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Complex Potentials and Integral Equations for Thermoelasticity Problem of Orthotropic Cracked Plate

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Abstract
In the present article, the plane thermoelasticity problem of an infinite orthotropic cracked plate, is studied. The interaction between thermal and mechanical field, in a plate containing a crack and a heat source is investigated. The plate is submitted to normal stresses at infinity, where a heat flow is also activated. Applying the complex potentials method and the technique of Cauchy integrals, a system of singular integral equations is obtained. These equations, allow the determination of the temperature's distribution, but also of the stress state at any point of the medium.

Keywords- Thermoelasticity, Orthotropic plate, Crack, Complex potentials, Integral equations.

1. Introduction
When discontinuities occur simultaneously with mechanical and thermal stresses, the mechanical properties of the materials seem to be touched. For instance, their defiance to loads seem to be weakened. Therefore, in recent years, the study of thermoelastic analysis of anisotropic cracked materials interests several researchers. Atkinson and Clements (1977) handled the thermoelastic problem of an anisotropic medium containing a crack. Tsai (1986) investigated the thermal stress issue for a couple of cracks in an orthotropic plate, implementing the methods of the so-called Fourier transforms and finite Hilbert transforms. Similar problems were solved by Boaxing and Xiangzhou (1994) for two or three cracks. Later, Chen (2005) determined the stress intensity factors for a crack in an infinite orthotropic medium. More recently Ding et al. (2015), Itou (2014) and Wu et al. (2022) investigated thermoselasticity problems in orthotropic cracked materials.

This article is concerned with the formulation of the state equations of thermoelasticity issue for an orthotropic infinite cracked plate, using complex potentials and the technique of singular integral equations. Note that the particular technique has been already used in many cases, e.g. for handling plane problems in the field of elasticity and thermoelasticity for cracked bodies (see Bardzokas et al., 1989; Bardzokas et al., 1996; Chan and Koshkin, 2019; Chen, 2004, 2014; Liolios and Exadaktylos, 2006; Savruk and Zelenyal, 1986; Telichev, 2016). Moreover, mathematical backround of this paper can be applied in other sciences (Chalikias et al., 2016, 2020, 2021; Lalou et al., 2016). In the recent years, singular integral equations, have been implemented to elasticity issues in anisotropic cracked materials by Morini et al. (2013). In previous works (see, e.g., Bardzokas and Lalou, 2009; Lalou, 2012), the proposed method has been used to study thermoelastic analysis of isotropic medium. The innovation of the present work is the implementation of the underlying method, for handling the thermoelasticity issue, in an orthotropic cracked medium. Although many researchers have studied thermoelasicity problems in orthotropic materials, and many others have applied singular integral equations to elasticity problems, this article presents the pioneer implementation of the technique for handling a thermoelasticity problem in orthotropic cracked material.

2. The Problem and Fundamental Relations

2.1 Fundamental Relations

The thermal field can be represented by the next differential equation

$$k_{11}\frac{\partial^2 T}{\partial x^2}+k_{22}\frac{\partial^2 T}{\partial y^2}+k_{33}\frac{\partial^2 T}{\partial z^2}=cp\frac{\partial T}{\partial t}-Q, \tag{1}$$

where, c is the specific heat of the body, p expresses its density and Q is the quantity of heat which is radiated from the unit volume per unit of time. The generalized Hooke's law is

$$\begin{aligned}\varepsilon_{xx}&=c_{12}\sigma_{xx}+c_{12}\sigma_{yy}+c_{16}\sigma_{xy}+a_1T,\\ \varepsilon_{yy}&=c_{12}\sigma_{xx}+c_{22}\sigma_{yy}+c_{26}\sigma_{xy}+a_2T,\\ \gamma_{xy}&=c_{16}\sigma_{xx}+c_{26}\sigma_{yy}+c_{66}\sigma_{xy}-2a_6T.\end{aligned} \tag{2}$$

Under the assumption of observing stationary temperature, the temperature function $T(x,y)$ could be expressed via the following

$$T(x,y)=F(z_3)+\overline{F(z_3)}=2ReF(z_3), \tag{3}$$

where, F is analytical function of the complex variable $z_3=x_1+\mu x_2$. Parameter μ_3 is one of the roots of the characteristic equation

$$\lambda_{22}\mu^2+2\lambda_{12}\mu\quad+\lambda_{11}=0. \tag{4}$$

Under the condition that coefficients c_{ij}, a_i remain constant, stresses can be expressed by complex potential $\Phi(z_1), \Psi(z_2), F(z_3)$ as (see, e.g. Ding and Li, 2015)

$$\begin{aligned}\sigma_x&=2Re[\mu_1^2\Phi(z_1)+\mu_2^2\,\Psi(z_2)+n_0\mu_3^2F(z_3)],\\ \sigma_y&=2Re[\Phi(z_1)+\ \Psi(z_2)+n_0F(z_3)],\\ T_{xy}&=-2Re\left[\mu_1\ \Phi(z_1)+\mu_2\ \ \Psi(z_2)+n_0\mu_3\ F(z_3)\right],\end{aligned} \tag{5}$$

where, $z_1=x+\mu_1y$, $z_2=x+\mu_2y$ and $\mu_1, \mu_2, \overline{\mu_1}, \overline{\mu_2}$ of the characteristic function

$$a_{11}\mu^4-2a_{16}\mu^3+(2a_{12}+a_{66})\mu^2-2a_{26}\mu\quad+a_{22}=0.$$

The latter function corresponds to the following biharmonic equation

$$a_{22}\frac{\partial^4 U}{\partial x^4}-2a_{26}\frac{\partial^4 U}{\partial x^3\partial y}+(2a_{12}+a_{66})\frac{\partial^4 U}{\partial x^2\partial y^2}-2a_{16}\frac{\partial^4 U}{\partial x\ \ \partial y^3}+a_{11}\frac{\partial^4 U}{\partial y^4}=0 \tag{6}$$

where,

$$n_0=-\frac{a_1\mu_3^2+2a_6\mu_3\ +a_2}{(\mu_3\ -\mu_1\ )(\mu_3\ -\mu_2\ )(\mu_3\ -\overline{\mu_1})(\mu_3\ -\overline{\mu_2})},$$

$$\sigma_{xx}=\frac{\partial^2 U}{\partial y^2},\sigma_{yy}=\frac{\partial^2 U}{\partial x^2},\sigma_{xy}=-\frac{\partial^2 U}{\partial x\partial y}$$

and U denotes the Airy's function.

In the case where at point (x_0, y_0) inside the orthotropic medium a thermal source of power p_0 exists, the complex potentials in the region enclosing this point take on the form

$$\varphi(z_1) = a'_0(z_1 - t_1)ln(z_1 - t_1),$$
$$\psi(z_2) = \beta'_0(z_2 - t_2)ln(z_2 - t_2),$$
$$\psi(z_3) = m_0(z_3 - t_3)ln(z_3 - t_3), \tag{7}$$

where,

$$m_0 = -\frac{q_0}{4\pi\sqrt{k_{11}k_{22}}}, \qquad t_j = x_0 + \mu_j y_0,$$

while

$$\Phi(z_1) = \varphi'(z_1), \Psi(z_2) = \psi'(z_2), F(z_3) = \psi'(z_3).$$

Note that the coefficients a'_0, β'_0 are given from the following relations

$$a'_0 = \frac{m \quad - n\mu_2}{\mu_1 - \mu_2}, \beta'_0 = -\frac{m \quad - n\mu_1}{\mu_1 - \mu_2},$$
$$Im(m(\mu_1 + \mu_2) - n\mu_1\mu_2 - m\frac{\lambda_0}{c_{11}}) = 0,$$
$$Im(m\mu_1\mu_2 + n\mu_1\mu_2(\mu_1 + \mu_2) - m_0(a_1\mu_3 - (\mu_3 - \mu_2 - \mu_1))/c_{11}) = 0,$$
$$\lambda_0 = \frac{a_1\mu_3^2 + 2a_6\mu_3 + a_2}{(\mu_3 - \overline{\mu_1})(\mu_3 - \overline{\mu_2})}.$$

It is straightforward that if the reference system is rotated through an angle a, the components of the stresses tensor in the new system, is connected to the respective components of the original system with the relations

$$\sigma_n + \sigma_s = \sigma_x + \sigma_y,$$
$$\sigma_n - \sigma_s + 2i\sigma_t = e^{-2ia}(\sigma_x - \sigma_y + 2i\tau_{xy}) \tag{8}$$

2.2 The General Framework of the Problem

We take into account an infinite orthotropic plate *S*, which contains a crack *l*. This crack is supposed to be affected by mechanical and thermal field. The plate is submitted to the stresses $\sigma_x^\infty, \sigma_y^\infty, \tau_{xv}^\infty$ at infinity, and is affected by homogeneous thermal flow q_∞. Besides these loading conditions, a heat source q_0 is acting at point a. Furthermore, we assume that the temperature T^+, T^- and the normal and shear stresses $\sigma_n^\pm - i\sigma_t^\pm$ on the edges of the crack, are pre-determined. (we use symbol +(-) for the upper (lower) edge). The underlying orthotropic plate is illustrated at Figure 1.

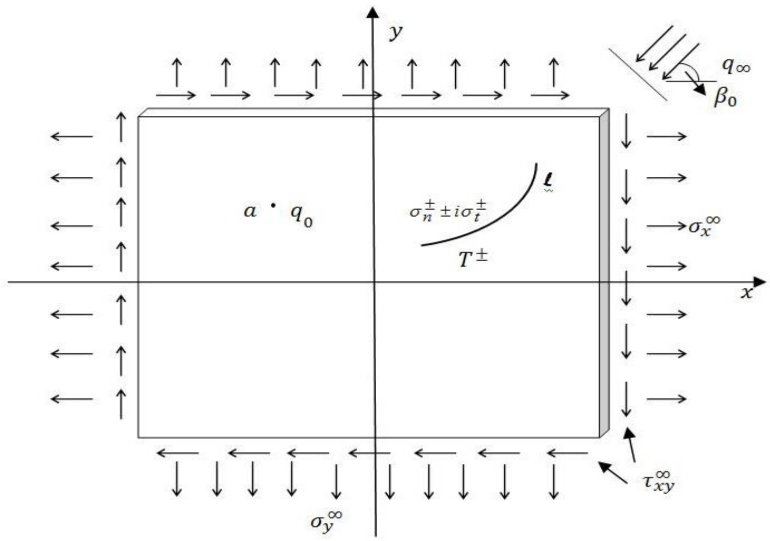

Figure 1. Infinite orthotropic plate containing a heat source and a crack.

3. Formation of the State Equations

For handling the issue of thermoconductivity of a multiconnected cracked body, the temperature could be represented via the following

$$T(x,y) = T_0\ (x,y) + T_*\ (x,y), \quad (9)$$

where, $T_0\ (x,y)$ is the proffered thermal field which is supposed to be pre-specified, and $T_*\ (x,y)$ is the sought after thermal field which occurs because of the existence of discontinuities at the body.

The thermal potential $f(z_3)$, $[T(x,y) = f(z_3) + \overline{f(z_3)} = 2Ref(z_3)]$, describing the thermal field $T(x,y)$ is given by

$$f(z_3) = f_0(z_3) + f_*(z_3), \quad (10)$$

where,

$$f_0(z_3) = m_0(1 + \ln(z_3 - a_3)) + \frac{(cos\beta_0 + \overline{\overline{\mu_3}}sin\beta_0)z_3}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^\infty$$

and

$$f_*(z_3) = \frac{1}{2\pi i}\int_{l_0}\frac{\gamma(\tau_3)}{\tau_3 - z_3}d\tau_3,\ \gamma(\tau_3) = \tau^{(1)}(\tau_3) + i\gamma^{(2)}(\tau_3)$$

with γ the density of the Cauchy integral along the boundary l. The limit values of $f(z)$ together with the boundary l, can be expressed by the aid of Sohotsky-Plemelj formulae as

$$f^\pm(t_3) = m_0(1 + \ln(t_3 - a_3)) + \frac{(cos\beta_0 + \overline{\mu_3}sin\beta_0)t_3}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^\infty \pm \frac{1}{2}\gamma(t_3) + \frac{1}{2\pi i}\int_{l_3}\frac{\gamma(\tau_3)}{\tau_3 - t_3}d\tau_3. \quad (11)$$

Therefore

$$T^\pm(t_3) = f^\pm(t_3) + \overline{f^\pm(t_3)}$$

or

$$T^{\pm}(t_3) = Re\left[m_0(1+\ln(t_3-a_3)) + \frac{(cos\beta_0+\overline{\mu_3}sin\beta_0)t_3}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^{\infty}\right] \pm \frac{1}{2}\gamma(t_3)$$
$$+\frac{1}{2\pi i}\int_{l_3}\frac{\gamma(\tau_3)}{\tau_3-t_3}d\tau_3 \pm -\frac{1}{2}\overline{\gamma(t_3)} - \frac{1}{2\pi i}\int_{l_3}\frac{\overline{\gamma(t_3)}}{\tau_3-t_3}\overline{d\tau_3}. \quad (12)$$

Subtracting relation (12) by parts, we obtain

$$T^{+}(t_3) - T^{-}(t_3) = \gamma(t_3) + \overline{\gamma(t_3)} = 2Re\big(\gamma(t_3)\big) = 2\gamma^{(1)}(t_3). \quad (13)$$

Adding relation (12) by parts

$$T^{+}(t_3) + T^{-}(t_3) - 2Re\left[m_0(1+\ln(t_3-a_3)) + \frac{(cos\beta_0+\overline{\overline{\mu_3}}sin\beta_0)t_3}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^{\infty}\right] = Re\left(\frac{2}{\pi i}\int_{l_3}\frac{\gamma(\tau_3)}{\tau_3-t_3}d\tau_3\right). \quad (14)$$

In order to define the stress-strain field, the complex potentials $\Phi_0^{\pm}(z), \Psi_0^{\pm}(z)$ are utilized. Taking into consideration relations (7) complex potentials are given

$$\Phi_0\ (z_1) = \Gamma + a'_0(1+\ln(z_1-a_1)) + \Phi(z_1), \quad (15)$$

$$\Psi_0\ (z_2) = \Gamma' + b'_0(1+\ln(z_1-a_2)) + \Psi(z_2), \quad (16)$$

where,

$$\Phi(z_1) = \frac{1}{2\pi i}\int_{l_1}\frac{\varphi(\tau_1)}{\tau_1-z_1}d\tau_1,$$
$$\Psi(z_2) = \frac{1}{2\pi i}\int_{l_2}\frac{\psi(\tau_2)}{\tau_2-z_2}d\tau_2,$$
$$\mu_1^2\Gamma + \overline{\mu_1^2\Gamma} + \mu_2^2\Gamma' + \overline{\mu_2^2\Gamma'} = \sigma_x^{\infty},$$
$$\Gamma + \overline{\Gamma} + \Gamma' + \overline{\Gamma'} = \sigma_y^{\infty},$$
$$\mu_1\ \Gamma + \overline{\mu_1\ \Gamma} + \mu_2\ \Gamma' + \overline{\mu_2\ \Gamma'} = -\tau_{xv}^{\infty}.$$

The limited values of the defined functions $\Phi(z_1), \Psi(z_2)$, on crack l, according to Sohotsky- Plemelj formulae take the following form

$$\Phi^{\pm}(t_1) = \pm\frac{1}{2}\varphi(t_1) + \frac{1}{2\pi i}\int_{l_1}\frac{\varphi(\tau_1)}{\tau_1-z_1}d\tau_1\,, \tau_1 \in l_1, \quad (17)$$

$$\Psi^{\pm}(t_2) = \pm\frac{1}{2}\psi(t_2) + \frac{1}{2\pi i}\int_{l_2}\frac{\psi(\tau_2)}{\tau_2-z_2}d\tau_2\,, \tau_2 \in l_2. \quad (18)$$

Taking into consideration relations (8) we have

$$\sigma_n + i\sigma_t = \frac{1}{2}\big(\sigma_x+\sigma_y\big) - \frac{1}{2}e^{-2ia}\big(\sigma_x - \sigma_y + 2i\tau_{xy}\big). \quad (19)$$

Substituting relation (5) into relation (19) we get

$$\sigma_x + i\sigma_t = Re\{(1+\mu_1^2)\Phi_0\ (z_1) + (1+\mu_2^2)\Psi_0\ (z_2) + n_0(\mu_3^2+1)f(z_3)\}$$
$$-\frac{\overline{dt}}{dt}Re\{(\mu_1^2-1)\Phi_0\ (z_1) + (1+\mu_2^2)\Psi_0\ (z_2) + n_0(\mu_3^2-1)f(z_3)\}$$
$$-2iRe\{\mu_1\Phi_0\ (z_1) + \mu_2\Psi_0\ (z_2) + n_0\mu_3 f(z_3)\}, \quad (20)$$

where,

$$\frac{\overline{dt}}{dt} = e^{-2i\theta}$$

or

$$2(\sigma_n + i\sigma_t) = (1+\mu_1^2)\Phi_0\ (z_1) + (1+\bar{\mu}_1^2)\overline{\Phi_0\ (z_1)} + (1+\mu_2^2)\Psi_0\ (z_2) + (1+\bar{\mu}_2^2)\overline{\Psi_0\ (z_2)} + n_0(\mu_3^2+1)f(z_3) + \overline{n_0}(\bar{\mu}_3^2+1)\overline{f(z_3)}$$
$$-\frac{\overline{dt}}{dt}\Big\{(\mu_1^2-1)\Phi_0\ (z_1) + (\bar{\mu}_1^2-1)\overline{\Phi_0\ (z_1)} + (\mu_2^2-1)\Psi_0\ (z_2) + (\bar{\mu}_2^2-1)\overline{\Psi_0\ (z_2)} + n_0(\mu_3^2-1)f(z_3) + \overline{n_0}(\bar{\mu}_3^2-1)\overline{f(z_3)}\Big\}$$
$$-2iRe\Big\{\mu_1\Phi_0\ (z_1) + \overline{\mu_1}\overline{\Phi_0\ (z_1)} + \mu_2\Psi_0\ (z_2) + \overline{\mu_2}\overline{\Psi_0\ (z_2)} + n_0\mu_3 f(z_3) + \overline{n_0\mu_3}\overline{f(z_3)}\Big\}. \quad (21)$$

Moreover, we derive

$$2(\sigma_n + i\sigma_t) = (1+\mu_1^2)\Phi_0\ (z_1) + (1+\bar{\mu}_1^2)\overline{\Phi_0\ (z_1)} + (1+\mu_2^2)\Psi_0\ (z_2) + (1+\bar{\mu}_2^2)\overline{\Psi_0\ (z_2)} + n_0(\mu_3^2+1)f(z_3) + \overline{n_0}(\bar{\mu}_3^2+1)\overline{f(z_3)}$$
$$+\frac{\overline{dt}}{dt}\Big\{(1+\mu_1\ i)^2\Phi_0\ (z_1) + (1+\overline{\mu_1}i)^2\overline{\Phi_0\ (z_1)} + (1+\mu_2\ i)^2\Psi_0\ (z_2) + (1+\overline{\mu_2}i)^2\overline{\Psi_0\ (z_2)} + n_0(1+\mu_3\ i)^2 f(z_3) + \overline{n_0}(1+\overline{\mu_3}i)^2\overline{f(z_3)}\Big\}. \quad (22)$$

The boundary stress condition along the lips of the crack, substituting the complex potentials by relations (11), (17) and (18), becomes

$$(1+\mu_1^2)\Phi^{\pm}(t_1) + (1+\bar{\mu}_1^2)\overline{\Phi^{\pm}(t_1)} + (1+\mu_2^2)\Psi^{\pm}(t_2) + (1+\bar{\mu}_2^2)\overline{\Psi^{\pm}(t_2)} + n_0(\mu_3^2+1)f_*(z_3) + \overline{n_0}(\bar{\mu}_3^2+1)\overline{f_*(z_3)}$$
$$+\frac{\overline{dt}}{dt}\Big\{(1+\mu_1\ i)^2\Phi^{\pm}(t_1) + (1+\overline{\mu_1}i)^2\overline{\Phi^{\pm}(t_1)} + (1+\mu_2\ i)^2\Psi^{\pm}(t_2) + (1+\overline{\mu_2}i)^2\overline{\Psi^{\pm}(t_2)} + n_0(1+\mu_3\ i)^2 f_*(z_3) + \overline{n_0}(1+\overline{\mu_3}i)^2\overline{f_*(z_3)}\Big\}, \quad (23)$$

where,

$$q^{\pm}(t) = \sigma_n^{\pm} + i\sigma_t^{\pm}$$
$$-\frac{1}{2}\Big\{(1+\mu_1^2)\Gamma + \big(1+\overline{\mu_1^2)\Gamma} + (1+\mu_1^2)a'_0(1+\ln(t_1-a_1)) + (1+\bar{\mu}_1^2)a'_0(1+\ln(\overline{t_1}-\overline{a_1})) + (1+\mu_2^2)\Gamma' + (1+\overline{\mu_2^2)\Gamma'}\Big\}$$
$$+(1+\mu_2^2)b'_0(1+\ln(t_2-a_2)) + (1+\bar{\mu}_2^2)b'_0(1+\ln(\overline{t_2}-\overline{a_2})) + n_0(\mu_3^2+1)m_0(1+\ln(t_3-a_3))$$
$$+n_0(\mu_3^2+1)\frac{(cos\beta_0 + \overline{\overline{\mu_3}}sin\beta_0)t_3}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^{\infty} + \overline{n_0}(\bar{\mu}_3^2+1)m_0(1+\ln(\overline{t_3}-\overline{a_3}))$$
$$+\overline{n_0}(\bar{\mu}_3^2+1)\frac{(cos\overline{\beta_0} + \overline{\mu_3}sin\overline{\beta_0})\overline{t_3}}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^{\infty}\}$$

$$-\frac{1}{2}\frac{\overline{dt}}{dt}\{(1+\mu_1^2)\Gamma + \big(1+\overline{\mu_1^2)\Gamma} + (1+\mu_1^2)a'_0(1+\ln(t_1-a_1)) + (1+\bar{\mu}_1^2)a'_0(1+\ln(\overline{t_1}-\overline{a_1}))$$

$$+(1+\mu_2^2)\Gamma' + (1+\overline{\mu_2^2)\Gamma'}\Big\} + (1+\mu_2^2)b'_0(1+\ln(t_2-a_2)) + (1+\bar{\mu}_2^2)b'_0(1+\ln(\overline{t_2}-\overline{a_2}))$$

$$+n_0(1+\mu_3\ i)^2 m_0(1+\ln(t_3-a_3))+\overline{n_0}(1+\overline{\mu_3}i)^2 m_0(1+\ln(\overline{t_3}-\overline{a_3}))$$

$$+n_0(1+\mu_3\ i)^2\frac{(cos\beta_0+\mu_3 sin\beta_0)t_3}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^\infty++\overline{n_0}(1+\overline{\mu_3}i)^2\frac{(cos\overline{\beta_0}+\mu_3 sin\overline{\beta_0})\overline{t_3}}{2\sqrt{k_{11}k_{22}}Im\mu_3}q^\infty.$$

Based on relations

$$z_k=x+\mu_k y, k=1,2,3, z=x+iy$$

we have

$$t_1=\frac{1}{2}[(1-i\mu_1)t+(1+i\mu_1)\bar{t}],$$
$$t_2=\frac{1}{2}[(1-i\mu_2)t+(1+i\mu_2)\bar{t}],$$
$$t_3=\frac{1}{2}[(1-i\mu_3)t+(1+i\mu_3)\bar{t}].$$

After derivation these relations become

$$\frac{dt_1}{dt}=\frac{1}{2}\left[(1-i\mu_1)+(1+i\mu_1)\frac{\overline{dt}}{dt}\right],$$
$$\frac{dt_2}{dt}=\frac{1}{2}\left[(1-i\mu_2)+(1+i\mu_2)\frac{\overline{dt}}{dt}\right],$$
$$\frac{dt_3}{dt}=\frac{1}{2}\left[(1-i\mu_3)+(1+i\mu_3)\frac{\overline{dt}}{dt}\right],$$
$$\frac{dt_1}{\overline{dt}}=\frac{1}{2}\left[(1-i\mu_1)\frac{dt}{\overline{dt}}+(1+i\mu_1)\right],$$
$$\frac{dt_2}{\overline{dt}}=\frac{1}{2}\left[(1-i\mu_2)\frac{dt}{\overline{dt}}+(1+i\mu_2)\right],$$
$$\frac{dt_3}{\overline{dt}}=\frac{1}{2}\left[(1-i\mu_3)\frac{dt}{\overline{dt}}+(1+i\mu_3)\right].$$

Taking the conjugate expression and multiplying by $\frac{\overline{dt}}{dt}$ equation (23) becomes

$$(1-i\mu_1)\left[(1-i\mu_1)+(1+i\mu_1)\frac{\overline{dt}}{dt}\right]\Phi^\pm(t_1)+(1-i\overline{\mu_1})\left[(1-i\overline{\mu_1})+(1+i\overline{\mu_1})\frac{\overline{dt}}{dt}\right]\overline{\Phi^\pm(t_1)}$$
$$+(1-i\mu_2)\left[(1-i\mu_2)+(1+i\mu_2)\frac{\overline{dt}}{dt}\right]\Psi^\pm(t_2)+(1-i\overline{\mu_2})\left[(1-i\overline{\mu_2})+(1+i\overline{\mu_2})\frac{\overline{dt}}{dt}\right]\overline{\Psi^\pm(t_2)}$$
$$+n_0(1+\mu_3\ i)\frac{1}{2}\left[(1-i\mu_3)+(1+i\mu_3)\frac{\overline{dt}}{dt}\right]f_*(t_3)+\overline{n_0}(1+\overline{\mu_3\ i})\frac{1}{2}\left[(1-i\overline{\mu_3})+(1+i\overline{\mu_3})\frac{\overline{dt}}{dt}\right]\overline{f_*(t_3)}$$
$$=2\frac{\overline{dt}}{dt}\overline{q^\pm(t)}. \tag{27}$$

Multiplying equation (23) with $(1-i\overline{\mu_2})$ and equation (27) with $(1-i\overline{\mu_2})$ and subtracting by parts, the following equation is derived

$$(\mu_1-\overline{\mu_2})\left[(1-i\mu_1)+(1+i\mu_1)\frac{\overline{dt}}{dt}\right]\Phi^\pm(t_1)+(\overline{\mu_1}-\overline{\mu_2})\left[(1-i\overline{\mu_1})+(1+i\overline{\mu_1})\frac{\overline{dt}}{dt}\right]\overline{\Phi^\pm(t_1)}$$

$$+(\mu_2-\overline{\mu_2})\left[(1-i\mu_2)+(1+i\mu_2)\frac{\overline{dt}}{dt}\right]\Psi^{\pm}(t_2)+n_0(\mu_3-\overline{\mu_2})\left[(1-i\mu_3)+(1+i\mu_3)\frac{\overline{dt}}{dt}\right]f_*^{\pm}(t_3)$$
$$+\overline{n_0}(\overline{\mu_3}-\overline{\mu_2})\left[(1-i\overline{\mu_3})+(1+i\overline{\mu_3})\frac{\overline{dt}}{dt}\right]\overline{f_*^{\pm}(t_3)}=p^{\pm}(t), \quad (28)$$

where,

$$p^{\pm}(t)=-i(1-\overline{\mu_2}i)q^{\pm}(t)+i\frac{\overline{dt}}{dt}(1+\overline{\mu_2}i)\overline{q^{\pm}(t)}.$$

Equation (28) based on equations (25) and (26) becomes

$$2(\mu_1-\overline{\mu_2})\frac{dt_1}{dt}\Phi^{\pm}(t_1)+2(\overline{\mu_1}-\overline{\mu_2})\frac{\overline{dt_1}}{dt}\overline{\Phi^{\pm}(t_1)}+2(\mu_2-\overline{\mu_2})\frac{dt_2}{dt}\Psi^{\pm}(t_2)$$
$$+2n_0(\mu_3-\overline{\mu_2})\frac{dt_3}{dt}f_*^{\pm}(t_3)+2\overline{n_0}(\overline{\mu_3}-\overline{\mu_2})\frac{\overline{dt_3}}{dt}\overline{f_*^{\pm}(t_3)}=p^{\pm}(t) \quad (29)$$

Recalling (29) and replacing the limited values of functions $\Phi(z_1)$ and $\Psi(z_2)$ from (17) and (18), we obtain

$$(\mu_1-\overline{\mu_2})\frac{dt_1}{dt}\frac{2}{\pi i}\int_{l_1}\frac{\varphi(\tau_1)}{\tau_1-t_1}d\tau_1-(\overline{\mu_1}-\overline{\mu_2})\frac{\overline{dt_1}}{dt}\frac{2}{\pi i}\int_{l_1}\frac{\overline{\varphi(\tau_1)}}{\overline{\tau_1}-\overline{t_1}}\overline{d\tau_1}+(\mu_2-\overline{\mu_2})\frac{dt_2}{dt}\frac{2}{\pi i}\int_{l_2}\frac{\psi(\tau_2)}{\tau_2-t_2}d\tau_2$$
$$+n_0(\mu_3-\overline{\mu_2})\frac{dt_3}{dt}\frac{2}{\pi i}\int_{l_1}\frac{\gamma(\tau_3)}{\tau_3-t_3}d\tau_3-n_0(\overline{\mu_3}-\overline{\mu_2})\frac{\overline{dt_3}}{dt}\frac{2}{\pi i}\int_{l_1}\frac{\overline{\gamma(\tau_3)}}{\overline{\tau_3}-\overline{t_3}}\overline{d\tau_3}=p^{+}(t)+p^{-}(t)=2p_1(t). \quad (30)$$

Subtracting expressions in (29) by parts produces the equation

$$(\mu_1-\overline{\mu_2})2\frac{dt_1}{dt}\varphi(t_1)+(\overline{\mu_1}-\overline{\mu_2})2\frac{\overline{dt_1}}{dt}\overline{\varphi(t_1)}+(\mu_2-\overline{\mu_2})2\frac{dt_2}{dt}\psi(t_2)+2n_0(\mu_3-\overline{\mu_2})\frac{dt_3}{dt}\gamma(t_3)$$
$$+n_0(\overline{\mu_3}-\overline{\mu_2})2\frac{\overline{dt_3}}{dt}\overline{\gamma(t_3)}=p^{+}(t)-p^{-}(t)=2p_2(t). \quad (31)$$

Solving equation in terms of $\psi(t_2)$ we obtain

$$\psi(t_2)=\frac{1}{\mu_2-\overline{\mu_2}}\frac{dt}{dt_2}p_2(t)-\frac{\mu_1-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{dt_1}{dt_2}\varphi(t_1)-\frac{\overline{\mu_1}-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{\overline{dt_1}}{dt_2}\overline{\varphi(t_1)}-n_0\frac{\mu_3-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{dt_3}{dt_2}\gamma(t_3)$$
$$-\overline{n_0}\frac{\overline{\mu_3}-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{\overline{dt_3}}{dt_2}\overline{\gamma(t_3)}. \quad (32)$$

By integrating, we get

$$\frac{1}{2\pi i}\int_{l_2}\frac{\psi(\tau_2)}{\tau_2-z_2}d\tau_2=\frac{1}{\mu_2-\overline{\mu_2}}\frac{1}{2\pi i}\int_{l}\frac{p_2(\tau)}{\tau_2-z_2}d\tau$$
$$-\frac{\mu_1-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{1}{2\pi i}\int_{l_1}\frac{\varphi(\tau_1)}{\tau_2-z_2}d\tau_1-\frac{\overline{\mu_1}-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{1}{2\pi i}\int_{l_1}\frac{\overline{\varphi(\tau_1)}}{\tau_2-z_2}\overline{d\tau_1}-n_0\frac{\mu_3-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{1}{2\pi i}\int_{l_3}\frac{\gamma(\tau_3)}{\tau_2-z_2}d\tau_3$$
$$-\overline{n_0}\frac{\overline{\mu_3}-\overline{\mu_2}}{\mu_2-\overline{\mu_2}}\frac{1}{2\pi i}\int_{l_3}\frac{\overline{\gamma(\tau_3)}}{\tau_2-z_2}\overline{d\tau_3}\ . \quad (33)$$

Substituting (33) into (30) we get

$$\frac{\mu_1-\overline{\mu_2}}{\pi i}\frac{dt_1}{dt}\int_{l_1}\frac{\varphi(\tau_1)}{\tau_1-t_1}d\tau_1-\frac{\overline{\mu_1}-\overline{\mu_2}}{\pi i}\frac{\overline{dt_1}}{dt}\int_{l_1}\frac{\overline{\varphi(\tau_1)}}{\overline{\tau_1}-\overline{t_1}}\overline{d\tau_1}-\frac{\mu_1-\overline{\mu_2}}{\pi i}\frac{dt_2}{dt}\int_{l_1}\frac{\varphi(\tau_1)}{\tau_2-t_2}d\tau_1$$

$$-\frac{\overline{\mu_1}-\overline{\mu_2}}{\pi i}\frac{dt_2}{dt}\int_{l_1}\frac{\overline{\varphi(\tau_1)}}{\tau_2-t_2}\overline{d\tau_1}-n_0\frac{\mu_3-\overline{\mu_2}}{\pi i}\frac{dt_2}{dt}\int_{l_1}\frac{\gamma(\tau_3)}{\tau_2-t_2}d\tau_3-\overline{n_0}\frac{\overline{\mu_3}-\overline{\mu_2}}{\pi i}\frac{dt_2}{dt}\int_{l_1}\frac{\overline{\gamma(\tau_3)}}{\tau_2-t_2}\overline{d\tau_3}$$
$$+n_0\frac{\mu_3-\overline{\mu_2}}{\pi i}\frac{dt_3}{dt}\int_{l_3}\frac{\gamma(\tau_3)}{\tau_3-t_3}d\tau_3-n_0\frac{\overline{\mu_3}-\overline{\mu_2}}{\pi i}\frac{\overline{dt_3}}{dt}=p_1(t)-\frac{dt_2}{dt}\frac{1}{\pi i}\int_{l}\frac{p_2(\tau)}{\tau_2-t_2}d\tau,\tau\in l,t_1\in l_1,t_2\in l_2,t_3\in l_3.$$

4. Conclusion

A problem of plane thermoelasticity of an orthotropic plate, containing a crack and submitted to thermal and mechanical field, is studied. Using the Complex potentials and the method of singular integral equations, a system of state equations of thermoelasticity is formulated. The unknowns of equations (14) and (34) are the stresses and the temperature at a specific point of the plate. It is evident that evaluating quantitatively the system of equations, we can determine the distribution of stresses and temperature at every point of the plate.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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