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Higher-Order Wolfe Type Symmetric Fractional Programming Problem Under Generalized Assumptions

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Abstract
The Wolfe-type model over arbitrary cones is a new sort of model that we introduce in this article. We defend duality theorems under more generalized higher-order assumptions in the section after this. We identify a function that only belongs to the class of generalize higher order pseudoinvex functions, not the class of generalize higher order invex functions. In the last section, we added a conclusion for future prosecutions for the researchers. Additionally, compared to earlier results in the literature, all of the results in this research are more broadly applicable.

Keywords- Higher-order, Generalized invexity/ pseudoinvexity, Arbitrary cones, Wolfe model, Fractional programming.

1. Introduction

The hypothesis of duality is significant in numerical programming and is fruitful both analytically and practically aspects. Duality as practised in our day to day life means the mode of adjustment of two contrary or subsidiary parts via which they combine into a whole. Duality theory harmonizes to restricted optimization problems. According to the duality principle, if a primal or dual problem has a solution, there will inevitably be a dual or primordial problem with a solution, and the best solution for both problems is the same. The issue of streamlining an objective function of several elements subject to restrictions on the elements is known as the numerical programming. The optimize problem is called nonlinear optimize problem if the objective function or restrictions are nonlinear. Nonlinear optimize problem plays a crucial role in such various areas management sciences, engineering, operations research, quality control and computer science. If there is cooperation between the objective functions, we usually aimed at optimizing all the objective functions at the identical time (Bazarra and Goode, 1973; Brumelle, 1981; Dorn, 1960).

Duality of "non-differentiable second order Wolfe-type symmetric" primal and dual programming problems were defined by Dubey et al. (2021) and Yang et al. (2003). They also established the weak, strong, and strict converse duality theorems. They additionally examined symmetric minimax blended number base and dual issues. Jayswal et al. (2014) studied a pair of "Wolfe-type non-differentiable multi objective second-order symmetric" duality projects including two pieces capacities and duality theorems for this pair with generalized conditions. Information envelopment analysis, charge programming, hazard analysis, and portfolio hypothesis are all applications of fractional programming (see, for example, Ying, 2012; Antczak, 2015; Khalil and Abdullah, 2018; Dubey and Mishra, 2019; Dubey et al., 2020; Kumar et al., 2021; Kaur and Sharma, 2022).

Suneja et al. (2018), introduced higher-order Schaible type dual model and derived duality theorems under cone convex and other related functions. Kapoor et al. (2017), analysed the results of duality relationship of Wolfe and Mond-Weir type models by using higher-order cone–convex, (K_1, K_2) - pseudoconvexity/quasiconvexity assumptions. Chaudhary and Sharma (2019), another class of summed up preinvex set esteemed maps is presented and its portrayal as far as their contingent epi-subordinates is acquired. Under the assumption of higher-order convexity or pseudoconvexity, Dubey and Mishra (2020) focused on " non-differentiable Mond-Weir higher-order fractional symmetric duality over arbitrary cones" type problems. On the basis of above assumptions, we can also explain the duality relations.

This paper is coordinated as follows: In segment 1 and 2, we give some brief introductory information and definitions about the paper. In section 3, we formulate a higher-order symmetric primal dual model of the Wolfe type over cones and examine the duality theorems of weak duality, strong duality, and strict converse duality on the concept of "higher-order fractional symmetric duality" over arbitrary cones. We include the paper's conclusion in the last section.

2. Preliminaries and Definitions

Let $K_1 \subseteq R^n$ & $K_2 \subseteq R^m$ be open sets, $f(x_{11}, x_{12})$ real valued function be differentiable and specified on $K_1 \times K_2$. Let $\mathcal{H} : K_1 \times K_2 \times R^n \to R$ be differentiable function, $\rho \in R$ and $\phi : K_1 \times K_2 \to R^n$.

Definition 2.1 If $f(x_{11}, x_{12})$ is higher order $(V, \rho, \theta) -$ invex at $u_{11} \in K_1$ for fixed $u_{12} \in K_2$' w.r.t. $\mathcal{H}$, $\exists\ \eta_1$, ρ and θ such that, $\forall\ x_{11} \in K_1$ and $s \in R^m$, we have

$$[f(x_{11}, u_{12}) - f(u_{11}, u_{12})] \geq \eta_1^T(u_{11}, u_{12})\left[\nabla_{x_{11}} f(u_{1.1}, u_{12}) + \nabla_s \mathcal{H}(u_{1.1}, u_{12}, s)\right] + \mathcal{H}(u_{1.1}, u_{12}, s) - s^T \nabla_s \mathcal{H}(u_{11}, u_{12}, s) + \rho \|\theta(x_{11}, u_{11})\|^2.$$

Definition 2.2 If $f(x_{11}, x_{12})$ is higher order (V, ρ, θ)-invex at $u_{12} \in K_2$ for fixed $u_{11} \in K_1$ w.r.t. $\mathcal{H}$, $\exists\ \eta_2$, ρ and ϕ such that, $\forall\ x_{12} \in K_2$ and $r \in R^n$, we have

$$[f(u_{11}, x_{12}) - f(u_{11}, u_{12})] \geq {\eta_2}^T(x_{12}, u_{12})\left[\nabla_{x_{12}} f(u_{1.1}, u_{12}) + \nabla_r \mathcal{H}(u_{1.1}, u_{12}, r)\right] + \mathcal{H}(u_{1.1}, u_{12}, r) - r^T \nabla_r \mathcal{H}(u_{11}, u_{12}, r) + \rho \|\phi(x_{12}, u_{12})\|^2.$$

Definition 2.3 According to Gao (2016) if $f(x_{11}, x_{12})$ is higher order (V, ρ, θ)-pseudoinvex at $u_{11} \in K_1$ for fixed $u_{12} \in K_2$ w.r.t. $\mathcal{H}$, $\exists\ \eta_1$, ρ and θ such that, $\forall\ x_{11} \in K_1$ and $s \in R^m$, we have

$$\eta_1^T(u_{11}, u_{12})\left[\nabla_{x_1} f(u_{11}, u_{12}) + \nabla_s \mathcal{H}(u_{11}, u_{12}, s)\right] + \mathcal{H}(u_{11}, u_{12}, s) - s^T \nabla_s \mathcal{H}(u_{11}, u_{12}, s) + \rho \|\theta(x_{11}, u_{11})\|^2 \geq 0$$
$$= [f(x_{11}, u_{12}) - f(u_{11}, u_{12})] \geq 0.$$

Definition 2.4 According to Ying (2016) if $f(x_{11}, x_{12})$ is higher order (V, ρ, θ)-pseudoinvex at $u_{12} \in K_2$ for fixed $u_{11} \in K_1$ w.r.t. $\mathcal{H}$, $\exists$ η_2, ρ and ϕ such that, $\forall$ $x_{12} \in K_2$ and $r \in R^n$, we have

$$\eta_2{}^T(x_{12}, u_{12})\left[\nabla_{x_{12}} f(u_{11}, u_{12}) + \nabla_r \mathcal{H}(u_{11}, u_{12}, r)\right] + \mathcal{H}(u_{11}, u_{12}, r)$$

$$-r^T \nabla_r \mathcal{H}(u_{11}, u_{12}, r) + \rho\|\phi(x_{12}, u_{12})\|^2 \geq 0 \quad \Rightarrow [f(u_{11}, x_{12}) - f(u_{11}, u_{12})] \geq 0.$$

Definition 2.5 The C^* is the +tive polar cone of $C \subseteq R^s$ by Jayswal et al. (2014)

$$C^* = \{x_{12} \in R^s : x_{11}{}^T x_{12} \geq 0\}.$$

Demonstrates the example of function that is higher-order (V, ρ, θ) − pseudoinvex but not (V, ρ, θ) − invex, as below:

Example 2.1 Let the function $f: X \times X \rightarrow R$, where $X = [0, 1]$. Consider

$$f(x_{11}, x_{12}) = 3\,x_{11} + 2\,x_{12}.$$

Let $\eta : X \times X \rightarrow R$, given by

$$H(u_{11}, u_{12}, r) = (u_{11} + u_{12})r.$$

Consider the function $\theta(x_{11}, u_{11}) = x_{11}\,u_{11}$ and $\rho = 1$.

At this point, we assert that f is a higher-order (V, ρ, θ) − pseudoinvex function w.r.t. $\mathcal{H}$ & at the point u_{11} i.e., it is sufficient to prove that

$$\eta^T(x_{11}, u_{11})\left[\nabla_{x_{11}} f(u_{11}, u_{12}) + \nabla_r \mathcal{H}(u_{11}, u_{12}, r)\right] + \mathcal{H}(u_{11}, u_{12}, r) - r^T \nabla_r \mathcal{H}(u_{11}, u_{12}, r) + \rho\|\phi(x_{11}, u_{11})\|^2 \geq 0$$

$$\Rightarrow f(x_{11}, u_{11}) - f(u_{11}, u_{12}) \geq 0.$$

Let

$$\psi_1 = \eta^T(x_{11}, u_{11})\left[\nabla_{x_{11}} f(u_{11}, u_{12}) + \nabla_r \mathcal{H}(u_{11}, u_{12}, r)\right] + \mathcal{H}(u_{11}, u_{12}, r) - r^T \nabla_r \mathcal{H}(u_{11}, u_{12}, r) + \rho\|\phi(x_{11}, u_{11})\|^2.$$

Substituting the values of f, $\eta^T, \theta, \mathcal{H}(u_{11}, u_{12}, r)$ & ρ, we get

$$\psi_1 = (x_{11} - u_{11})[3 + (u_{11} + u_{12}) + (u_{11} - u_{12})r - (u_{11} - u_{12})r + \rho x_{11}{}^2 u_{11}{}^2].$$

At the point $u_{11} = 0$, the above expression follows that

$$\psi_1 = 3x_{11} + x_{11}u_{12}, \;\; \forall\, x_{11}, u_{12} \in [0, 1].$$

Thus, $\psi_1 \geq 0$, $\forall\, x_{11}, u_{12} \in [0, 1]$, (see Figure 1).

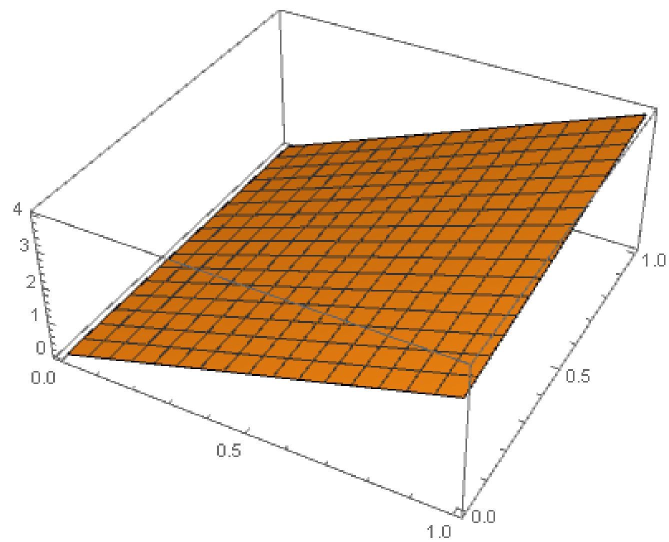

Figure 1. $\psi_1 = 3x_{11} + x_{11}u_{12}$.

Next, consider

$$\psi_2 = f(x_{11}, u_{12}) - f(u_{11}, u_{12}).$$

Putting the values of function f, we find that

$$\psi_2 = 3\,x_{11} + 2\,u_{12} - 3\,u_{11} - 2\,u_{12}.$$

The aforementioned statement reduces to for $u_{11} = 0$, we have

$$\psi_2 = 3\,x_{11}.$$

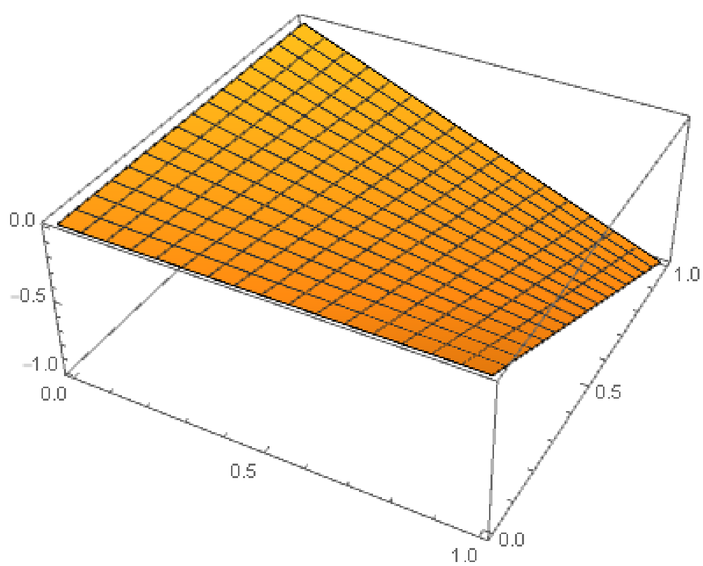

Figure 2. $\psi_2 = 3\,x_{11}$.

Thus, $\psi_2 \geq 0, \forall\, x_{11} \in [0,1]$, (see Figure 2).

Hence result shows, f is higher-order $(V,\rho,\theta)-$ pseudoinvex function at $u_{11} = 0 \in [0,1]$.

Next, we assert that f is not higher-order $(V,\rho,\theta)-$ invex function w.r.t. $\mathcal{H}$ & u_{11}, i.e.

$$f(x_{11},u_{12}) - f(u_{11},u_{12}) - \eta^T(x_{11},u_{11})\left[\nabla_{x_{11}} f(u_{11},u_{12}) + \nabla_r \mathcal{H}(u_{11},u_{12},r)\right] - \mathcal{H}(u_{11},u_{12},r) + r^T\nabla_r\mathcal{H}(u_{11},u_{12},r) - \rho\|\phi(x_{11},u_{11})\|^2 < 0.$$

Let

$$\psi_3 = ff(x_{11},u_{12}) - f(u_{11},u_{12}) - \eta^T(x_{11},u_{11})\left[\nabla_{x_{11}} f(u_{11},u_{12}) + \nabla_r \mathcal{H}(u_{11},u_{12},r)\right] - \mathcal{H}(u_{11},u_{12},r) + r^T\nabla_r\mathcal{H}(u_{11},u_{12},r) - \rho\|\phi(x_{11},u_{11})\|^2.$$

Putting values and simplifying at the point $u_{11} = 0$, we get

$$\psi_3 = -x_{11}u_{12} \ngeq 0, \forall\, x_{11}, u_{12} \in [0,1].$$

Thus, f is not higher-order $(V,\rho,\theta)-$ invex function at $u_{11} = 0 \in [0,1]$. Hence above example shows that the f is higher order $(V,\rho,\theta)-$ pseudoinvex function w.r.t. $\mathcal{H}$ and at the point $u_{11} = 0 \in [0,1]$ but not higher order $(V,\rho,\theta)-$ invex at the same point.

3. Higher-Order Fractional Symmetric Duality over Arbitrary Cones

In this section, we discuss following primal-dual model as:

Primal Problem (HWFP):

$$\text{Minimize}\quad \frac{f(x_{11},x_{12}) - x_{12}{}^T\left[\nabla_{x_{12}} f(x_{11},x_{12}) + \nabla_r\mathcal{H}(x_{11},x_{12},r)\right] + \mathcal{H}(x_{11},x_{12},r) - r^T\nabla_r\mathcal{H}(x_{11},x_{12},r)}{g(x_{11},x_{12}) - x_{12}{}^T\left[\nabla_{x_{12}} g(x_{11},x_{12}) + \nabla_r\mathcal{G}(x_{11},x_{12},r)\right] + \mathcal{G}(x_{11},x_{12},r) - r^T\nabla_r\mathcal{G}(x_{11},x_{12},r)}$$

Subject to

$$\Big[\Big(f(x_{11},x_{12}) - x_{12}{}^T\left[\nabla_{x_2} f(x_{11},x_{12}) + \nabla_r\mathcal{H}(x_{11},x_{12},r)\right] + \mathcal{H}(x_{11},x_{12},r) - r^T\nabla_r\mathcal{H}(x_{11},x_{12},r)\Big)\Big(\nabla_{x_{12}} g(x_{11},x_{12}) + \nabla_r\mathcal{G}(x_{11},x_{12},r)\Big)\Big(g(x_{11},x_{12}) - x_{12}{}^T\left[\nabla_{x_{12}} g(x_{11},x_{12}) + \nabla_r\mathcal{G}(x_{11},x_{12},r)\right] + \mathcal{G}(x_{11},x_{12},r) - r^T\nabla_r\mathcal{G}(x_{11},x_{12},r)\Big)\Big(\nabla_{x_{12}} f(x_{11},x_{12}) + \nabla_r\mathcal{H}(x_{11},x_{12},r)\Big)\Big] \in C_2{}^*,$$

$$x_{11} \in C_1,$$

Dual Problem (HWFD):

$$\text{Maximize}\quad \frac{f(u_{11},u_{12}) - u_{11}{}^T\left[\nabla_{x_{11}} f(u_{1.1},u_{12}) + \nabla_s\phi(u_{1.1},u_{12},s)\right] + \phi(u_{1.1},u_{12},s) - s^T\nabla_s\phi(u_{11},u_{12},s)}{g(u_{11},u_{12}) - u_{11}{}^T\left[\nabla_{x_{11}} g(u_{1.1},u_{12}) + \nabla_s\psi(u_{1.1},u_{12},s)\right] + \psi(u_{1.1},u_{12},s) - s^T\nabla_s\psi(u_{11},u_{12},s)}$$

Subject to

$$-\Big[\Big(f(u_{11},u_{12}) - {u_{11}}^{T}[\nabla_{x_{11}} f(u_{11},u_{12}) + \nabla_s \phi(u_{1.1},u_{12},s)] + \phi(u_{1.1},u_{12},s) - s^T\nabla_s\phi(u_{1.1},u_{12},s)\Big)\Big(\nabla_{x_{11}} g(u_{11},u_{12}) + \nabla_s\psi(u_{11},u_{12},s)\Big)\Big(g(u_{11},u_{12}) - {u_{11}}^{T}[\nabla_{x_1} g(u_1,u_2) + \nabla_s\psi(u_{11},u_{12},s)] + \psi(u_{11},u_{12},s) - s^T\nabla_s\psi(u_{11},u_{12},s)\Big)\Big(\nabla_{x_{11}} f(u_{11},u_{12}) + \nabla_s\phi(u_{11},u_{12},s)\Big)\Big] \in {C_1}^*,$$

$$u_{12} \in C_2,$$

where, $f: S_1 \times S_2 \to R$ and $g: S_1 \times S_2 \to R_+\backslash\{0\}$ are differentiable functions and r, s are vector in R^m and R^n, respectively.

The above problem can be rewritten as equivalently:

(EHWFP) Min. w

Subject to

$$\Big(f(x_{11},x_{12}) - {x_{12}}^{T}[\nabla_{x_{12}} f(x_{11},x_{12}) + \nabla_r \mathcal{H}(x_{11},x_{12},r)] + \mathcal{H}(x_{11},x_{12},r) - r^T\nabla_r\mathcal{H}(x_{11},x_{12},r)\Big) - w\Big(g(x_{11},x_{12}) - {x_{12}}^{T}[\nabla_{x_{12}} g(x_{11},x_{12}) + \nabla_r\mathcal{G}(x_{11},x_{12},r)] + \mathcal{G}(x_{11},x_{12},r) - r^T\nabla_r\mathcal{G}(x_{11},x_{12},r)\Big) = 0, \quad (1)$$

$$-\Big[\Big(\nabla_{x_{12}} f(x_{11},x_{12}) + \nabla_r\mathcal{H}(x_{11},x_{12},r)\Big) - w\Big(\nabla_{x_{12}} g(x_{11},x_{12}) + \nabla_r\mathcal{G}(x_{11},x_{12},r)\Big)\Big] \in {C_1}^*, \quad (2)$$

$$x_{11} \in C_1. \quad (3)$$

For dual problem, we have

(EHWFD) Min. t

Subject to

$$\Big(f(u_{11},u_{12}) - {u_{11}}^{T}\Big(\nabla_{x_{11}} f(u_{1.1},u_{12}) + \nabla_s\phi(u_{1.1},u_{12},s)\Big) + \phi(u_{11},u_{12},s) - s^T\nabla_s\phi(u_{11},u_{12},s)\Big) - t\Big(g(u_{11},u_{12}) - {u_{11}}^{T}\Big(\nabla_{x_{11}} g(u_{1.1},u_{12}) + \nabla_s\psi(u_{1.1},u_{12},s)\Big) + \psi(u_{1.1},u_{12},s) - s^T\nabla_s\psi(u_{11},u_{12},s)\Big) = 0, \quad (4)$$

$$\Big[\Big(\nabla_{x_{11}} f(u_{1.1},u_{12}) + \nabla_s\phi(u_{1.1},u_{1.2},s)\Big) - t\Big(\nabla_{x_{11}} g(u_{11},u_{12}) + \nabla_s\psi(u_{11},u_{12},s)\Big)\Big] \in {C_2}^*, \quad (5)$$

$$u_{12} \in C_2. \quad (6)$$

Let P^o & Q^o be sets of feasible solutions of (EHWFP) & (EHWFD), respectively. Derive the duality relations under aforesaid conditions.

Theorem 3.1 (Weak Duality Theorem). Let $(x_{11,}x_{12}, w, r) \in P^o$ & $(u_{11}, u_{12}, t, s) \in Q^o$. Let

- **(i)** $f(.,u_{12}) - tg(.,u_{12})$ be higher order $(V,\rho,\theta)-$invex at u_{11} for fixed u_{12} w.r.t. $\phi(u_{11},u_{12},s) - t\varphi(u_{11},u_{12},s)$ & η_1,
- **(ii)** $-f(x_{11,}.) + wg(x_{11,}.)$ be higher order $(V,\rho,\theta)-$invex at x_{12} for fixed $x_{11,}$ w.r.t. $-\mathcal{H}(x_{11,}x_{12},r) + w\,\mathcal{G}(x_{11,}x_{12},r)$ & η_2,
- **(iii)** $\eta_1(x_{11,},u_{11}) + u_{11} \in C_1$ & $\eta_2(u_{12},x_{12}) + x_{12} \in C_2$,
- **(iv)** $g(x_{11,},u_{12}) > 0$,
- **(v)** $\rho \geq 0$.

Then, $w \geq t$.

Proof: By (i), we have

$$f(x_{11}, u_{12}) - tg(x_{11}, u_{12}) - \left(f(u_{1.1}, u_{12}) - tg(u_{1.1}, u_{12})\right) \geq \eta_1{}^T(x_{11}, u_{11}) \left[\nabla_{x_{11}} f(u_{11}, u_{12}) - tg(u_{1.1}, u_{12}) + \nabla_s \phi(u_{1.1}, u_{12}, s) - t\psi(u_{11}, u_{12}, s)\right) + \left(\phi(u_{1.1}, u_{12}, s) - t\psi(u_{1.1}, u_{12}, s)\right) - s^T \nabla_s\left(\phi(u_{11}, u_{12}, s) - t\psi(u_{11}, u_{12}, s)\right) + \rho\|\theta(x_{11}, u_{11})\|^2. \quad (7)$$

Next, by hypothesis (ii), gives

$$-f(x_{11}, u_{12}) + wg(x_{11}, u_{12}) - \left(-f(x_{11,}x_{12}) + wg(x_{11,}x_{12})\right) \geq \eta_2{}^T(u_{12}, x_{12}) \left[\nabla_{x_{12}}\left(-f(x_{11,}x_{12}) + wg(x_{11,}x_{12})\right) + \nabla_r\left(-\mathcal{H}(x_{11,}x_{12}, r) + w\,\mathcal{G}(x_{11,}x_{12}, r)\right)\right] + \left(-\mathcal{H}(x_{11,}x_{12}, r) + w\,\mathcal{G}(x_{11,}x_{12}, r)\right) - r^T\nabla_r\left(-\mathcal{H}(x_{11,}x_{12}, r) + w\,\mathcal{G}(x_{11,}x_{12}, r)\right) + \rho\left\|\theta(x_{11,}x_{12})\right\|^2 \quad (8)$$

On adding inequalities (7) and (8), we obtain that

$$f(x_{11}, u_{12}) - tg(x_{11}, u_{12}) - \left(f(x_{11}, u_{12}) - tg(x_{11}, u_{12})\right) - f(x_{11}, u_{12}) + wg(x_{11}, u_{12}) — \left(f(x_{11,}x_{12}) + wg(x_{11,}x_{12})\right) \geq \eta_1{}^T(x_{1.1}, u_{12})[\nabla_{x_{11}}\left(f(u_{1.1}, u_{12}) - tg(u_{1.1}, u_{12})\right) + \nabla_s\left(\phi(u_{1.1}, u_{12}, s) - t\psi(u_{11}, u_{12}, s)\right)] + \eta_2{}^T(u_{12}, x_{12})\left[\nabla_{x_{12}}\left(-f(x_{11,}x_{12}) + wg(x_{11,}x_{12})\right) + \nabla_r\left(-\mathcal{H}(x_{11,}x_{12}, r) + w\,\mathcal{G}(x_{11,}x_{12}, r)\right)\right] + \left(-\mathcal{H}(x_{11,}x_{12}, r) + w\,\mathcal{G}(x_{11,}x_{12}, r)\right) - r^T\nabla_r\left(-\mathcal{H}(x_{11,}x_{12}, r) + w\,\mathcal{G}(x_{11,}x_{12}, r)\right) + \left(\phi(u_{11}, u_{12}, s) - t\psi(u_{11}, u_{12}, s)\right) - s^T\nabla_s\left(\phi(u_{11}, u_{12}, s) - t\psi(u_{11}, u_{12}, s)\right) + \rho\|\theta(x_{11}, u_{11})\|^2. \quad (9)$$

From dual constraints (5) and hypothesis (iii), we get

$$\left(\eta(x_{11,}, u_{11}) + u_{11}\right)^T[\nabla_{x_{11,}} f(u_{11}, u_{12}) + \nabla_s\phi(u_{11}, u_{12}, s) - t\nabla_{x_{11,}} g(u_{11}, u_{12}) + \nabla_s\psi(u_{11}, u_{12}, s)] \geq 0,$$

Or

$$\eta_1{}^T(x_{11,}, u_{11})\left[\nabla_{x_{11}} f(u_{11}, u_{12}) + \nabla_s\phi(u_{11}, u_{12}, s) - t\left(\nabla_{x_{11}} g(u_{11}, u_{12}) + \nabla_s\psi(u_{11}, u_{12}, s)\right)\right] \geq -u_{11}{}^T\left[\nabla_{x_{11}} f(u_{11}, u_{12}) + \nabla_s\phi(u_{11}, u_{12}, s) - t\left(\nabla_{x_{11}} g(u_{11}, u_{12}) + \nabla_s\psi(u_{11}, u_{12}, s)\right)\right]. \quad (10)$$

Similarly, inequality (2) and hypothesis (iii), we get

$$-\left(\eta_2^T(u_{12}, x_{12}) + x_{12}\right)^T\left[\nabla_{x_{12}} f(x_{11,}x_{12}) + \nabla_r\mathcal{H}(x_{11,}x_{12}, r) - w\left(\nabla_{x_{12}} g(x_{11,}x_{12}) + \nabla_r\,\mathcal{G}(x_{11,}x_{12}, r)\right)\right] \geq 0,$$

Or

$$-\eta_2{}^T(u_{12}, x_{12})\left[\nabla_{x_{12}} f(x_{11,}x_{12}) + \nabla_r\mathcal{H}(x_{11,}x_{12}, r) - w\left(\nabla_{x_{12}} g(x_{11,}x_{12}) + \nabla_r\,\mathcal{G}(x_{11,}x_{12}, r)\right)\right] \geq x_{12}{}^T\left[\nabla_{x_{12}} f(x_{11,}x_{12}) + \nabla_r\mathcal{H}(x_{11,}x_{12}, r) - w\left(\nabla_{x_{12}} g(x_{11,}x_{12}) + \nabla_r\,\mathcal{G}(x_{11,}x_{12}, r)\right)\right]. \quad (11)$$

From inequalities (9)−(11) and using hypothesis (v), we find that

$\Big[f(x_{11},u_{12}) - tg(x_{11},u_{12}) - \big(f(u_{11},u_{12}) - tg(u_{11},u_{12})\big) - \big(\phi(u_{11},u_{12},s) - t\psi(u_{11},u_{12},s)\big) - f(x_{11},u_{12}) + wg(x_{11},u_{12}) - \big(-f(x_{11},x_{12}) + wg(x_{11},x_{12})\big) - \big(-\mathcal{H}(x_{11},x_{12},r) + w\,\mathcal{G}(x_{11},x_{12},r)\big)\Big] \geq -u_{11}{}^{T}\Big[\nabla_{x_{11}} f(u_{11},u_{12}) + \nabla_s \phi(u_{11},u_{12},s) - t\big(\nabla_{x_{11}} g(u_{11},u_{12}) + \nabla_s \psi(u_{11},u_{12},s)\big)\Big] + x_{12}{}^{T}\Big[\nabla_{x_{12}} f(x_{11},x_{12}) + \nabla_r H(x_{11},x_{12},r) - w\big(\nabla_{x_{12}} g(x_{11},x_{12}) + \nabla_r G(x_{11},x_{12},r)\big)\Big].$

Using equation (1) and (4), we obtain that

$$(w - t)\ g(x_{11},u_{12}) \geq 0. \tag{12}$$

Using (iv), we get

$$w \geq t.$$

Hence, the results.

Remark 3.1 The preceding weak duality theorem for symmetric primal-duality pair (EHWFP) and (EHWFD) can also be obtained under the aforementioned conditions because every convex function is pseudoconvex (Jayswal and Prasad, 2014).

Theorem 3.2 (Weak Duality Theorem). Let $(x_{11},x_{12},w,r) \in P^o$ & $(u_{11},u_{12},t,s) \in Q^o$. Let

- **(i)** $f(.,u_{12}) - tg(.,u_{12})$ 'be higher order (V,ρ,θ) –'pseudoinvex at u_{11} for fixed u_{12} w.r.t. $\phi(u_{11},u_{12},s) - t\varphi(u_{11},u_{12},s)$ and η_1,
- **(ii)** $-f(x_{11},.) + wg(x_{11},.)$ be higher order (V,ρ,θ) –'pseudoinvex at x_{12} for fixed x_{11} w.r.t. $-\mathcal{H}(x_{11},x_{12},r) + w\,\mathcal{G}(x_{11},x_{12},r)$ and η_2,
- **(iii)** $\eta_1(x_{11},u_{11}) + u_{11} \in C_1$ and $\eta_2(u_{12},x_{12}) + x_{12} \in C_2$,
- **(iv)** $g(x_{11},u_{12}) > 0$,
- **(v)** $\rho \geq 0$.

  Then, $w \geq t$.

Proof: This proof is on same pattern as theorem 3.1.

Theorem 3.3 (Strong Duality Theorem). Assume that if f & g be differentiable functions and $(\bar{x}_{11},\bar{x}_{12},\bar{w},\bar{r})$ be an optimal solution of (EHWFP). Consider

- **(i)** $\nabla_{rr}[\mathcal{H}(\bar{x}_{11},\bar{x}_{12},\bar{r}) - \bar{w}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})]$ is non-singular,
- **(ii)** $\big(\bar{x}_{11}{}^{T}\nabla_{x_{11}} g(\bar{x}_{11},\bar{x}_{12}) - \bar{x}_{12}{}^{T}\nabla_{x_{12}} g(\bar{x}_{11},\bar{x}_{12})\big) f(\bar{x}_{11},\bar{x}_{12}) + \big(\bar{x}_{12}{}^{T}\nabla_{x_{12}} f(\bar{x}_{11},\bar{x}_{12}) - \bar{x}_{11}{}^{T}\nabla_{x_{11}} f(\bar{x}_{11},\bar{x}_{12})\big) g(\bar{x}_{11},\bar{x}_{12}) = 0,$
- **(iii)** $\nabla_{x_{11}}\mathcal{H}(\bar{x}_{11},\bar{x}_{12},0) = \nabla_{x_{11}}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},0) = 0, \nabla_{x_{12}}\mathcal{H}(\bar{x}_{11},\bar{x}_{12},0) = \nabla_{x_{12}}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},0) = 0,$
  $\mathcal{H}(\bar{x}_{11},\bar{x}_{12},0) = \mathcal{G}(\bar{x}_{11},\bar{x}_{12},0) = 0,\ \nabla_s\phi(\bar{x}_{11},\bar{x}_{12},0) = \nabla_s\varphi(\bar{x}_{11},\bar{x}_{12},0) = 0,$
  $\phi(\bar{x}_{11},\bar{x}_{12},0) = \varphi(\bar{x}_{11},\bar{x}_{12},0) = 0,\ \mathcal{H}(\bar{x}_{11},\bar{x}_{12},0) = \nabla_r\,\mathcal{G}(\bar{x}_{11},\bar{x}_{12},0) = 0.$

There exists $\bar{r} = 0$ such that $\big((\bar{x}_{11},\bar{x}_{12},\bar{w},\bar{r}) = 0\big) \in Q^o$ and objective values of (EHWFP) & (EHWFD) are identical. In addition, $(\bar{x}_{11},\bar{x}_{12},\bar{w},\bar{s}) = 0$ is an optimal solution of (EHWFD), if presumptions of Weak Duality (3.1 or 3.2) are true.

Proof: Since $(\bar{x}_{11},\bar{x}_{12},\bar{w},\bar{r})$ is an optimal solution of (EHWFP), $\alpha,\beta,\mu\in R$, $x_2\in C_2$ in such a way that the following Fritz John (John, 1948) prerequisites are met at $(\bar{x}_{11},\bar{x}_{12},\bar{w},\bar{r})$:

$$\Big[\beta\left(\nabla_{x_{11}}f(\bar{x}_{11},\bar{x}_{12})-\bar{w}\nabla_{x_{11}}g(\bar{x}_{11},\bar{x}_{12})\right)+(\Upsilon-\beta\bar{x}_{12})^T\left(\nabla_{x_{12}x_{11}}f(\bar{x}_{11},\bar{x}_{12})-\bar{w}\nabla_{x_{12}x_{11}}g(\bar{x}_{11},\bar{x}_{12})\right)+ \left(\Upsilon-\beta\bar{x}_{12}-\frac{\beta\bar{r}}{2}\right)^T\left(\nabla_{x_{11}}\mathcal{H}(\bar{x}_{11},\bar{x}_{12},\bar{r})-\bar{w}\nabla_{x_{11}}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})\right)-\mu\Big]=0, \tag{13}$$

$$(\Upsilon-\beta\bar{x}_{12}-\beta\bar{r})^T\left(\nabla_{rr}\mathcal{H}(\bar{x}_{11},\bar{x}_{12},\bar{r})-\bar{w}\nabla_{rr}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})\right)+\left(\Upsilon-\beta\bar{x}_{12}\frac{\beta\bar{r}}{2}\right)^T\left\{\nabla_{x_{12}}\{\nabla_{rr}\mathcal{H}(\bar{x}_{11},\bar{x}_{12},\bar{r})- \bar{w}\nabla_{rr}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})\}\right\}=0, \tag{14}$$

$$\Upsilon^T\left[\nabla_{x_{12}}f(\bar{x}_{11},\bar{x}_{12})-\bar{w}\nabla_{x_{12}}g(\bar{x}_{11},\bar{x}_{12})+\nabla_r\mathcal{H}(\bar{x}_{11},\bar{x}_{12},\bar{r})-\bar{w}\nabla_r\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})\right]=0, \tag{15}$$

$$\alpha-\beta\left[g(\bar{x}_{11},\bar{x}_{12})-\bar{x}_{12}{}^T\left(\nabla_{x_{12}}g(\bar{x}_{11},\bar{x}_{12})+\nabla_r\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})+\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})-\bar{r}^T\nabla_r\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})\right)\right]- \Upsilon\left(\nabla_{x_{12}}g(\bar{x}_{11},\bar{x}_{12})+\nabla_r\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})\right)=0, \tag{16}$$

$$(\Upsilon-\beta\bar{x}_{12}-\beta\bar{r})^T\left(\nabla_{rr}\mathcal{H}(\bar{x}_{11},\bar{x}_{12},\bar{r})-\bar{w}\nabla_{rr}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})\right)=0, \tag{17}$$

$$\mu^T\bar{x}_{11}=0, \tag{18}$$

$$(\alpha,\beta,\Upsilon)\neq 0,\ (\alpha,\beta,\Upsilon)\geq 0. \tag{19}$$

Since $\nabla_{rr}\mathcal{H}(\bar{x}_{11},\bar{x}_{12},\bar{r})-\bar{w}\nabla_{rr}\mathcal{G}(\bar{x}_{11},\bar{x}_{12},\bar{r})$ is non-singular and use inequality (17) then

$$\Upsilon=\beta(\bar{x}_{12}+\bar{r}). \tag{20}$$

The remaining part of the proof is conducted in a manner similar to that used by Ying (2012) to prove the strong duality theorem.

Theorem 3.4 (Strict Converse Duality). Assume optimal solution of (EHWFD) be $(\bar{u}_1,\bar{u}_2,\bar{t},\bar{s})$, where f and g are differentiable. Suppose that

(i) $\nabla_{ss}[\phi(\bar{u}_{11},\bar{u}_{12},\bar{s})-t\,\psi(\bar{u}_{11},\bar{u}_{12},\bar{s})]$ is non-singular,

(ii) $\nabla_{x_1}\phi(\bar{u}_{11},\bar{u}_{12},0)=\nabla_{x_1}\psi(\bar{u}_{11},\bar{u}_{12},0)=0,\quad \nabla_s\phi(\bar{u}_{11},\bar{u}_{12},0)=\nabla_s\psi(\bar{u}_{11},\bar{u}_{12},0)=0,$
$\mathcal{H}(\bar{u}_{11},\bar{u}_{12},0)=\mathcal{G}(\bar{u}_{11},\bar{u}_{12},0)=0,\quad \phi(\bar{u}_{11},\bar{u}_{12},0)=\psi(\bar{u}_{11},\bar{u}_{12},0)=0,$
$\nabla_{x_{12}}\phi(\bar{u}_{11},\bar{u}_{12},0)=\nabla_{x_{12}}\psi(\bar{u}_{11},\bar{u}_{12},0)=0,\quad \nabla_r\mathcal{H}(\bar{u}_{11},\bar{u}_{12},0)=\nabla_r\mathcal{G}(\bar{u}_{11},\bar{u}_{12},0)=0,$

(iii) $\left(\bar{u}_{11}{}^T\nabla_{x_{11}}g(\bar{u}_{11},\bar{u}_{12})-\bar{u}_{12}{}^T\nabla_{x_{11}}g(\bar{u}_{11},\bar{u}_{12})\right)f(\bar{u}_{11},\bar{u}_{12})+\left(\bar{u}_2{}^T\nabla_{x_1}f(\bar{u}_{11},\bar{u}_{12})-\bar{u}_{11}{}^T\nabla_{x_{11}}f(\bar{u}_{11},\bar{u}_{12})\right)g(\bar{u}_{11},\bar{u}_{12})=0.$

Then, $\exists\ \bar{s}=0$ such that $(\bar{u}_{11},\bar{u}_{12},\bar{t},\bar{s})\in P^o$ and objective values are equal. In addition, $(\bar{u}_{11},\bar{u}_{12},\bar{t},\bar{s})=0$, is an optimal solution of (EHWFP), if presumptions of Weak Duality (3.1 or 3.2) are true.

Proof: The confirmation conforms to Theorem 3.3 because of the symmetric programming problem.

4. Conclusion

We discussed duality hypotheses under $(V, \rho, \theta) -$ invexity conditions over arbitrary requirements in this article as it relates to the "higher-order fractional dual symmetric problem. Multiobjective symmetric higher-order dual projects with cone imperatives and cone functions can be reached through the current study. In addition, this research can be expanded to multiobjective symmetric higher-order dual projects over complicated spaces. One could consider this to be the analysts' upcoming project.

Conflicts of Interest

The authors declare no conflict of interest.

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