under feasible parameter settings, it is observed that:

$\frac{\partial^2 TP_m}{\partial G^2} < 0$, as profit decreases with excessive green investment due to diminishing returns
$\frac{\partial^2 TP_m}{\partial T_m^2} < 0$, since longer cycle times increase holding and spoilage costs.
$\frac{\partial^2 TP_m}{\partial A^2} < 0$, due to the diminishing marginal effect of advertising on demand.
$\frac{\partial^2 TP_m}{\partial p^2} < 0$, reflecting price sensitivity of downstream demand.

$\frac{\partial^2 TP_m}{\partial x^2} < 0$, due to the diminishing efficiency of preservation efforts.

Cross-partial derivatives such as $\frac{\partial^2 TP_m}{\partial G\, \partial A}$ and $\frac{\partial^2 TP_m}{\partial G\, \partial x}$ also support the interaction effects but do not violate the condition of negative semi-definiteness.

Hence, the manufacturer's total profit function is strictly concave in its decision variables over the feasible region. This guarantees a unique optimal solution exists for the manufacturer in the Stackelberg game and simplifies the analysis for backward induction.

5. Solution Algorithm

5.1 Solution Methodology for the Retailer

Step 1: Using MATLAB to solve the retailer's optimization problem. Initialize the parameters.

Step 2: Defining step sizes for the retailer's decision variables: Let the step sizes be $\Delta A = x_1$, $\Delta T = x_2$, $\Delta p = x_3$

Step 3: Initializing decision variables: Set $A = 0$, $T = 1$, $p = 200$, and total profit $TP_r = 0$

Step 4: For a given manufacturer strategy $(G, w, s,)$ and fixed A, solving the retailers' problems:

- Computing t_2, S, B, and Q
- Evaluating the following cost components: Holding Cost $(HC,)$ Shortage and Lost Sale Cost $(SC,\ LSC)$, Advertisement and Preservation Cost $(AC,\ PC)$, Transportation and Deterioration Cost $(TC,\ DC)$, Interest Costs $(IP,\ IE)$.
- Computing the retailer's total profit:
  $TP_r(p, A, T) = \text{Sales Revenue} - (\text{HC} + \text{SC} + \text{LSC} + \text{AC} + \text{PC} + \text{TC} + \text{DC} + \text{IP} - \text{IE})$

Step 5: Updating $A \leftarrow A + \Delta A$, repeat Step 4. For each updated A, evaluating optimal values of p and T_r to maximize TP_r.

Step 6: Storing the optimal values (A^*, T^*, p^*) , and the corresponding $TP_r *$

Step 7: Stop.

5.2 Solution Methodology for the Manufacturer

Step 1: Using MATLAB to solve the manufacturer's optimization problem. Initialize all necessary model parameters such as production cost, emission rates, incentive rates, etc.

Step 2: Defining step sizes for the manufacturer's decision variables: Let the step sizes be $\Delta G = x_1$, $\Delta T_m = x_2$.

Step 3: Initializing decision variables: Set $G = 0$, $T_m = 1$, and total profit $TP_m = 0$.

Step 4: For a fixed retailer response $(p, A, T,)$ and given G, solving the manufacturers' profit functions:

- Computing relevant elements: Demand function, cycle time effects, Quantity ordered by retailer, Carbon emission and associated penalties, Transfer price and subsidy incentives.
- Evaluating the following cost components: Production and Setup Cost (PC, SC), Green Technology Investment Cost (GTC), Transportation Cost (TC), Carbon Emission Cost (CEC), Inventory Holding Cost (HC).
- Computing the manufacturer's total profit:

$TP_m(G, T_m) = \text{Revenue from Wholesale} - (\text{PC} + \text{SC} + \text{GTC} + \text{CEC} + \text{HC} + \text{TC})$.

Step 5: Updating $G \leftarrow G + \Delta G$, and repeat Step 4. For each updated G, evaluate optimal values of T_m that maximize TP_m.

Step 6: Storing the optimal values (G^*, T_m^*) , and the corresponding TP_m^*.

Step 7: Stop.

6. Numerical Example

This section provides a numerical example to demonstrate the proposed model.

6.1 Manufacturer

The parameter values from the previous production models (Giri et al., 2018; Ruidas et al., 2022) are set as follows: the production cost coefficient κ is \$920, the green production cost sensitivity γ is 0.85, the advertising impact coefficient α is 1.25, and the preservation effort impact coefficient β is 0.45. The selling price per unit P is \$800, the government subsidy per unit R is \$100, the basic production cost P_{co} is \$30, and the variable production cost component P_{c1} is \$4. The setup cost S_c is \$200, the holding cost rate h_c is 0.05, the deterioration rate d_c is 0.03 per month, and the carbon tax per unit E_{tax} is \$2, s_r is \$80, the green technology cost factor G is \$2, $x = 0.05$. The recycling cost per unit C_r is \$0.5, and the cycle time is. The preservation effectiveness decay rate θ is 0.02, the advertisement cost per unit A_d is \$0.5, and $A = \$10$ $p = \$150$ per unit, $e_s = 1500$ per unit, $e_p = 20$ units/unit, from holding $e_h = 4$ units/unit/month, and $e_r = 8$ units/unit, $w = 2.85943$, $s = 3.2$, and ζ is 0.3. The green investment g_i is \$350, $m = 0.02$, $k = 400$ units, and $n = 2$, $d = 50$ units/day, $c_q = \$50$, the transportation cost per unit c_t is \$0.0002, p_c is \$0.2, and $\phi = 0.01$, $\eta = 10$, $\eta_1 = 5$, $r = 2.5$, the effective interest rate on advance payment i_e is 0.15, the manufacturer's investment interest rate r_1 is 2%, and the wholesale price under advance payment w_1 is \$150. Using these parameters and applying Stackelberg game theory, the manufacturer determines the optimal strategy, anticipating the retailer's response. The results are: $TP_m = \$5{,}3897$, $t_2 = 0.8001$, $Q_2 = 596.1075$, and the subsidy is \$58561.

6.2 Retailer

The parameter values from the previous production models (Giri et al., 2018; Giri & Dash, 2022) are set as follows: the fixed advertisement cost A is \$20. The production cost coefficients for two different scenarios are $\kappa_1 = 920$ \$/unit and $\kappa_2 = 930$ \$/unit. The green production cost sensitivities are $\gamma_1 = 0.85$ and $\gamma_2 = 0.9$, respectively. The advertising impact coefficients are $\alpha_1 = 1.25$ and $\alpha_2 = 1.3$, $\beta_1 = 0.45$ and $\beta_2 = 0.5$. The deterioration rate τ is 0.03 per month, the spoilage rate ρ is 0.02, and $l = 8$ months.

The product's lifetime months. The holding cost for the retailer h_r is \$0.5 per unit per month, while the holding cost for the owner (e.g., manufacturer or warehouse) h_o is \$0.4 per unit per month. The carbon emission cost per unit ϕ is \$0.1, and the demand-dependent deterioration cost C_d is \$0.3 per unit. The preservation effort effectiveness decay rate θ is 0.6, and the shortage time $s = 6$ days.

The advertising responsiveness coefficient λ is 0.3, and the greenness level G is fixed at 2. The setup cost C_0 is \$1500, and the transportation cost per shipment C_t is \$50.

The per-unit production cost C_p is \$10, and p_r is \$400. The per-unit transportation cost C_t is \$0.1. The effective interest rate for the retailer's advance payment i_e is 0.12, and the investment interest rate for the manufacturer i_p is 0.15. The carbon tax or emission penalty per unit C_b is \$0.5.

The spoilage rate parameter δ is 0.5 and $L = 8$ days, while the green production rate m is 0.25, $w = 0.5$. The financing interest rate for the retailer $q = \%5$.

These parameter values are employed to evaluate the effectiveness of the green supply chain coordination strategy under different scenarios. The aim is to optimize key decisions such as price, advertisement level, preservation effort, and ordering policies for the manufacturer and retailer while minimizing environmental impact and maximizing total profit. Applying Stackelberg game theory to model the leader-follower interaction, the retailer responds optimally to the manufacturer's strategy. The results are: $TP_r = \$54449$, $Q = 120.9413$, $S = 95.3552$, $t_1 = 0.35$, $t_2 = 4.5674$, the total cycle length $T_r = 7$.

7. Sensitivity Analysis

To conduct the sensitivity analysis, we employed MATLAB (R2014a). The contour plots were generated to visually interpret how the retailer's and manufacturer's total profit functions respond to variations in key decision variables, such as cycle time**,** selling price**,** advertising investment**,** and green investment**.** These simulations involved numerically evaluating the profit functions over a grid of feasible values for each variable pair while holding others constant, ensuring computational rigor and reproducibility.

Table 3. Sensitivity analysis for the manufacturer.

Parameter	Change	Q_2	Carbon tax	Carbon emission	Total profit
α	-30%	557	8.2275	58759.7363	59630.4023
	-20%	578	8.2617	59003.6515	56480.6325
	-10%	590	8.2800	59134.0965	54796.8979
	10%	599	8.2949	59241.1140	53415.7341
	20%	601	8.2977	59261.0425	53158.5513
	30%	602	8.2992	59271.6953	53021.0742
β	-30%	598	8.2927	59225.2332	53620.6835
	-20%	597	8.2918	59218.8252	53703.3830
	-10%	597	8.2908	59211.7176	53795.1112
	10%	595	8.2885	59195.0899	54009.7040
	20%	594	8.2871	59185.3912	54134.8746
	30%	594	8.2856	59174.6337	54273.7106
γ	-30%	597	8.2915	59216.8103	53729.3866
	-20%	597	8.2910	59212.7370	53781.9542
	-10%	597	8.2904	59208.4166	53837.7121
	10%	596	8.2890	59198.9734	53959.5845
	20%	595	8.2883	59193.8178	54026.1224
	30%	595	8.2876	59188.3493	54096.6982
κ	-30%	598	8.2931	59227.8623	53586.7539
	-20%	598	8.2920	59219.8528	53690.1205
	-10%	597	8.2908	59211.8434	53793.4871
	10%	595	8.2886	59195.8248	54000.2203
	20%	595	8.2875	59187.8155	54103.5868
	30%	594	8.2864	59179.8063	54206.9533
A	-30%	597	8.2914	59215.7108	53745.0762
	-20%	597	8.2908	59211.4860	53799.0997
	-10%	596	8.2902	59207.5429	53849.4890
	10%	596	8.2892	59200.3242	53941.6517
	20%	596	8.2888	59196.9858	53984.2366
	30%	595	8.2883	59193.7971	54024.8886
G	-30%	598	8.2927	59224.7810	58848.9199
	-20%	597	8.2917	59217.6718	57404.6677
	-10%	597	8.2907	59210.6951	55753.9065
	10%	596	8.2888	59197.0753	51833.6803
	20%	595	8.2878	59190.4083	49564.5238
	30%	594	8.2869	59183.8243	47089.4970
p	-30%	592	8.2834	59158.8379	25677.5743
	-20%	594	8.2861	59178.0672	35029.3984
	-10%	595	8.2881	59192.5596	44442.3595
	10%	597	8.2910	59212.8297	63380.7581
	20%	598	8.2920	59220.1570	72886.1950
	30%	598	8.2929	59226.2287	82407.8358
x	-30%	606	8.2655	59031.0262	53940.0811
	-20%	602	8.2736	59088.6288	53925.6720
	-10%	599	8.2817	59146.2314	53911.2628

Table 3 continued…

	10%	593	8.2978	59261.4367	53882.4446
	20%	590	8.3059	59319.0393	53868.0355
	30%	587	8.3139	59376.6420	53853.6264
S_c	-30%	596	8.2897	59203.8341	53956.8537
	-20%	596	8.2897	59203.8341	53936.8537
	-10%	596	8.2897	59203.8341	53916.8537
	10%	596	8.2897	59203.8341	53876.8537
	20%	596	8.2897	59203.8341	53856.8537
	30%	596	8.2897	59203.8341	53836.8537
P_{co}	-30%	596	8.2897	59203.8341	59656.8537
	-20%	596	8.2897	59203.8341	57736.8537
	-10%	596	8.2897	59203.8341	55816.8537
	10%	596	8.2897	59203.8341	51976.8537
	20%	596	8.2897	59203.8341	50056.8537
	30%	596	8.2897	59203.8341	48136.8537
e_s	-30%	596	8.2267	58753.8341	53896.9167
	-20%	596	8.2477	58903.8341	53896.8957
	-10%	596	8.2687	59053.8341	53896.8747
	10%	596	8.3107	59353.8341	53896.8327
	20%	596	8.3317	59503.8341	53896.8117
	30%	596	8.3527	59653.8341	53896.7907
e_p	-30%	596	7.7521	55363.8341	53897.3914
	-20%	596	7.9313	56643.8341	53897.2122
	-10%	596	8.1105	57923.8341	53897.0329
	10%	596	8.4690	60483.8341	53896.6745
	20%	596	8.6482	61763.8341	53896.4953
	30%	596	8.8274	63043.8341	53896.3160
s	-30%	596	8.2897	59203.8341	52140.0199
	-20%	596	8.2897	59203.8341	52725.6312
	-10%	596	8.2897	59203.8341	53311.2425
	10%	596	8.2897	59203.8341	54482.4650
	20%	596	8.2897	59203.8341	55068.0762
	30%	596	8.2897	59203.8341	55653.6875
m	-30%	596	8.3017	59203.8341	53896.8418
	-20%	596	8.2944	59203.8341	53896.8490
	-10%	596	8.2912	59203.8341	53896.8523
	10%	596	8.2891	59203.8341	53896.8544
	20%	596	8.2888	59203.8341	53896.8547
	30%	596	8.2886	59203.8341	53896.8548

Table 4. Sensitivity analysis for retailer.

Parameter	Variation (%)	S	Q	t_2	B	TP
κ_1	-30%	99.87	139	3.7337	39.21	36638.14
	-20%	99.87	148	3.3261	48.26	41802.22
	-10%	99.87	157	3.0061	57.15	46637.54
	10%	99.87	175	2.5361	74.65	55325.83
	20%	99.87	183	2.3585	83.32	59179.74
	30%	99.87	192	2.2075	91.96	62707.16
κ_2	-30%	96.91	163	2.7483	65.94	52765.78
	-20%	97.89	164	2.7483	65.94	52225.59
	-10%	98.88	165	2.7483	65.94	51685.4
	10%	100.85	167	2.7483	65.94	50605.03
	20%	101.84	168	2.7483	65.94	50064.84
	30%	102.83	169	2.7483	65.94	49524.65
λ	-30%	105.69	220	1.9021	114.45	67211.27
	-20%	103.19	197	2.1777	93.83	61612.65
	-10%	101.31	180	2.46	78.19	56143.71

Table 4 continued…

	10%	98.72	155	3.0418	56.06	46689.08
	20%	97.78	146	3.3398	47.92	42751.62
	30%	97.01	138	3.6418	41.07	39278.67
α_1	-30%	99.87	503	1.0452	403.23	-21607.38
	-20%	99.87	327	1.389	227.53	79995.48
	-10%	99.87	225	1.9228	125.42	75293.21
	10%	99.87	131	4.0137	30.81	29967.3
	20%	99.87	108	5.9192	8.49	15295.38
	30%	99.87	78	8.7042	-21.9	5811.58
α_2	-30%	143.14	209	2.7483	65.94	27452.05
	-20%	120.31	186	2.7483	65.94	39948.89
	-10%	107.29	173	2.7483	65.94	47078.12
	10%	95.63	162	2.7483	65.94	53465.42
	20%	93.21	159	2.7483	65.94	54789.06
	30%	91.83	158	2.7483	65.94	55544.17
β_1	-30%	99.87	136	3.8921	36.2	34881.16
	-20%	99.87	145	3.4623	45	39964.35
	-10%	99.87	155	3.0829	54.84	45406.38
	10%	99.87	178	2.4537	78.5	57074.06
	20%	99.87	193	2.1944	92.77	63024.01
	30%	99.87	209	1.9664	109.02	68740.7
β_2	-30%	96.29	162	2.7483	65.94	53100.5
	-20%	97.31	163	2.7483	65.94	52543.58
	-10%	98.49	164	2.7483	65.94	51896.66
	10%	101.46	167	2.7483	65.94	50272.34
	20%	103.31	169	2.7483	65.94	49258.43
	30%	105.46	171	2.7483	65.94	48080.68
γ_1	-30%	99.87	152	3.1959	51.65	43677.67
	-20%	99.87	156	3.0384	56.16	46114.3
	-10%	99.87	161	2.8893	60.92	48606.19
	10%	99.87	171	2.6149	71.23	53721.32
	20%	99.87	177	2.4887	76.83	56322.23
	30%	99.87	183	2.3694	82.75	58933.06
γ_2	-30%	98.18	164	2.7483	65.94	52067.21
	-20%	98.71	165	2.7483	65.94	51778.84
	-10%	99.27	165	2.7483	65.94	51471.91
	10%	100.5	166	2.7483	65.94	50797.49
	20%	101.18	167	2.7483	65.94	50427.39
	30%	101.9	168	2.7483	65.94	50033.46
G_r	-30%	97.16	140	3.5643	42.72	40147.08
	-20%	98.07	149	3.2316	50.69	44134.27
	-10%	98.97	157	2.9658	58.41	47792.59
	10%	100.75	174	2.5668	73.3	54210.76
	20%	101.63	182	2.4127	80.52	57004.59
	30%	102.49	190	2.2803	87.63	59539.63
c_d	-30%	99.87	166	2.7483	65.94	52780.04
	-20%	99.87	166	2.7483	65.94	52235.1
	-10%	99.87	166	2.7483	65.94	51690.16
	10%	99.87	166	2.7483	65.94	50600.27
	20%	99.87	166	2.7483	65.94	50055.33
	30%	99.87	166	2.7483	65.94	49510.39
c_p	-30%	99.87	166	2.7483	65.94	51147.85
	-20%	99.87	166	2.7483	65.94	51146.97
	-10%	99.87	166	2.7483	65.94	51146.09
	10%	99.87	166	2.7483	65.94	51144.34
	20%	99.87	166	2.7483	65.94	51143.46
h_r	-30%	99.87	140	2.7483	65.94	51135.73
	-20%	99.87	166	2.7483	65.94	51138.89
	-10%	99.87	166	2.7483	65.94	51142.05
	10%	99.87	166	2.7483	65.94	51148.37

Table 4 continued…

	20%	99.87	166	2.7483	65.94	51151.53
	30%	99.87	166	2.7483	65.94	51154.7
h_o	-30%	99.87	166	2.7483	65.94	51991.63
	-20%	99.87	166	2.7483	65.94	51709.49
	-10%	99.87	166	2.7483	65.94	51427.35
	10%	99.87	166	2.7483	65.94	50863.07
	20%	99.87	166	2.7483	65.94	50580.93
	30%	99.87	166	2.7483	65.94	50298.79
L	-30%	99.87	166	2.7483	65.94	44485.98
	-20%	99.87	166	2.7483	65.94	47260.66
	-10%	99.87	166	2.7483	65.94	49418.75
	10%	99.87	166	2.7483	65.94	52557.78
	20%	99.87	166	2.7483	65.94	53734.92
	30%	99.87	166	2.7483	65.94	54730.95

7.1 Sensitivity Analysis for the Retailer

This subsection presents a sensitivity analysis to examine the impact of key parameters on the retailer's optimal decisions and total profit. The results help to understand how variations in factors such as price, advertisement, and preservation investment influence the retailer's performance.

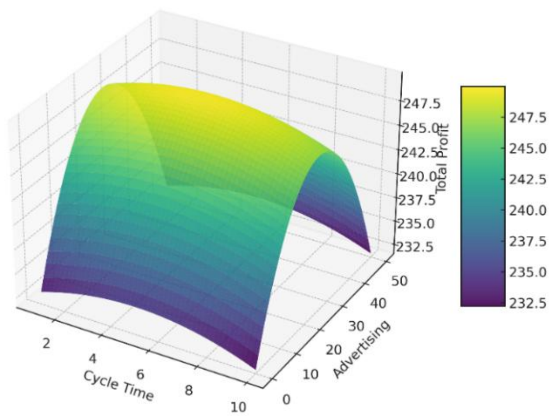

Figure 3. Retailer profit vs. cycle time and advertising.

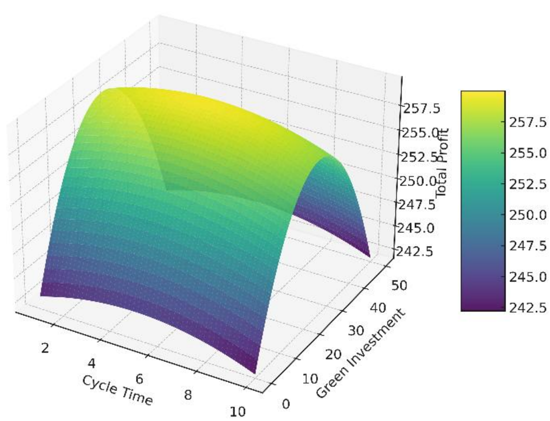

Figure 4. Retailer profit vs. cycle time and gren investment.

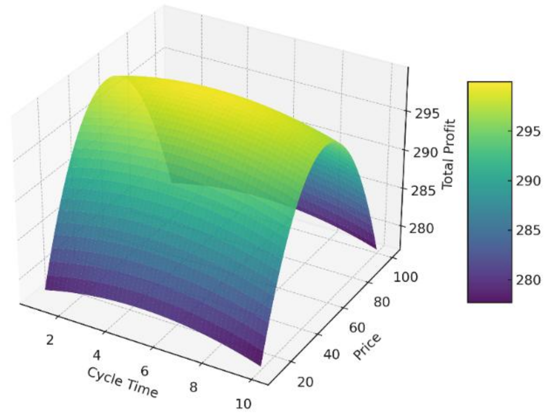

Figure 5. Retailer profit vs. cycle time and price.

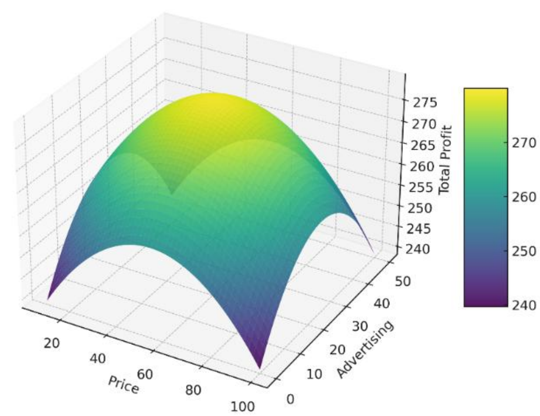

Figure 6. Retailer profit vs. price and advertising.

For the retailer, contour plots illustrate how profit varies with different combinations of decision variables. In **Figure 3**, profit is shown to increase with a balanced combination of cycle time and price. An excessively high price reduces demand, while a very short cycle time increases ordering costs, indicating the existence of an optimal pricing strategy that balances turnover and affordability. **Figure 4** presents profit as a function of cycle time and advertising, where increasing advertising generally enhances profit but shows diminishing returns beyond a threshold, especially at longer cycle times. **Figure 5** explores the effect of cycle time and green investment, revealing that moderate green investment levels improve profit when combined with optimized inventory cycles. **Figure 6** plots profit against price and advertising, showing that higher advertising levels support profitability at moderate prices but do not significantly increase profits at very high prices.

7.2 Managerial Insights of the Retailer

From the retailer's standpoint, the analysis suggests that a moderate pricing strategy combined with optimized cycle time leads to maximum profit, as supported by the shape of the contour in **Figure 3**. Overpricing or underpricing both result in profit loss. Retailers should thus avoid aggressive pricing tactics and instead focus on sustainable price-demand balance. Advertising is shown to be an effective demand stimulator, especially when the replenishment cycle is shorter, as depicted in **Figure 4**. However, excessive advertising without synchronized inventory planning leads to inefficiency and cost escalation. The optimal advertising level should be identified based on product life cycle and customer responsiveness.

Investment in green technology, as seen in **Figure 5**, is positively correlated with profit, but only up to a point. Retailers should treat green investment as a strategic necessity rather than a marketing gimmick. Investing moderately in green initiatives can improve customer loyalty and compliance with environmental norms, thereby enhancing profit margins. The interplay between price and advertising, shown in **Figure 6**, implies that these levers must be co-optimized rather than treated independently. Managers should use advertising to support moderately higher prices and maintain demand elasticity within an acceptable range.

7.3 Sensitivity Analysis for the Manufacturer

In **Figure 7**, the contour plot of cycle time vs. wholesale price shows that manufacturer profit increases with a moderate cycle time and wholesale price. Shorter cycles result in frequent setups and increased costs, while higher prices may reduce retailer demand. The plot confirms the need for a balanced pricing-production policy. **Figure 8** analyzes cycle time and advertising contribution**,** where profit improves with increased advertising support at lower cycle times, reflecting the manufacturer's role in boosting market demand through brand investment.

Figure 9 explores the interaction between cycle time and green investment, where profitability peaks when the manufacturer maintains a reasonable production rhythm and invests adequately in sustainable technologies. The plot shows that while green investment enhances brand value and reduces regulatory risk, the returns diminish after a certain point. **Figure 10**, which maps wholesale price and advertising contribution**,** reveals that profitability increases when the manufacturer shares advertising costs and maintains a competitive wholesale rate. This encourages retailers to engage more actively in promotional efforts, thereby expanding market reach and volume.

7.4 Managerial Insights of the Manufacturer

The contour plots for the manufacturer offer several actionable insights. **Figure 7** reveals that manufacturer profit is maximized by maintaining a balanced cycle time and wholesale price**.** Overextending the cycle increases holding costs and potential spoilage, while overly high wholesale prices could strain retailer relationships and reduce order volumes. Managers should thus target a moderate price point that fosters long-term collaboration with downstream partners.

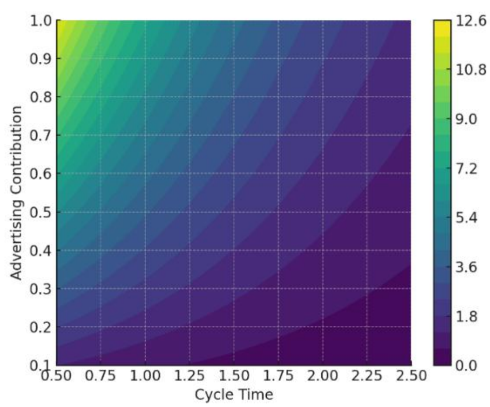

Figure 7. Manufacturer’s profit vs. cycle time and advertisement.

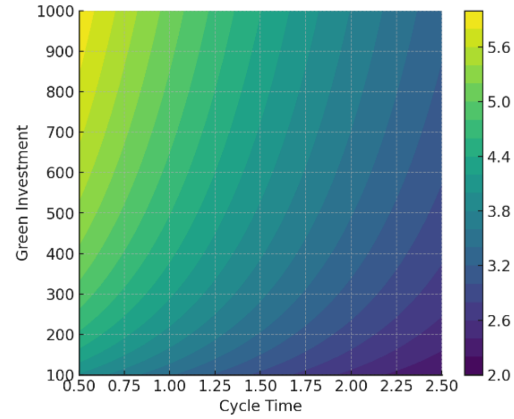

Figure 8. Manufacturer's total profit vs. cycle time and gren investment.

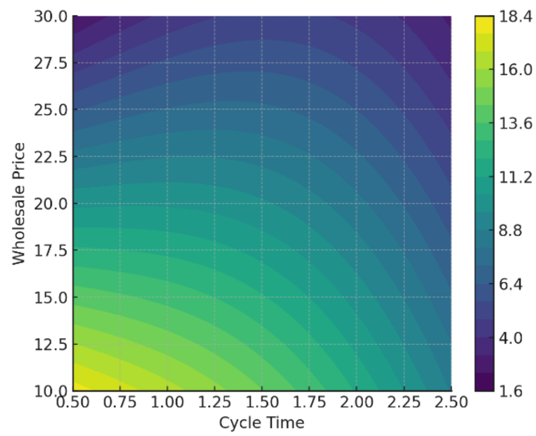

Figure 9. Manufacturer's profit vs. cycle time and wholesale price.

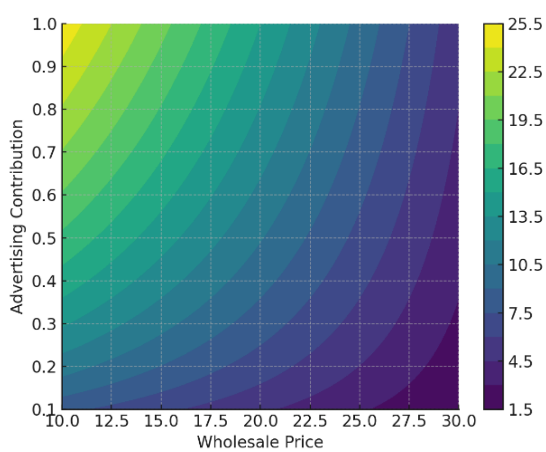

Figure 10. Manufacturer's total profit wholesale price and advertising.

Figure 8 shows that advertising contribution from the manufacturer is most beneficial when production cycles are short. This is because shorter lead times can respond to demand surges generated by advertising more efficiently. Manufacturers are encouraged to engage in joint promotional campaigns with retailers, ensuring the timing and budget of advertisements align with supply capabilities.

As per **Figure 9**, green technology investment significantly boosts profitability up to a saturation point. This supports the adoption of energy-efficient machinery, biodegradable packaging, and carbon-reducing production practices. However, beyond the optimal level, the cost outweighs the marginal gain. Therefore, manufacturers should evaluate the ROI of green investments periodically to ensure sustainability does not erode economic viability.

Figure 10 further implies that manufacturers should partially subsidize advertising efforts to increase product awareness and consumer acceptance, especially for green or premium products. When coupled with competitive pricing, this strategy enhances both market share and retailer loyalty. Overall, the analysis advocates for integrated decision-making, where manufacturers align cycle time, pricing, green investment, and co-marketing strategies to improve both operational efficiency and environmental performance.

8. Conclusion

This study develops a sustainable coordination framework for a two-echelon supply chain comprising a green manufacturer and a retailer managing a dual-warehouse system. The model integrates environmentally sensitive demand, product deterioration, advertising, preservation technologies, and carbon emissions into a hierarchical decision-making structure under a Stackelberg game. The comprehensive analysis explores how pricing, promotional strategies, and environmental investments jointly influence supply chain performance.

The findings reveal that green efforts by the manufacturer, such as cleaner technologies and environmental practices, lead to increased demand, reduced carbon tax burden, and improved total profit. However, these benefits are contingent upon managing production costs and emission intensity effectively. The study confirms that environmentally responsible production is not only critical for sustainability but also strategically advantageous in enhancing market competitiveness and profitability.

From the retailer's perspective, investment in advertising and promotional efforts significantly boosts demand and profitability, despite higher operational costs. Retailers benefit when manufacturers adopt green initiatives, suggesting strong upstream-downstream synergy in environmentally conscious supply chains. Moreover, optimal pricing is crucial; while higher prices can initially increase revenue, excessive pricing deters demand and reduces profit, underscoring the importance of finding a strategic price point. Similarly, appropriate inventory cycle length and preservation techniques are essential to balance holding costs with product freshness, particularly for perishable items.

Overall, the study demonstrates that coordinated strategies focused on environmental responsibility, demand stimulation, and cost management can enhance both economic and ecological outcomes in supply chains. The model provides valuable insights for policymakers and practitioners aiming to align profitability with sustainability goals in a dynamic and environmentally sensitive market.

Future research can expand on this framework by considering uncertainties in demand, incorporating government cap-and-trade regulations, or utilizing AI-driven tools for real-time optimization of supply chain decisions.

Conflict of Interest
All authors declare no conflicts of interest in this paper.

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During the preparation of this work, the author(s) used generative AI to improve the article's language. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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