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# Multicomponent Stress-Strength Reliability under Competing Risks

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**Abstract**
In this work, we focus on the evaluation and statistical inference of the multicomponent reliability parameter of an s-out-of-k:G system under a competing risk setting with the risks playing the role of multiple stresses that put under pressure the operation of the system. The problem is expressed through multistate systems using a generalized unified distribution family for both stress and strength. The distributions involved are assumed to belong to two special subclasses of the well-known Lehmann Alternative Family of distributions. The originality of the work lies on the fact that it brings within the above two distribution subclasses, several distributions popular in reliability theory and at the same time investigates the reliability parameter under the competing risks framework. Several probabilistic properties including the probability of failure due to specific cause are presented. The applicability of the proposed methodology is explored via two representative examples based on the Power and Lomax distribution.

**Keywords-** Multicomponent strength-stress parameter, Reliability analysis, Competing risks, Class of distributions, Lahmann alternative family.

## 1. Introduction

Reliability analysis is a cornerstone of modern engineering design, ensuring that components and systems can withstand uncertain operational environments and maintain functionality throughout their intended lifetimes. Among the classical reliability characteristics is the so-called strength–stress reliability parameter R which has been widely employed to evaluate reliability by considering the probability that the inherent strength of a component $X$ exceeds the applied stress $Y$ (see e.g. Birnbaum et al., 1961; Johnson, 1988). This framework provides an intuitive yet rigorous way to quantify reliability:
$R = P(X > Y)$.

When only a single stress factor is considered, the model captures essential aspects of failure risk. However, in practice one rarely encounters a single source of stress. Indeed, most technical systems typically operate under multiple, frequently interacting stresses, like thermal, mechanical, electrical, environmental etc., that jointly contribute to degradation and eventual failure of the system.

Such systems with multiple stresses can be viewed as competing risks so that the corresponding models could provide a natural extension. Indeed, the competing risks framework acknowledges that failure can arise from one of several possible stress factors, with each stress acting as a potential "failure cause." The overall reliability is then determined by taking into consideration these stresses and the system strength (see for instance Crowder, 2001; Lawless, 2003). Competing risks remains in the center of attention of many researchers for several years. Indeed, for instance, Dui et al. (2024) explore system resilience in a multi-state context with competing risks and maintenance while Dimitrakopoulou et al. (2025) explored the connection between the competing risks problem and the multi-state system methodology. Wang and Yan (2025) discuss semi-parametric estimation under hierarchical Archimedean copulas and Fayomi et al. (2025) combine Bayesian and classical inference under censoring using specific distributions. The Bayesian perspective for inferential statistics was also examined by Llopis-Cardona et al. (2021). The semi-Markov modelling was recently investigated by Barbu et al. (2017) and also by Garcia-Maya et al. (2022) who analysed competing risks via the semi-Markov phase-type distribution with some identifiability issues studied in Lindqvist (2023). For related references the interested reader may refer to Gaynor et al. (1993), Andersen et al. (2002), Lau et al. (2009), Wei et al. (2018), Schuster et al. (2020), and Karagrigoriou et. al (2025).

The significance of addressing this issue is primarily theoretical since the combination of strength–stress reliability with competing risks provides a unified framework to analyze which stress dominates the failure process and enhancing interpretability beyond a single aggregated failure probability. At the same time, from a practical perspective, many safety-critical systems such as mechanical, electrical or aerospace face uncertain environments with various competing stresses/risks. It is thus crucial to consider more complex modeling techniques to ensure a reliable design, risk assessment, and preventive maintenance strategies.

This study addressing this important issue by developing a generalized strength–stress reliability framework under for multiple stresses under competing risks. The proposed approach is presented under a general class of distributions which are frequently encountered in reliability settings. The present analysis allows not only the estimation of overall reliability and the inference for the parameters involved but also the allocation of failure probabilities to individual stresses, offering actionable insights for system designers and operators.

The remainder of the paper is organized as follows: The reliability parameter is discussed in Section 2 while Section 3 is devoted to the reliability evaluation and probabilistic results under the competing risks setting for the single, multiple and heterogeneous cases as well as for the cases of systems in series and parallel. Generalizations of the previous theoretical results are presented in Sub section 3.4 which are further extended in Sub section 3.5, to two special subclasses of the well-known Lehmann Alternative family of distributions (Lehmann, 1953). Section 4 deals with the statistical inference and more specifically with point and interval estimation for the distributional parameters involved. Section 5 provides applications while some concluding remarks are stated in Section 6.

## 2. Reliability Parameter

One of the popular concepts in reliability theory is the multicomponent stress-strength reliability parameter or simply the reliability parameter which measures the reliability associated with a model of *k* components, when at least s components simultaneously survive a common random stress *Y* which acts independently of the strength. The definition of the reliability parameter of an *s*-out-of-*k* system is due to Bhattacharyya and Johnson (1974) and given by

$$R_{s,k} = \mathbb{P}(\text{at least } s \text{ of the } X_1, \dots, X_k \text{ exceed } Y) = \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} (1 - F_X(x))^i (F_X(x))^{k-i} dF_Y(x) \tag{1}$$

where, the k components $X_1, \dots, X_k$ have a common distribution $F_X(\cdot)$ and the stress variable $Y$, which is independent of the $X's$, has a cdf $F_Y(\cdot)$.

In reliability analysis, the maximum order statistic plays a central role when the system of interest continues to operate as long as at least one component is functioning—capturing the notion of system survival under best-case scenarios. This is particularly useful for modeling parallel-type configurations or backup-based systems, where failure only occurs after all the elements failed. Along with the stress-strength framework, this provides a general tool for assessing the capacity of a system to resist external challenges over time.

To model such scenarios effectively, it is important to consider families of distributions closed under the operation of taking the maximum. Specifically, we focus on the class of distributions $\mathcal{G}_{\max}$ consisting of distributions $G$ that satisfy the property:

$$G(\cdot;a_j) = \left(G(\cdot;1)\right)^{a_j}, \quad a_j > 0 \tag{2}$$

Suppose that $G(\cdot;a_j)$ is absolutely continuous with respect to the Lebesgue measure. In this case, there exists an associated density function which, as characterized by the relation in Equation (2), takes the form

$$g(\cdot;a_j) := a_j\left(G(\cdot;1)\right)^{a_j-1} g(\cdot;1) \tag{3}$$

This structure includes several common lifetime models, such as the power function, Type I extreme value, and discrete analogs like the Bernoulli in binary failure settings.

A key advantage of the family in Equations (2) and (3) lies in its closure under the maximum operation, as formalized below.

**Proposition 1 (Closure Under Maximum)** *Let $T_1, \dots, T_k$ be independent random variables such that $T_j \sim G(t;a_j)$ with $G \in \mathcal{G}_{max}$. Then the maximum random variable,*

$$T_{max} = T_{(k)} = max(T_1, \dots, T_k),$$

*has a distribution that belongs to the class $G_{max}$, with cumulative distribution function:*

$$G^{(k)}(t;a_1,\dots,a_k) = \left(G(t;1)\right)^{a_0}, \quad where \;\; a_0 = \sum_{j=1}^{k} a_j \tag{4}$$

*Furthermore, observe that the distribution of the r-th order statistic $T_{(r)}$ in this setting can be expressed, following results analogous to those by Balasubramanian et al. (1991), as:*

$$G^{(r)}(t) = \sum_{i=0}^{k-r} (-1)^{k-r-i} \binom{k-i-1}{k-r-i}\binom{k}{k-i} (G(t;1))^{\alpha_0}.$$

## 3. Reliability Expressions under Competing Risks

In this section, we derive closed-form expressions for the reliability of multicomponent systems subject to competing risks, under the stress-strength framework. We assume that all components $X_i$ and competing risks $Y_j$ follow a common lifetime distribution belonging to the class $\mathcal{G}_{\max}$, with possibly different shape parameters.

We begin with the simplest case involving a single component under competing risks, and then generalize to multiple-component systems under both series and partial-redundancy configurations.

### 3.1 Single Component under Competing Risks

We first consider a system consisting of a single component $X_1$, subject to $m$ competing risks $Y_1, Y_2, \dots, Y_m$, where each $Y_j$ has a shape parameter $a_{2j}$. The strength variable $X_1$ is assumed to follow a distribution from the same class, with shape parameter $a_1$.

**Theorem 1** *Let $X_1 \sim G(x; a_1)$ and $Y_j \sim G(x; a_{2j})$ for $j = 1,2, \ldots, m$, where all variables are independent with $G$ belonging to the distribution class given in Equation* (2). *If $Y_{(m)} = max\{Y_1, \ldots, Y_m\}$, then the stress-strength reliability is given by*

$$R_{X_1} = \mathbb{P}(X_1 > Y_{(m)}) = \frac{a_1}{a_1 + a_m} \tag{5}$$

*where,* $a_m = \sum_{j=1}^{m} a_{2j}$.

**Proof.** From the definition of multicomponent stress-strength reliability in Equation (1) and Proposition 1, we have

$$R_{X_1} = \mathbb{P}(X_1 > \max\{Y_1, \ldots, Y_m\})$$

$$= \int_0^\infty [1 - (G(x;1))^{a_1}] \cdot a_m [G(x;1)]^{a_m - 1} g(x;1)\, dx.$$

Let $u = 1 - G(x;1)$. The above integral becomes

$$a_m \int_0^1 [1 - u^{a_1}] u^{a_m - 1}\, du$$

$$= a_m \int_0^1 u^{a_m - 1} - u^{a_1 + a_m - 1}\, du$$

$$= a_m \left[\frac{1}{a_m} - \frac{1}{a_1 + a_m}\right]$$

$$= \frac{a_1}{a_1 + a_m}.$$

### 3.2 General $(s, k)$ System under Competing Risks

We now consider a more general configuration in which the system functions if at least $s$ out of the $k$ components are operational. This setting includes both series and parallel systems as special cases. Theorem 1 above can be considered as a special case of the Theorem below:

**Theorem 2** *Let $X_i \sim G(x; a_1)$ for $i = 1, \ldots, k$ and $Y_j \sim G(x; a_{2j})$ for $j = 1, \ldots, m$, with all variables mutually independent. Under the assumption that the system functions if at least $s$ components survive longer than the maximum competing risk $Y_{(m)}$, the stress-strength reliability is*

$$R_{(s,k)} = \frac{a_m}{a_1} \sum_{i=s}^{k} \binom{k}{i} B\left(i + 1, k - i + \frac{a_m}{a_1}\right) \tag{6}$$

*where,* $a_m = \sum_{j=1}^{m} a_{2j}$ and $B(a, b)$ denotes the Euler Beta function.

**Proof.** The reliability is defined as the probability that at least $s$ components out of $k$ survive beyond the minimum of the competing risks. Following the structure of the binomial survival model, we get

$$R_{(s,k)} = \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty [1 - (G(x;1))^{a_1}]^i [G(x;1)]^{a_1(k-i)} a_m [G(x;1)]^{a_m - 1} g(x;1)\, dx.$$

Using the substitution $u = G(x;1)$, the integral becomes a weighted Beta integral, yielding the expression in Equation (6).

These results provide compact expressions for system reliability using Beta functions, assuming independent and identically distributed strength variables and independent not necessarily identically distributed random variables which is typical in competing risks which usually experience different rates of failure.

The assumption of equal shape parameters within the $X$-group simplifies derivations and facilitates closed-form results.

### 3.3 Extensions to Heterogeneous Subgroup Structures

The results derived in the previous section assume a homogeneous structure where all component lifetimes share a common shape parameter within the $X$-group. However, practical systems often involve heterogeneous subsystems composed of components with different reliability characteristics. For example, different types of devices, materials, or redundancies may coexist within the same system, leading to natural partitions among components.

In this section, we extend the $(s, k)$-system reliability formulation to handle such heterogeneity. Specifically, we consider systems where the $k$ components are divided into $r$ distinct subgroups. Within each subgroup, components share a common shape parameter, but this parameter may differ across groups. The system is assumed to function if a minimum number of components survive in each group (for series systems), or if at least one group satisfies its minimum operational requirement (for parallel systems). These configurations generalize both the fully homogeneous $(s, k)$ model and includes the classic series/parallel systems as special cases.

#### 3.3.1 Series System with Heterogeneous Subgroups

Consider a system with $k$ components where $X_1, \dots, X_{k_1}$ are drawn from $G(\cdot; a_{11}), X_{k_1+1}, \dots, X_{k_1+k_2}$ from $G(\cdot; a_{12})$, and so on, with the last $k_r$ components following $G(\cdot; a_{1r})$. Each subgroup $i$ must have at least $s_i$ components operational for the overall system to function, with the constraints $s_1 + \dots + s_r = s$ and $k_1 + \dots + k_r = k, s < k$.

**Theorem 3** *Let the $k$ components be partitioned into $r$ subgroups with distributions from the class given in Equation (2) and each subgroup $j$ characterized by a shape parameter $a_{1j}$. Let $Y_1, \dots, Y_m$ denote the competing risks, each from the same class with respective shape parameters $a_{2i}$, $i = 1, \dots, m$. Then, the multicomponent reliability index of the series-type system that operates if at least $s_i$ components out of $k_i$ function for all $i = 1, \dots, r$ is*

$$R_{(s,k)} = \prod_{j=1}^{r} \sum_{i=s_j}^{k_j} \binom{k_j}{i} B\left(i + 1, k_j - i + \frac{a_m}{a_{1j}}\right) \tag{7}$$

*where, $a_m = \sum_{j=1}^{m} a_{2j}$.*

#### 3.3.2 Parallel System with Heterogeneous Subgroups

In contrast to the previous configuration, suppose the system operates if at least one subgroup satisfies its internal reliability requirement, that is, if at least one of the $r$ subgroups functions. This setup is a heterogeneous parallel system with internal group-wise reliability thresholds.

**Theorem 4** *Let the partition and component distributions be as defined above. Then the reliability of the system under this parallel operational rule is given by*

$$R_{(s,k)} = \sum_{j=1}^{r} \sum_{i=s_j}^{k_j} \binom{k_j}{i} B\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right) \tag{8}$$

*where*, $a_m = \sum_{j=1}^{m} a_{2j}$.

These models are particularly useful in real-world applications where systems are usually composed of multiple subsystems or modules with unique reliability profiles. In aerospace systems, for example, different components like avionics, propulsion, and structural support can experience different failure behavior due to differences in design, utilization, or exposure to the environment. Similarly, in large power grids or communication networks, reliability is typically dependent on the synchronized performance of heterogeneous submodules, each built up by different technologies and redundancies. By allowing group-wise reliability requirements and substitute shape parameters, the introduced expressions in Equations (7) and (8) are a general-purpose model for characterizing such heterogeneous systems with modular architecture and partial redundancy. This makes the analysis more realistic and aligned with the system engineers' and reliability analysts' practical problems.

### 3.4 Probabilistic Theorems

This section presents two key theorems that extend and generalize the reliability results established in the previous sections. The first theorem provides a closed-form expression in Equation (9) for the probability that a component lasts longer than the $r$-th ordered stress in the presence of multiple competing risks. The second theorem provides in Equation (10) the probability that a specific component (stress) is the cause of failure. These results are derived by applying the class $\mathcal{G}_{\max}$ and include the earlier theorems as particular cases.

**Theorem 5** *Let* $X \sim G(x; a_1)$ *and* $Y_{(r)}$ *denote the* $r^{th}$ *order statistic among* $k$ *independent* $Y_j \sim G(x; a_{2j})$ *stress variables, where* $G(\cdot;\cdot) \in \mathcal{G}_{max}$, $j = 2, \ldots, m$. *Then,*

$$\mathbb{P}(X > Y_{(r)}) = \sum_{i=0}^{m-r} (-1)^{k-r-i} \binom{k-i-1}{k-r-i} \binom{k}{k-i} \left[\frac{a_1}{a_1 + a_m}\right] \tag{9}$$

*where*, $a_m = \sum_{j=1}^{m} a_{2j}$.

**Proof.** We start from the definition of reliability for the $r$-th order statistic:

$$\mathbb{P}(X > Y_{(r)}) = \int (1 - G(x;1)^{a_1})\, dG_{Y_{(r)}}(x)$$

The pdf of the $r$-th order statistic is:

$$f_{Y_{(r)}}(x) = \sum_{i=0}^{m-r} (-1)^{m-r-i} \binom{m-i-1}{m-r-i} \binom{m}{m-i} a_m G(x;1)^{a_m - 1} g(x;1).$$

Thus,

$$\mathbb{P}\left(X > Y_{(r)}\right) =$$

$$= \int_0^\infty (1 - G(x;1)^{a_1}) \sum_{i=0}^{m-r} (-1)^{m-r-i} \binom{m-i-1}{m-r-i} \binom{m}{m-i} a_m G(x;1)^{a_m-1} g(x;1)\, dx$$

$$= \sum_{i=0}^{m-r} (-1)^{m-r-i} \binom{m-i-1}{m-r-i} \binom{m}{m-i} a_m \int_0^1 (1-u)^{a_1} u^{a_m - 1}\, du$$

$$= \sum_{i=0}^{m-r} (-1)^{m-r-i} \binom{m-i-1}{m-r-i} \binom{m}{m-i} a_m \left[\frac{1}{a_m} - \frac{1}{a_1 + a_m}\right]$$

$$= \sum_{i=0}^{m-r} (-1)^{m-r-i} \binom{m-i-1}{m-r-i} \binom{m}{m-i} \left[\frac{a_1}{a_1+a_m}\right].$$

Note that when $r = m$ the above Theorem reduces to Equation (5) in Theorem 1.

In reliability theory, an important question is the probability that system failure can be attributed to a particular component when there are multiple independent risks. The next result establishes a closed-form expression for the probability that failure is caused by the $i$-th component.

**Theorem 6** *Let $X \sim G(x; a_1)$ and $Y_j \sim G(x; a_{2j})$ for $j = 1, \dots, m$, all independent, where $G(\cdot;\cdot) \in \mathcal{G}_{max}$. Then the probability of failure being caused by component/stress $i$ is*:

$$\mathbb{P}(\text{failure due to cause } i) = \mathbb{P}(X < Y_i \ \text{ and } \ Y_j < X \ \ \forall j \neq i) = \frac{a_1 a_{2i}}{(a_m - a_{2i} + a_1)(a_m + a_1)} \quad (10)$$

*where*, $a_m = \sum_{j=1}^{m} a_{2j}$ and $a_{m-i} = \sum_{\substack{j=1 \\ j\neq i}}^{m} a_{2j}$.

**Proof.** The event of failure due to component $i$ is:

$$\mathbb{P}(X < Y_i \text{ and } Y_j < X \ \ \forall j \neq i)$$

$$= \int_0^\infty (1 - G(x;1)^{a_{2i}}) G(x;1)^{a_{m-i}} a_1 G(x;1)^{a_1 - 1} g(x;1)\, dx$$

$$= a_1 \int_0^\infty (1 - G(x;1)^{a_{2i}}) G(x;1)^{a_{m-i} + a_1 - 1} g(x;1)\, dx$$

$$= a_1 \int_0^1 (1 - u^{a_{2i}}) u^{a_m - a_{2i} + a_1 - 1} du$$

$$= a_1 \left[\frac{1}{a_m - a_{2i} + a_1} - \frac{1}{a_m + a_1}\right]$$

$$= \frac{a_1 a_{2i}}{(a_m - a_{2i} + a_1)(a_m + a_1)}.$$

According to the proceeding result, the probability of no failure can be obtained which is in accordance with Theorem 1. Indeed,

**Corollary 1** *The probability of no failure is*:

$$\mathbb{P}(\text{No failure}) = \frac{a_1}{a_m + a_1}$$

**Proof.** The probability of no failure is given by

$$\mathbb{P}(\text{No failure}) = \int_0^\infty G(x;1)^{a_m} a_1 G(x;1)^{a_1 - 1} g(x;1)\, dx.$$

Letting $u = G(x;1)$ the result is immediate.

## 3.5 Subclasses of the Lehmann Alternative Family

The family $G(\cdot; a_j)$ defined in Equation (2) can be viewed as a subclass of the well-known Lehmann alternative family (Lehmann, 1953), where each cumulative distribution function $F_j$ of the family, is given by

$$F_j = f_j(F)$$

with pdf given by
$f_j(x) = x^{a_j}$, and $F = F(x; 1)$.

In general, the functions $f_j$ are continuous, non-decreasing, and defined on the interval $[0,1]$, satisfying the boundary conditions $f_j(0) = 0$ and $f_j(1) = 1$, for $j = 1,2, \dots, m$. Thus, the class given in Equation (2) forms a subclass within the Lehmann alternative family which according to Equation (4), is closed under maxima, a feature not generally shared by the full Lehmann family. This additional property makes the subclass particularly useful for modeling reliability systems and competing risks.

Complementary to the above subclass which is closed under maxima, one could consider another subclass within the Lehmann alternative family, defined by
$\mathcal{G}_{\min} = \{f_j(x) = 1 - (1 - x)^{a_j}$, with $F = F(x; 1)\}$.

The extra feature of the above $\mathcal{G}_{\min}$ subclass is that it is closed under minima, making it suitable for systems where failure or degradation is driven by the weakest component/stress. Both $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ subclasses are widely applicable in statistical inference for reliability analysis and modeling of competing risks, due to their respective closure properties under extrema.

The second subclass is given below:
Let

$$\mathcal{G}_{\min} = \left\{G \mid G(\cdot; a_j) = 1 - (1 - G(\cdot; 1))^{a_j}, \quad a_j > 0\right\} \tag{11}$$

with corresponding pdf given by

$$g(\cdot; a_j) := a_j (1 - G(\cdot; 1))^{a_j - 1} g(\cdot; 1) \tag{12}$$

The class $\mathcal{G}_{\min}$ includes many well-known distributions that follow the structure given in Equations (11) and (12). This common form helps bring together different classical distributions into one general family. Some examples are the Geometric, Exponential, Weibull, Pareto, truncated Erlang, truncated Exponential, and Kumaraswamy distributions, all of which can be written in the form shown in $\mathcal{G}_{\min}$.

The theorems given in the previous section can also be formulated under the setting of the new subclass $\mathcal{G}_{\min}$, as defined in Equations (11) and (12). This subclass, which is closed under minima, leads to analogous but inverted reliability expressions due to its structural properties.

**Theorem 7** *Let $X \sim G(x; a_1)$, and let $Y_{(r)}$ denote the $r$-th order statistic among independent variables $Y_j \sim G(x; a_j)$, where $G(\cdot;\cdot) \in \mathcal{G}_{min}$. Then,*

$$\mathbb{P}(X > Y_{(r)}) = \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{m-i-1}{m-r} \binom{m}{m-i} \left[\frac{a_m}{a_1 + a_m}\right]$$

*where*, $a_m = \sum_{j=1}^{m} a_j$.

**Proof.** We start with the definition of the reliability for the $r$-th order statistic:
$\mathbb{P}(X > Y_{(r)}) = \int (1 - G(x; 1))^{a_1} \, dG_{Y_{(r)}}(x)$.

The pdf of the $r$-th order statistic is given by:

$$f_{Y_{(r)}}(x) = \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{m-i-1}{m-r} \binom{m}{m-i} a_m (1 - G(x; 1))^{a_m - 1} g(x; 1).$$

Then,

$$\mathbb{P}(X > Y_{(r)}) = \int_0^\infty (1 - G(x;1))^{a_1} \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{m-i-1}{m-r}\binom{m}{m-i} \times a_m(1 - G(x;1))^{a_m-1}\, g(x;1)\, dx$$

$$= \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{m-i-1}{m-r}\binom{m}{m-i} a_m \int_0^1 u^{a_1} u^{a_m-1}\, du$$

$$= \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{m-i-1}{m-r}\binom{m}{m-i} a_m \left[\frac{1}{a_1+a_m}\right]$$

$$= \sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{m-i-1}{m-r}\binom{m}{m-i} \left[\frac{a_m}{a_1+a_m}\right].$$

For $r = 1$, the stress-strength reliability associated with the minimum order statistic reduces to the case of a single component, and is given by

$$\mathbb{P}(X > Y_{(1)}) = \frac{a_m}{a_1+a_m}.$$

For the subclass $\mathcal{G}_{\min}$ we are able to derive expressions for the probability that failure occurs due to some given component. This probability, often referred to as the probability of failure due to cause $i$, characterizes the situation where the lifetime of component $i$ exceeds that of the system, while all other components survive beyond the system failure time. The following theorem provides this probability in closed form under the assumption that all variables follow distributions in $\mathcal{G}_{\min}$.

**Theorem 8** *Let $X \sim G(x; a_1)$ and $Y_j \sim G(x; a_{2j})$ for $j = 1, \dots, m$, all independent, where $G(\cdot;\cdot) \in \mathcal{G}_{min}$. Then the probability that the failure is caused by component i is given by*:

$$\mathbb{P}(\text{failure due to cause } i) = \mathbb{P}(X < Y_i \text{and} Y_j > X \ \forall j \neq i) = \frac{a_1(a_m - a_{2i})}{(a_m+a_1)(a_{2i}+a_1)},$$

*where*, $a_m = \sum_{j=1}^m a_{2j}$ and $a_{m-i} = \sum_{\substack{j=1\\ j\neq i}}^m a_{2j}$.

**Proof.** $\mathbb{P}(X < Y_i \text{ and } Y_j > X \ \forall j \neq i)$

$$= \int_0^\infty (1 - G(x;1))^{a_{2i}}[1 - (1 - G(x;1))^{a_{m-i}}] a_1 (1 - G(x;1))^{a_1-1} g(x;1)\, dx$$

$$= a_1 \int_0^\infty (1 - G(x;1))^{a_{2i}+a_1-1}[1 - (1 - G(x;1))^{a_m - a_{2i}}] g(x;1)\, dx$$

$$= a_1 \int_0^1 u^{a_{2i}+a_1-1}(1 - u^{a_m - a_{2i}})\, du \qquad (\text{with } u = 1 - G(x;1))$$

$$= a_1 \left[\frac{1}{a_{2i}+a_1} - \frac{1}{a_m+a_1}\right] = \frac{a_1(a_m - a_{2i})}{(a_m+a_1)(a_{2i}+a_1)}.$$

## 4. Point and Interval Estimation in Component and System Reliability

For estimating the unknown parameter $a$ of a distribution $G(\cdot; a)$ belonging to the $\mathcal{G}_{\max}$ class, one can consider a random sample from $G(\cdot; a)$ and apply the classical maximum likelihood estimation (MLE) method. The Theorem below provides the expression for the MLE of $a$ together with the associated asymptotic theory.

**Theorem 9** *Let* $Z_1, \dots, Z_n \overset{i.i.d.}{\sim} G(\cdot; a)$, *with* $G(\cdot;\cdot) \in \mathcal{G}_{max}$. *Then*:

1) *The maximum likelihood estimators (MLEs) are*
$\hat{a} = -\left(\frac{1}{n}\sum_{i=1}^{n} \ln G(Z_i; 1)\right)^{-1}$.

2) *Provided that regularity assumptions (smoothness, identifiability, finite information) hold, the estimators satisfy*
$\sqrt{n}\,(\hat{a} - a) \overset{d}{\to} N(0, \mathcal{I}(a)^{-1})$

*where, the Fisher information for a single observation is* $\mathcal{I}(a) = 1/a^2$. *Consequently*,
$\mathrm{Var}(\hat{a}) \approx \frac{a^2}{n}$.

**Proof.** The likelihood of the above sample is

$$\mathcal{L}(a) = \prod_{i=1}^{n} [a\, G(Z_i; 1)^{a-1} g(X_i; 1)].$$

Taking the logarithm, differentiating with respect to $a$ and setting the derivative equal to 0, leads to the desired result.

As for the asymptotic distribution let us consider a single random variable $Z \sim G(\cdot; a)$, where the log-likelihood is
$\ln\mathcal{L}_X(a) = \ln a + (a-1)\ln G(X; 1) + \ln g(X; 1)$.

First and second derivatives are
$\frac{\partial \ln\mathcal{L}_X}{\partial a} = \frac{1}{a} + \ln G(X; 1), \quad \frac{\partial^2 \ln\mathcal{L}_X}{\partial a^2} = -\frac{1}{a^2}$.

Taking expectations, the Fisher information per observation is
$\mathcal{I}(a) = \mathbb{E}\left[-\frac{\partial^2 \ln\mathcal{L}_X}{\partial a^2}\right] = 1/a^2$.

By standard MLE theory, we have
$\mathrm{Var}(\hat{a}) = \frac{a^2}{n}$,

and the asymptotic normality results follow directly.

Consequently, confidence intervals for $R$ can be constructed via the delta method or through parametric bootstrap techniques, offering practical tools for system reliability assessment in real-world applications.

## 4.1 Delta Method Confidence Interval for a Single Component System

**Theorem 10** *Let* $a_1$ *be the shape parameter corresponding to a strength component, and let* $a_m = \sum_{j=1}^{m} a_{2j}$ *denote the total shape parameter corresponding to the maximum of* $m$ *independent stress components. The reliability index (see Theorem 1) is defined as* $R = \frac{a_1}{a_1 + a_m}$.

*Then an* $(1-\alpha)\times 100\%$ *confidence interval for* $R$ *is*:

$$\hat{R} \pm z_{\alpha/2} \cdot \sqrt{\frac{1}{(\hat{a}_1+\hat{a}_m)^4}\left[\hat{a}_m^2 \cdot \frac{\hat{a}_1^2}{n_1} + \hat{a}_1^2 \cdot \sum_{j=1}^{m} \frac{\hat{a}_{2j}^2}{n_{2j}}\right]},$$

*where,* $\hat{a}_m = \sum_{j=1}^{m} \hat{a}_{2j}$, *and* $\hat{R} = \hat{a}_1/(\hat{a}_1+\hat{a}_m)$.

**Proof.**

$$\mathrm{Var}(\hat{a}_1) = \frac{a_1^2}{n_1}, \quad \mathrm{Var}(\hat{a}_{2j}) = \frac{a_{2j}^2}{n_{2j}}.$$

Applying the delta method in the Reliability parameter we have that

$$\mathrm{Var}(\hat{R}) = \left(\frac{\partial R}{\partial a_1}\right)^2 \mathrm{Var}(\hat{a}_1) + \sum_{j=1}^{m} \left(\frac{\partial R}{\partial a_{2j}}\right)^2 \mathrm{Var}(\hat{a}_{2j})$$

$$= \left(\frac{a_m}{(a_1+a_m)^2}\right)^2 \cdot \frac{a_1^2}{n_1} + \sum_{j=1}^{m} \left(\frac{-a_1}{(a_1+a_m)^2}\right)^2 \cdot \frac{a_{2j}^2}{n_{2j}}.$$

Equivalently,

$$\mathrm{Var}(\hat{R}) = \frac{1}{(a_1+a_m)^4}\left[a_m^2 \cdot \frac{a_1^2}{n_1} + a_1^2 \cdot \sum_{j=1}^{m} \frac{a_{2j}^2}{n_{2j}}\right].$$

By replacing the parameters with their estimates we obtain the desired results.

## 4.2 Delta Method Confidence Interval for the $(s, k)$ System

**Theorem 11** *Let* $a_1$ *be the common shape parameter associated with* $k$ *i.i.d. strength and let* $a_m = \sum_{j=1}^{m} a_{2j}$ *be the total shape parameter for the maximum of* $m$ *independent stress components (competing risks). The reliability index for a general* $(s, k)$ *system is:*

$$R = \frac{a_m}{a_1} \sum_{i=s}^{k} \binom{k}{i} B\left(i+1, k-i+\frac{a_m}{a_1}\right),$$

*where,* $B(\cdot,\cdot)$ *is the beta function.*
*Then an* $(1-\alpha)\times 100\%$ *confidence interval for* $R$ *is*:

$$\hat{R} \pm z_{\alpha/2} \cdot \sqrt{\left(\frac{\partial R}{\partial a_1}\right)^2 \frac{\hat{a}_1^2}{n_1} + \sum_{j=1}^{m} \left(\frac{\partial R}{\partial a_{2j}}\right)^2 \frac{\hat{a}_{2j}^2}{n_{2j}}},$$

*where,* $\hat{a}_m = \sum_{j=1}^{m} \hat{a}_{2j}$ *and* $\hat{R}$ *is obtained by substituting estimates into the expression of* $R$.

**Proof.** Let

$$g(a_1, a_m) = \frac{a_m}{a_1} \sum_{i=s}^{k} \binom{k}{i} B\left(i+1, k-i+\frac{a_m}{a_1}\right).$$

Define $u = a_m/a_1$ and write:

$$R = u \sum_{i=s}^{k} \binom{k}{i} B(i+1, k-i+u).$$

Using the chain rule, compute the derivatives:

$\frac{\partial R}{\partial a_1} = \frac{\partial R}{\partial u} \cdot \frac{\partial u}{\partial a_1} = \left[\sum_{i=s}^{k} \binom{k}{i} \left(B(i+1, k-i+u) + u \cdot \frac{\partial B(i+1,k-i+u)}{\partial u}\right)\right] \cdot \left(-\frac{a_m}{a_1^2}\right),$

$\frac{\partial R}{\partial a_{2j}} = \frac{\partial R}{\partial u} \cdot \frac{\partial u}{\partial a_{2j}} = \left[\sum_{i=s}^{k} \binom{k}{i} \left(B(i+1, k-i+u) + u \cdot \frac{\partial B(i+1,k-i+u)}{\partial u}\right)\right] \cdot \left(\frac{1}{a_1}\right),$

where,

$\frac{\partial B(i+1,k-i+u)}{\partial u} = B(i+1, k-i+u)[\psi(k-i+u) - \psi(i+1+k-i+u)],$

and $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ is the digamma function.

Substitute these into the delta method formula, we obtain:

$\text{Var}(\hat{R}) = \left(\frac{\partial R}{\partial a_1}\right)^2 \cdot \frac{a_1^2}{n_1} + \sum_{j=1}^{m} \left(\frac{\partial R}{\partial a_{2j}}\right)^2 \cdot \frac{a_{2j}^2}{n_{2j}}$ .

Using the estimates $\hat{a}_1$, $\hat{a}_{2j}$ and plug into the interval, we finally have:

$\hat{R} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(\hat{R})}.$

### 4.3 Confidence Intervals for Systems with Heterogeneous Subgroups

The two theorems in this section provide the expression for the confidence intervals for a system in series and a system in parallel with heterogeneous subgroups.

**Theorem 12** *Let the $k$ components be partitioned into $r$ subgroups, each with shape parameter $a_{1j}$ $(j = 1, \dots, r)$, and let $a_m = \sum_{j=1}^{m} a_{2j}$ be the total stress shape parameter. The system operates if at least $s_j$ of $k_j$ components function within each subgroup $j$, for all $j = 1, \dots, r$. The reliability index is:*

$R_{(s,k)} = \prod_{j=1}^{r} \sum_{i=s_j}^{k_j} {k_j}_i \; B\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right).$

*Then an $(1-\alpha) \times 100\%$ confidence interval for $R_{(s,k)}$ is*:

$\hat{R}_{(s,k)} \pm z_{\alpha/2} \cdot \sqrt{\left(\frac{\partial R}{\partial a_m}\right)^2 \cdot \frac{\hat{a}_m^2}{n_m} + \sum_{j=1}^{r} \left(\frac{\partial R}{\partial a_{1j}}\right)^2 \cdot \frac{\hat{a}_{1j}^2}{n_{1j}}}.$

**Proof.** Define the reliability index as:

$R = \prod_{j=1}^{r} R_j, \quad \text{where} R_j = \sum_{i=s_j}^{k_j} \binom{k_j}{i} B\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right).$

The partial derivatives with respect to $a_m$ and $a_{ij}$ respectively are:

$\frac{\partial R}{\partial a_m} = \sum_{j=1}^{r} \left(\frac{\partial R_j}{\partial a_m} \cdot \prod_{l \neq j} R_l\right), \quad \text{where} \quad \frac{\partial R_j}{\partial a_m} = \sum_{i=s_j}^{k_j} \binom{k_j}{i} \cdot \frac{1}{a_{1j}} \cdot B'\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right),$

$\frac{\partial R}{\partial a_{1j}} = \left(\frac{\partial R_j}{\partial a_{1j}} \cdot \prod_{l \neq j} R_l\right), \quad \text{where} \quad \frac{\partial R_j}{\partial a_{1j}} = \sum_{i=s_j}^{k_j} \binom{k_j}{i} \cdot \left(-\frac{a_m}{a_{1j}^2}\right) \cdot B'\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right).$

The delta method gives:

$\text{Var}(\hat{R}) = \left(\frac{\partial R}{\partial a_m}\right)^2 \cdot \frac{a_m^2}{n_m} + \sum_{j=1}^{r} \left(\frac{\partial R}{\partial a_{1j}}\right)^2 \cdot \frac{a_{1j}^2}{n_{1j}}.$

Substitute $\hat{a}_m, \hat{a}_1$ and obtain $\widehat{R}$ its interval and its variance.

**Theorem 13** *Consider the same partition of* $k$ *components into* $r$ *subgroups, each with shape parameter* $a_{1j}$, *and let* $a_m = \sum_{j=1}^{m} a_{2j}$. *The system functions if at least one subgroup satisfies its own internal reliability threshold* $s_j$ *out of* $k_j$. *The reliability index is:*

$$R_{(s,k)} = \sum_{j=1}^{r} \sum_{i=s_j}^{k_j} \binom{k_j}{i} B\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right).$$

*The* $(1-\alpha) \times 100\%$ *confidence interval for* $R_{(s,k)}$ *is given by*

$$\hat{R}_{(s,k)} \pm z_{\alpha/2} \cdot \sqrt{\left(\frac{\partial R}{\partial a_m}\right)^2 \cdot \frac{\hat{a}_m^2}{n_m} + \sum_{j=1}^{r} \left(\frac{\partial R}{\partial a_{1j}}\right)^2 \cdot \frac{\hat{a}_{1j}^2}{n_{1j}}}.$$

**Proof.** Define:

$$R = \sum_{j=1}^{r} R_j, \quad \text{where} R_j = \sum_{i=s_j}^{k_j} \binom{k_j}{i} B\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right).$$

Then:

$$\frac{\partial R}{\partial a_m} = \sum_{j=1}^{r} \sum_{i=s_j}^{k_j} \binom{k_j}{i} \cdot \frac{1}{a_{1j}} \cdot B'\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right),$$

$$\frac{\partial R}{\partial a_{1j}} = \sum_{i=s_j}^{k_j} \binom{k_j}{i} \cdot \left(-\frac{a_m}{a_{1j}^2}\right) \cdot B'\left(i+1, k_j - i + \frac{a_m}{a_{1j}}\right).$$

The delta method gives:

$$\text{Var}(\hat{R}) = \left(\frac{\partial R}{\partial a_m}\right)^2 \cdot \frac{a_m^2}{n_m} + \sum_{j=1}^{r} \left(\frac{\partial R}{\partial a_{1j}}\right)^2 \cdot \frac{a_{1j}^2}{n_{1j}}.$$

Substitute $\hat{a}_m$, $\hat{a}_{1j}$ to compute $\hat{R}$ and confidence intervals.

## 5. Applications

### 5.1 Reliability Behavior under Competing Risks

To demonstrate the applicability of the proposed methodology, two representative distribution families are considered: the Power function distribution from the class $\mathcal{G}_{\max}$, and the Lomax distribution from the class $\mathcal{G}_{\min}$. This helps to visualize how the reliability parameter $R = P(X > Y_{(r)})$ behaves under different scenarios involving competing risks.

We first consider the case where a strength variable $X$ and five stress variables (competing risks) $Y_j$, $j = 1,2,\dots,5$ follow Power function distributions, which belong to class $\mathcal{G}_{\max}$ with shape parameters $a_1 = 0.1,0.2,\dots,2$ and $a_{2i} \in \{0.2,0.6,0.9,1.5,4\}$ , $i = 1,2,\dots,5$ . The results can be established for multicomponents but here for convenience are presented for the single component case. Continuous lines in **Figure 1** represent the case where the true values of $a_1$ and $a_2$ are used, while dashed lines correspond to the case where these parameters are estimated, according to Theorem 9.

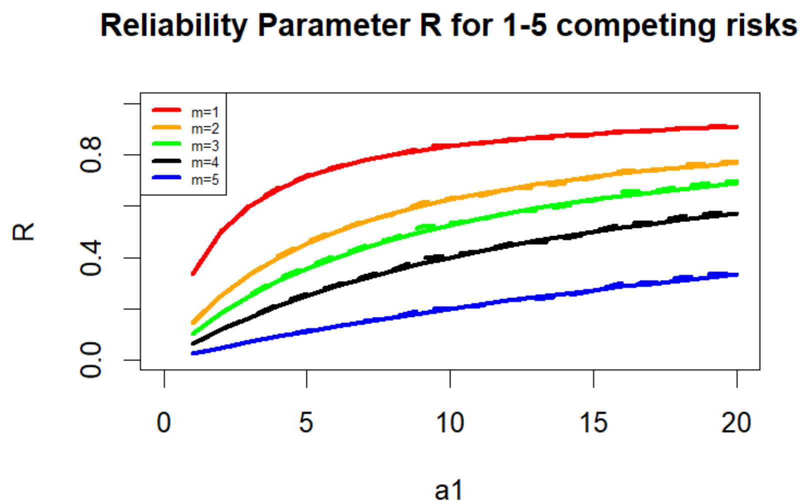


**Figure 1.** Reliability parameter $R$ for 1–5 competing risks using power function distributions ($\mathcal{G}_{\max}$), for $r = m$. Solid lines: true parameters; dashed lines: estimated parameters.

From **Figure 1**, we observe that the reliability $R$ increases with the strength parameter $a_1$. As expected, higher strength improves reliability, and more competing risks (higher $m$) result in lower reliability.

In contrast to the $\mathcal{G}_{\max}$ behavior, we now consider the Lomax distribution as a representative of the class $\mathcal{G}_{\min}$. More precisely, the strength variable $X$ follows a Lomax distribution with scale parameter 2 and varying shape parameters $a_1 = 0.1, 0.2, \ldots, 2$, while the five stress variables $Y$ also follow a Lomax distribution with the same scale parameter and shape parameters equal to 0.2,0.6,0.9,1.5 & 4. The effect of shape parameter variation on reliability is plotted for different numbers of competing risks $m = 1, 2 \ldots, 5$.

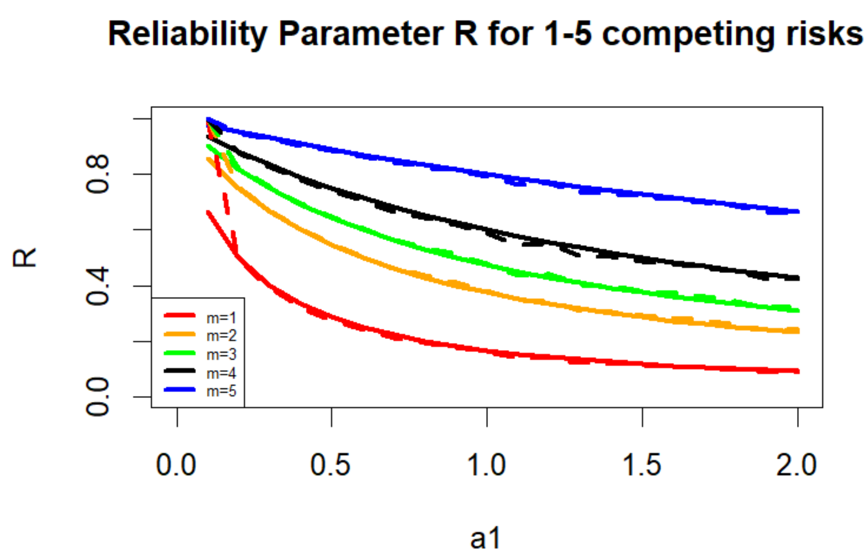


**Figure 2.** Reliability parameter $R$ using Lomax ($G_{\min}$) distributions. Strength $X \sim \text{Lomax}(2, a_1)$, Stress $Y \sim \text{Lomax}(2, a_{2j}), j = 1,2 \ldots, 5$ for various shape parameters and number of risks, $m$.

In **Figure 2**, we observe that reliability $R$ decreases as $a_1$ increases, which is expected under the $\mathcal{G}_{\min}$ family. Here, a higher shape parameter corresponds to lighter tails, implying lower probability of extreme values — and thus, a lower chance that $X$ exceeds $Y_{(r)}$. This is the reverse of the trend seen in $\mathcal{G}_{\max}$.

The reliability parameter based on the estimated parameters is very close to the one using the true values. This shows that the estimation method works well and gives reliable results. The minimal difference between the solid and dashed lines suggests that even if the parameters are estimated from data, the reliability remains accurate. This makes the method useful, especially when only sample data are available.

### 5.2 Probability of Failure Due to a Specific Cause

We also investigate the probability that the system fails due to a specific cause $i$ when five competing risks are invloved ($i = 1, 2, \dots, 5$), under both the Power function distribution from the $\mathcal{G}_{\max}$ class and the Lomax distribution from the $\mathcal{G}_{\min}$ class. The results are shown in **Figures 3** and **4** with the failure being due to risk $i = 1, 2, \dots, 5$.

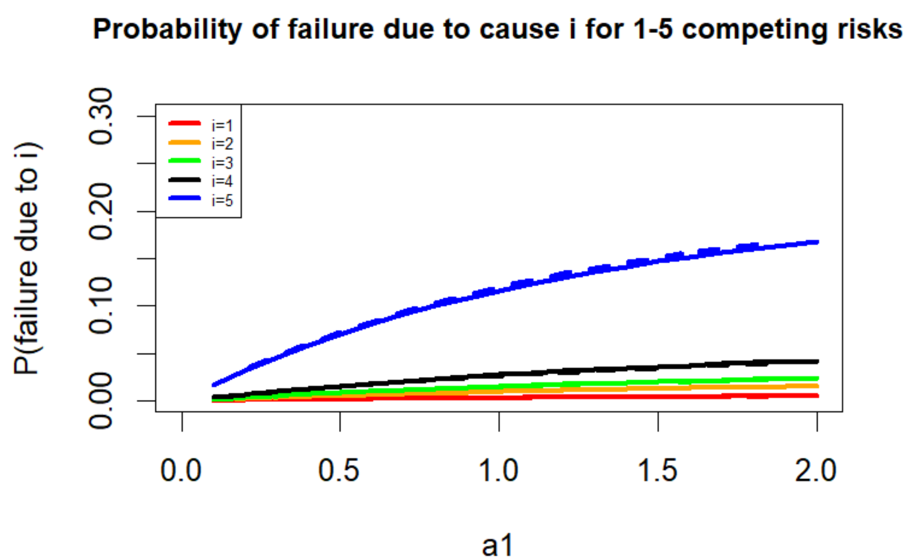


**Figure 3.** Probability of failure due to cause $i$ for $i = 1$ to 5 under the power function distribution ($\mathcal{G}_{\max}$).

For the Power function distribution (**Figure 3**), the failure probability for each cause increases as the strength shape parameter $a_1$ becomes larger. It reflects that smaller strength increases the likelihood of failure due to one of the risks. The last cause ($i = m$) tends to have the highest chance of causing failure, and this probability becomes smaller for smaller $i$.

On the other hand, in the case of the Lomax distribution (**Figure 4**), as the strength parameter $a_1$ increases, the failure probability due to each cause decreases. In this setting, the first risk ($i = 1$) has the highest probability of causing failure, which is different from what observed in the $\mathcal{G}_{\max}$ case.

The contrasting behavior between $\mathcal{G}_{\max}$ and $\mathcal{G}_{\min}$ highlights the importance of selecting appropriate distribution family according to the physical or operational nature of the system under investigation.

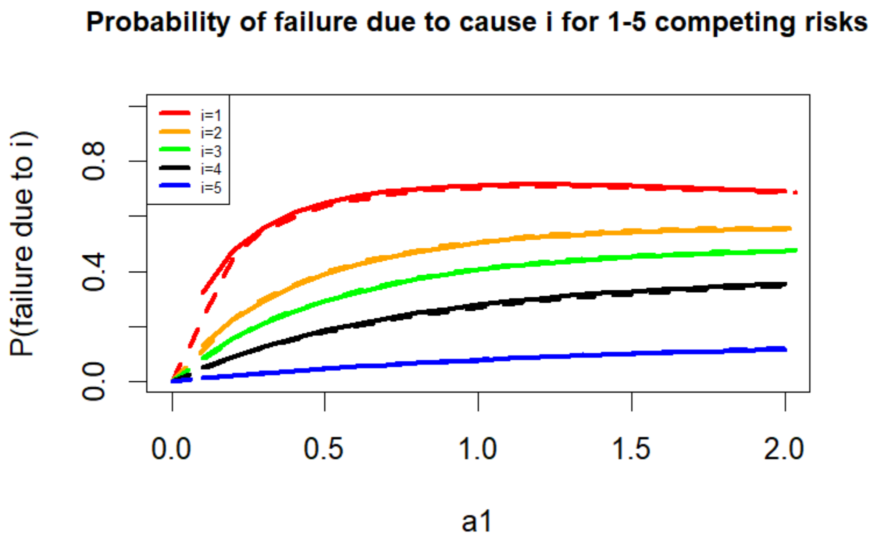


**Figure 4.** Probability of failure due to cause $i$ for $i = 1$ to 5 under the Lomax distribution ($\mathcal{G}_{\min}$).

## 6. Discussion and Conclusions

In this work, we focus on the evaluation and statistical inference of the multicomponent reliability parameter of an $s$-out-of-$k$:$G$ system under a competing risks setting with the risks playing the role of multiple stresses that put under pressure the operation of the system.

The distributions involved are assumed to belong to two special subclasses of the well known Lehmann Alternative Family of distributions. The subclasses are closed under extrema, and thus provide a great flexibility since the distribution members involved are frequently encountered in reliability theory and survival analysis.

The main contribution of this work lies on the fact that it brings within the above two distribution subclasses, several distributions popular in reliability theory and at the same time investigates the reliability parameter under the competing risks framework.

The results clearly show that the reliability parameter under all scenarios examined, depends on the parameters $a_1$ and $a_{21}, \dots, a_{2m}$ associated with the distributions of the strength and stress variables (the competing risks). In fact, the main results in Theorems 1 - 4 reveal that the reliability parameter is associated with the distributional parameters for the single and multiple component cases as well as for series and parallel systems with heterogeneous subgroups. The main results are complemented with generalizations that include the probability of failure due to any one of the causes/risks involved. In addition, this work provides statistical inference including point and confidence interval estimation and the relevant asymptotic theory.

The applications considered in this work confirm the theoretical results established and show a variety of practical implications. For instance, the reliability parameter based on the estimated parameters was found to be very close to the one using the true values. This shows that the estimation method works well and gives reliable results. Furthermore, the results clearly show that even if the parameters are estimated from data, the reliability remains accurate. This conclusion makes the method useful, especially when only sample data are available.

All the above results provide the background for optimizing maintenance decisions in various types of technical systems for general subclasses of the Lehmann Alternative Family of distributions and provide a valuable generalization in covering time to event analysis under the competing risks setting. At the same time new research directions are revealed which should be the focus of future work. Such problems include the extension of classical models to cover non-linear and non-monotonic strength-stress relations and the investigation of special families of distributions including mixed distributions. Another open problem that should be investigated in the future is the case of degradation-based reliability under competing risks where degradation models could be considered which over time, experience reduction in terms of the strength and accumulation in terms of the stress.

**Conflicts of Interest**

The authors confirm that there is no conflict of interest to declare for this publication.

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