International Journal of Mathematical, Engineering and Management Sciences
Vol. 11, No. 4, 1622-1638, 2026
https://doi.org/10.33889/IJMEMS.2026.11.4.066

Lie Symmetry Analysis and Invariant Solutions of the Diffusion Equation on the Sphere with Linear Reaction

Khalid Ali Alanezy
Department of Mathematics,
King Fahd University of Petroleum & Minerals, Dhahran, 31261, Saudi Arabia.
Corresponding author: alanezy@kfupm.edu.sa

(Received on February 7, 2026; Revised on April 5, 2026 & June 2, 2026; Accepted on July 10, 2026)

Abstract
This paper investigates the linear reaction–diffusion equation on the unit sphere by means of Lie point symmetry analysis. The determining equations show that the finite–dimensional Lie point symmetry algebra is five-dimensional and is generated by the three rotational Killing fields, time translation, and scaling, together with the infinite–dimensional superposition ideal. An optimal system of one–dimensional subalgebras is constructed and used, together with a two–stage reduction procedure based on commuting generators, to derive inequivalent similarity reductions and explicit invariant solution families. The stationary reductions lead to Legendre-type ordinary differential equations, and the global smoothness requirement on the sphere selects the polynomial branch corresponding to spherical harmonics and the associated eigenvalue quantization. In addition, mixed space–time reductions produce explicit non–stationary invariant patterns, including a locally defined family generated by a combined rotation–time–scaling symmetry. The results provide a symmetry-based framework that complements the classical spectral description of diffusion on the sphere.

Keywords- Lie symmetries, Spherical heat equation, Reaction–diffusion, Invariant solutions, Spherical harmonics.

1. Introduction and Motivation

Diffusion and reaction on curved geometric substrates arise naturally in physics, geoscience, chemistry, and biology. Examples include heat transport on planetary atmospheres and crusts, charge migration on spherical nanoparticles, and chemical kinetics on lipid membranes. In all such settings, the ambient geometry is not incidental: curvature modifies the diffusion operator, breaks Euclidean translational invariance, and imposes global topological constraints on admissible fields and boundary data. This geometric imprint is most transparent on the two-sphere $\mathbb{S}^2$, the canonical compact surface of constant positive curvature, where the Laplace–Beltrami operator contains curvature-dependent coefficients in standard spherical coordinates and its eigenspaces are spanned by spherical harmonics, with spectral and heat–kernel properties documented in Chavel (1984) and Grigor'yan (2009). In contrast to the flat case, symmetry and spectral structures are shaped by the isometry group SO(3) and by the compactness of the domain.

The Lie group approach to differential equations, initiated by S. Lie and developed through the modern calculus of prolongations, provides a systematic mechanism to uncover continuous symmetries of partial differential equations and to use them for reduction, classification, and exact solution construction. Standard references include the monographs by Bluman & Cole (1974), Bluman & Kumei (1989), Hydon (2000), Ibragimov (1993), Ibragimov (1999), Ovsiannikov (1982), and Olver (1986). In Euclidean space, the classical heat equation enjoys a rich Lie algebra encompassing space-time translations, rotations, Galilean boosts, and scaling, together with infinite-dimensional superposition symmetries stemming from linearity; see (Clarkson & Mansfield, 1994) for representative analyses and refinements. On a curved manifold, the landscape changes: point symmetries are constrained by the metric, and for heat-type equations they are generated by the homothetic algebra of the underlying geometry (Stephani, 1990). For $\mathbb{S}^2$ this means that only the rotational Killing fields and time translations survive as point symmetries; curvature eliminates Euclidean boosts and thereby narrows the algebraic structure available for reductions (Sepanski, 2007).

To better position the present contribution within the existing literature, **Table 1** summarizes the most relevant related works in terms of the governing equation, geometric setting, main analytical theme, and the specific gap addressed by the present study.

Table 1. Comparative literature review and motivation for the present work.

Author(s)	Equation / problem setting	Geometry	Main methodology / theme	Scope and relevance
Clarkson & Mansfield (1994)	$u_t = u_{xx} + f(u)$	Euclidean space	Classical and nonclassical symmetries	No curvature or spherical geometry
Koval & Popovych (2023)	Linear heat equation	Flat space	Generalized symmetries	No curved geometry analysis
Dicke & Veselić (2024)	Spherical heat equation	Sphere	Spectral/control analysis	No Lie symmetry reduction, no invariant solutions on $\mathbb{S}^2$
Raza et al. (2022)	Nonlinear partial differential equations	Flat space	Optimal systems and adjoint method	Not applied to curved manifolds
Present work	$u_t = \Delta_{\mathbb{S}^2} u + \alpha u, \alpha \in \mathbb{R} \setminus \{0\}$	Sphere $\mathbb{S}^2$	Full Lie symmetry + optimal system + reductions + invariant exact solutions	Provides a Lie-symmetry reduction study for the linear reaction--diffusion equation on the sphere

As shown in **Table 1**, the existing literature has developed three closely related directions: symmetry analysis of heat equations in flat spaces, general geometric classifications of heat-type equations on Riemannian manifolds, and spectral analysis of the spherical heat equation. However, a unified treatment combining a complete Lie point symmetry derivation, an explicit optimal-system construction, and systematic invariant solution generation for the linear reaction–diffusion equation on $\mathbb{S}^2$ is still lacking. The present work is intended to fill this gap.

Recent developments further highlight the sustained interest in the symmetry and analytical structure of heat-type equations and in the spherical heat equation itself; see, for example, recent work on point and generalized symmetries of heat equations, spherical heat-equation analysis, related spectral control questions, and adjoint classification (Dicke & Veselić, 2024; Koval & Popovych, 2023; Raza et al., 2022).

1.1 Main Objectives

Guided by these geometric considerations, this paper investigates the linear reaction–diffusion equation on the unit sphere,

$$u_t = \Delta_{\mathbb{S}^2} u + \alpha u, \qquad \alpha \in \mathbb{R} \setminus \{0\} \tag{1}$$

where, $\Delta_{\mathbb{S}^2}$ is the Laplace–Beltrami operator on $\mathbb{S}^2$. Our objective is to give a concise and geometry–based Lie symmetry treatment of Equation (1). By deriving and solving the determining equations, we show that the geometric part of the Lie point symmetry algebra is precisely $\mathfrak{so}(3)\oplus\mathbb{R}$, generated by the three rotational Killing fields and time translations; in the linear homogeneous case $f(u) = \alpha u$, this extends to include the u-scaling and the infinite-dimensional superposition ideal, in agreement with the general homothetic-algebra principle for heat equations on Riemannian manifolds. This identification makes explicit how curvature suppresses Euclidean scaling and Galilean symmetries and explains why no additional point symmetries arise beyond the isometries of the sphere together with time invariance. With the algebra in hand, we construct an optimal system of subalgebras and compute the associated similarity variables following standard Lie–reduction methodology (Bluman & Kumei, 1989; Hydon, 2000; Ovsiannikov, 1982). The resulting reductions yield curvature-adapted parabolic equations in one spatial and one time variable, such as axisymmetric flows that depend only on x and t, as well as stationary ordinary differential equations that encode time-independent patterns consistent with spherical symmetry. Classical connections between symmetry methods, separation of variables, and reductions of linear partial differential equations (PDEs) in mathematical physics are discussed in Miller (1977).

The scope of the paper is intentionally restricted to the linear, spatially uniform reaction term on the unit sphere. Nonlinear sources, anisotropic or spatially varying diffusivities, non-unit radii, and boundary-driven problems are not considered here. In this focused setting, our aim is to make transparent, through a direct Lie–symmetry analysis, the fact that the symmetry algebra on $\mathbb{S}^2$ reduces to $\mathfrak{so}(3)\oplus\mathbb{R}$, and to demonstrate how this algebra organizes the resulting reduced models and spectral solutions. This perspective complements traditional spectral and heat–kernel approaches by adopting a symmetry-first methodology: the geometry uniquely determines the available point symmetries, and the identified algebra then guides curvature–compatible reductions and the construction of explicit solutions.

Although the analysis is carried out for the linear reaction–diffusion equation on the unit sphere, the conclusions are not limited to this single model. The results illustrate a general principle for heat-type equations on curved manifolds: the ambient geometry strongly constrains the admissible point symmetries, and the resulting symmetry algebra organizes both the reduced equations and the structure of exact invariant solutions. In this sense, the present paper also serves as a model case for symmetry–based reduction of diffusion-type equations on other compact homogeneous manifolds and for related problems with modified source terms or variable geometric parameters.

From a physical perspective, such models arise naturally in processes occurring on curved surfaces, including heat transport on planetary atmospheres, diffusion on spherical biological membranes, and chemical reactions on nanoparticle surfaces. In these contexts, curvature plays a fundamental role in shaping transport mechanisms and pattern formation, and the present analysis provides explicit solution structures that can be used to interpret and predict such phenomena. From a mathematical standpoint, the work contributes to the theory of partial differential equations on manifolds by establishing a complete and explicit connection between the geometry of $\mathbb{S}^2$, its symmetry algebra, and the construction of invariant solutions. It also demonstrates how Lie symmetry methods can complement classical spectral approaches, offering an alternative and systematic framework for deriving reduced models, understanding eigenfunction structures, and generating exact solutions in curved settings.

For clarity, **Figure 1** summarizes the overall workflow of the present study, from model formulation and geometric setup to symmetry classification, reduction, and invariant solution construction.

1.2 Main Contribution

The main contributions of this work are as follows. First, we derive the full Lie symmetry algebra of the diffusion equation on the sphere with linear reaction, including the finite-dimensional geometric symmetries and the infinite-dimensional linear superposition ideal. Second, we construct an optimal system of one-dimensional subalgebras for the finite-dimensional algebra. Third, we obtain representative similarity reductions and explicit invariant solutions, emphasizing the role of spherical geometry in selecting regular global modes.

2. Mathematical Preliminaries and Geometric Setting

A symmetry analysis on a curved space must begin by fixing the geometry, the relevant differential operators, and the way transformations act on the variables. In this section, we record the geometric data of the two-sphere, introduce the Laplace–Beltrami operator that drives diffusion, and summarize the infinitesimal Lie framework used later to classify point symmetries of our heat-type model (1).

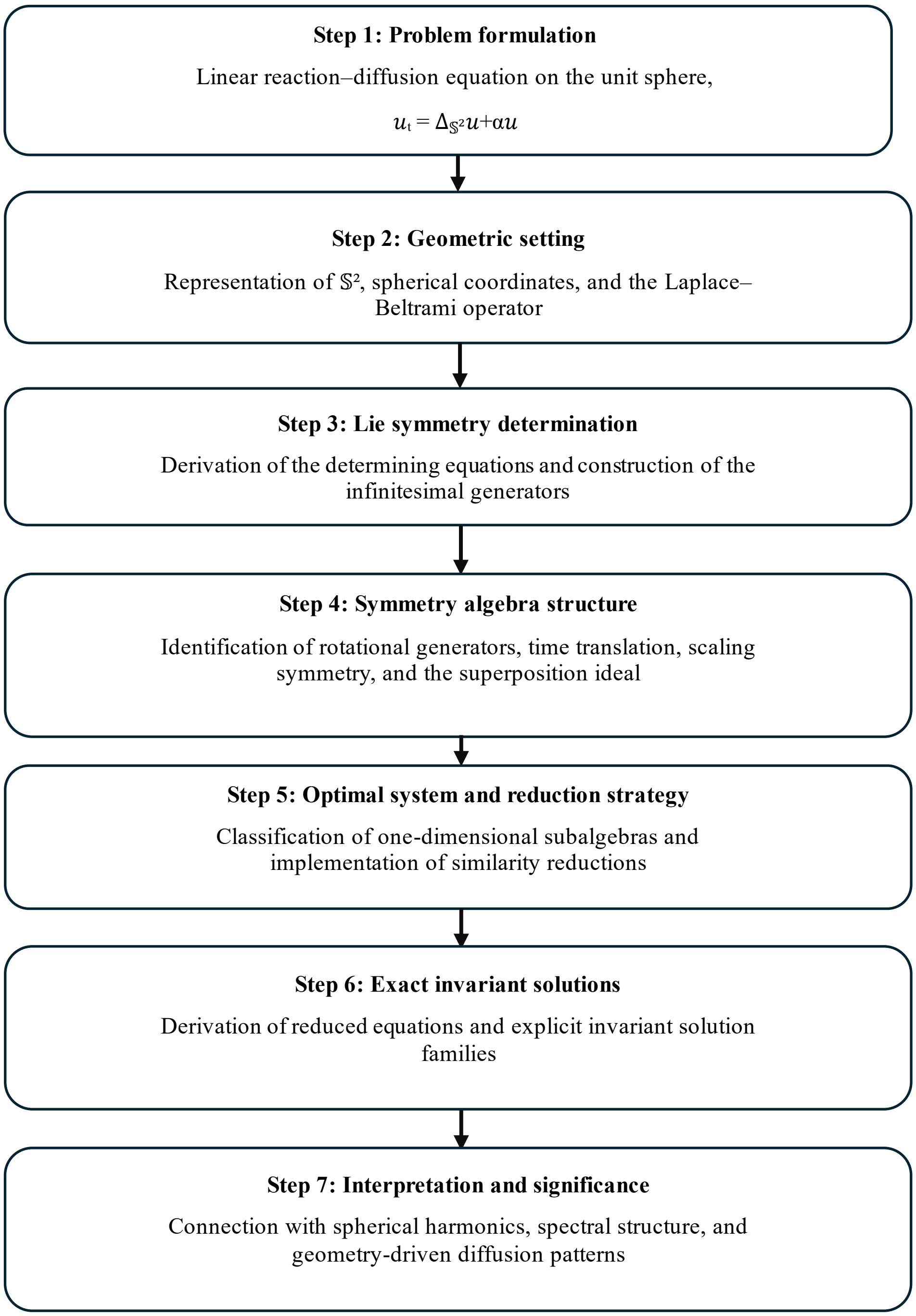

Figure 1. Research framework of this study.

Let (M, g) be a Riemannian manifold. Diffusion on (M, g) is governed by the Laplace–Beltrami operator

$$\Delta_M f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|}\, g^{ij}\, \partial_j f \right) \tag{2}$$

which reduces to the usual Laplacian in flat space (Chavel, 1984; Hydon, 2000). For the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, we use standard spherical coordinates (x, y), with colatitude $x \in (0, \pi)$ and azimuth $y \in [0, 2\pi)$. The induced metric

$$ds^2 = dx^2 + \sin^2 x\, dy^2 \tag{3}$$

has determinant $|g| = \sin^2 x$ and inverse $g^{ij} = \mathrm{diag}(1, \csc^2 x)$. Substitution into the definition of Δ_M yields the intrinsic spherical Laplacian

$$\Delta_{\mathbb{S}^2} = \partial_x^2 + \cot x\ \partial_x + \csc^2 x\ \partial_y^2 \tag{4}$$

the basic operator for isotropic diffusion on $\mathbb{S}^2$. The coefficients $\cot x$ and $\csc^2 x$ encode the curvature; they disappear in the flat limit. The isometry group of $\mathbb{S}^2$ is SO(3), generated by three Killing vector fields (infinitesimal rotations). These Killing fields form a copy of $\mathfrak{so}(3)$ and will reappear as spatial symmetry generators of the diffusion equations studied here (Stephani, 1990).

Substituting Equation (4) into Equation (1), we get the linear reaction–diffusion equation on the unit sphere,

$$u_t = u_{xx} + (\cot x)\, u_x + (\csc^2 x)\, u_{yy} + \alpha u, \qquad \alpha \in \mathbb{R}\backslash\{0\} \tag{5}$$

where, $u = u(x, y, t)$, x denotes colatitude, and y is azimuth. The term αu represents a spatially uniform linear reaction: positive α models amplification and negative α models attenuation. Equation (5) is a prototypical linear parabolic flow on $\mathbb{S}^2$ that couples intrinsic geometry, via $\Delta_{\mathbb{S}^2}$, to simple linear kinetics.

Lie's method seeks continuous transformations that map solutions to solutions. A one-parameter local group acts on the extended space (x, y, t, u) as

$$x^* = x + \varepsilon\xi, \quad y^* = y + \varepsilon\theta, \quad t^* = t + \varepsilon\tau, \quad u^* = u + \varepsilon\phi \tag{6}$$

where, ε is infinitesimal and $(\xi, \theta, \tau, \phi)$ are smooth functions. The associated infinitesimal generator is

$$X = \xi\partial_x + \theta\partial_y + \tau\partial_t + \phi\partial_u \tag{7}$$

Because a PDE involves derivatives of u, one extends X to its second prolongation $X^{[2]}$ by replacing ordinary derivatives with total derivatives; see (Bluman & Kumei, 1989; Hydon, 2000; Ovsiannikov, 1982) for a full account. If $\mathcal{F} = 0$ denotes the PDE under study, the invariance criterion is

$$X^{[2]}(\mathcal{F})|_{\mathcal{F}=0} = 0 \tag{8}$$

which produces a linear, overdetermined system of determining equations for the unknown components $(\xi, \theta, \tau, \phi)$. Solving that system yields all Lie point symmetries.

Once a generator X is known, its invariants are obtained from the characteristic system

$$\frac{dx}{\xi} = \frac{dy}{\theta} = \frac{dt}{\tau} = \frac{du}{\phi} \tag{9}$$

Solving this system provides similarity variables that collapse the number of independent variables and reduce the original PDE to a lower-dimensional equation. On $\mathbb{S}^2$, the most natural reductions are attached to the rotational Killing fields, producing axisymmetric flows, rotating patterns, and stationary states aligned with spherical harmonics. Such reductions lead to parabolic problems in one space and time, as well as to stationary ordinary differential equations (ODEs) on x or on rotational invariants, and they respect the underlying curvature.

Because $\mathbb{S}^2$ is compact and has no boundary, $\Delta_{\mathbb{S}^2}$ has a discrete spectrum with eigenfunctions given by spherical harmonics. This spectral structure is fully aligned with the symmetry picture: the SO(3) action organizes eigenspaces, while time translations reflect the evolution semigroup generated by $\Delta_{\mathbb{S}^2}$ (or $\Delta_{\mathbb{S}^2} + \alpha I$ in the linear reaction case). In later sections, we exploit both viewpoints, symmetry and spectrum, to derive reduced models and to interpret exact or modal solutions in a way that is consistent with the geometry.

Remark 1. The Cauchy problem associated with Equation (1) is well–posed in standard function spaces. Indeed, since the Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$ on the compact manifold $\mathbb{S}^2$ is self–adjoint with discrete spectrum, the operator $\Delta_{\mathbb{S}^2} + \alpha I$ generates a strongly continuous analytic semigroup on $\mathrm{L}^2(\mathbb{S}^2)$. It follows that, for any initial data $u_0 \in \mathrm{L}^2(\mathbb{S}^2)$, there exists a unique global solution $u(t)$ depending continuously on the initial data (Pazy, 1983). This well–posedness result provides the functional setting within which the invariant solutions derived below are interpreted.

3. Lie Symmetry Algebra and Classification for the Linear Reaction–Diffusion Equation on the Sphere

In this section, we derive the full Lie symmetry algebra for Equation (5) and prove that it splits into a finite geometric part and an infinite solution part. This algebraic structure reveals how curvature, compactness, and linearity combine to determine the transformation group of heat flow on the sphere. The complete classification is given in the main theorem.

On $\mathbb{S}^2$, there are no global translations or dilations; thus, the Galilean and spatial scaling symmetries of the planar heat equation are absent. Instead, isotropy of the round metric provides exactly three rotational Killing fields generating $\mathfrak{so}(3)$, and the time-autonomous structure contributes ∂_t. Because the equation is linear and homogeneous in u, two additional symmetry mechanisms appear: the scaling $u\partial_u$, and an infinite-dimensional superposition ideal $\psi\partial_u$, where each ψ solves Equation (5).

Theorem 3.1 (Full Lie symmetry algebra for the linear reaction–diffusion equation on $\mathbb{S}^2$).
Consider Equation (5):
$u_t = \Delta_{\mathbb{S}^2}u+\alpha u, \quad \Delta_{\mathbb{S}^2} = \partial_{xx} + \cot x\ \partial_x + \csc^2 x\ \partial_{yy}, \quad \alpha \in \mathbb{R} \setminus \{0\}.$

The Lie point symmetries of Equation (5) are generated by:

i. Finite-dimensional part (dimension 5):

$$L_1 = \cos y\, \partial_x - \cot x \sin y\, \partial_y \tag{10}$$

$$L_2 = \sin y\ \partial_x + \cot x \cos y\ \partial_y \tag{11}$$

$$L_3 = \partial_y \tag{12}$$

$$T = \partial_t \tag{13}$$

$$D = u\partial_u \tag{14}$$

namely the three rotations ($\mathfrak{so}(3)$), time translation, and u-scaling.

ii. Infinite-dimensional solution (superposition) symmetries $\mathfrak{g}_\infty$:

$$X_{\psi} = \psi(t,x,y)\partial_u, \quad \textit{for any solution} \quad \psi_t = \Delta_{\mathbb{S}^2}\psi + \alpha\psi \tag{15}$$

Proof. Let

$$X = \xi(x,y,t,u)\,\partial_x + \theta(x,y,t,u)\,\partial_y + \tau(x,y,t,u)\,\partial_t + \phi(x,y,t,u)\,\partial_u \tag{16}$$

be a Lie point symmetry generator and let

$$F := u_t - u_{xx} - (\cot x)\,u_x - (\csc^2 x)\,u_{yy} - \alpha u = 0 \tag{17}$$

Since $F = 0$ is an evolution equation (first order in t), the invariance criterion is imposed on the solution manifold by substituting Equation (5) whenever u_t occurs in prolonged expressions. Equivalently, one may use the characteristic

$$Q = \phi - \xi u_x - \theta u_y - \tau u_t \tag{18}$$

and apply $\mathrm{pr}^{(2)}X(F)|_{F=0} = 0$.

After substituting Equation (1) and splitting with respect to the independent jet variables, we obtain the following determining equations:

e1: $\xi_u = 0,$

e2: $\theta_u = 0,$

e3: $\tau_u = 0,$

e4: $\phi_{uu} = 0,$

e5: $\tau_x = 0,$

e6: $\tau_y = 0,$

e7: $\xi_x = 0,$

e8: $\theta_x + \csc^2 x\ \xi_y = 0,$

e9: $\theta_y + \cot x\ \xi = 0,$

e10: $\xi_t = 0,$

e11: $\theta_t = 0,$

e12: $a_x = 0,$

e13: $a_y = 0,$

e14: $a_t = 0,$

e15: $b_t - b_{xx} - \cot x\, b_x - \csc^2 x\ b_{yy} - \alpha b = 0,$

e16: $\tau_t = 0.$

By e4, the symmetry component is affine in u:

$$\phi(x,y,t,u) = a(x,y,t)\,u + b(x,y,t) \tag{19}$$

where, $a(x,y,t)$ and $b(x,y,t)$ are smooth coefficient functions to be determined.

From e_1–e_3, we have $\xi = \xi(x, y, t)$, $\theta = \theta(x, y, t)$, $\tau = \tau(x, y, t)$. From e_5, e_6, and e_{16} we obtain

$$\tau_x = \tau_y = \tau_t = 0 \Longrightarrow \tau = \text{constant} \tag{20}$$

Hence, the time-translation symmetry generator is $T = \partial_t$.

Equations e_7–e_9 are precisely the Killing system for the round metric $ds^2 = dx^2 + \sin^2 x\, dy^2$ on $\mathbb{S}^2$, and together with e_{10}–e_{11}, they imply that the spatial part is time-independent and equals a linear combination of the three rotational Killing fields L_1, L_2, and L_3.

Finally, e_{12}–e_{14} yield a = constant, giving the scaling generator $D = u\partial_u$. Equation e_{15} shows that b must satisfy the same linear PDE as u, hence for any solution ψ of Equation (1), we obtain a solution symmetry $X_\psi = \psi\partial_u$.

Therefore, the finite-dimensional Lie point symmetry algebra is $\{L_1, L_2, L_3, T, D\}$ and the infinite-dimensional ideal is $\{ X_\psi = \psi\partial_u : \psi_t = \Delta_{\mathbb{S}^2}\psi + \alpha\psi \}$.

Remark 2. When $\alpha \neq 0$, the constant choice $\psi \equiv 1$ fails, so ∂_u is not admitted, whereas time-dependent vertical shifts such as $e^{\alpha t}\partial_u$ are included in the ideal; since $\psi(t) = e^{\alpha t}$ solves the equation.

Remark 3 (Lie brackets). Among the finite-dimensional generators $\{L_1, L_2, L_3, T, D\}$, the only non–zero Lie brackets are those of the $\mathfrak{so}(3)$ triple. The elements T and D commute with all finite generators. Hence

$$\mathfrak{g}_{\text{fin}} \cong \mathfrak{so}(3) \oplus \langle \mathrm{T} \rangle \oplus \langle \mathrm{D} \rangle \tag{21}$$

With the generators defined in Theorem 3.1, the non–zero brackets are

$$[L_1, L_2] = -L_3, \qquad [L_2, L_3] = -L_1, \qquad [L_3, L_1] = -L_2 \tag{22}$$

i.e. $[L_i, L_j] = -\varepsilon_{ijk}\, L_k$, hence the commutator table is skew-symmetric. Since T and D act trivially on (x, y, t), they commute with the rotational Killing fields. Their action on solution symmetries $X_\psi = \psi\partial_u$ is

$$[L_i, X_\psi] = X_{L_i(\psi)}, \qquad [T, X_\psi] = X_{\psi_t}, \qquad [D, X_\psi] = -X_\psi \tag{23}$$

Thus, $\mathfrak{g}_\infty = \{X_\psi : \psi_t = \Delta_{\mathbb{S}^2}\psi + \alpha\psi\}$ is an Abelian ideal, confirming the semidirect-sum structure

$$\mathfrak{g} = (\mathfrak{g}_{\text{fin}} = \mathfrak{so}(3) \oplus \langle \mathrm{T} \rangle \oplus \langle \mathrm{D} \rangle) \ltimes \mathfrak{g}_\infty \tag{24}$$

The vector fields L_1, L_2, and L_3 are the rotational Killing fields of $(\mathbb{S}^2, g)$, encoding the isotropy of the round sphere. The generator $T = \partial_t$ reflects time homogeneity of the equation. For the homogeneous linear case $(u_t = \Delta u + \alpha u)$, the scaling $D = u\partial_u$ arises from homogeneity in u, while the infinite-dimensional ideal $X_\psi = \psi\partial_u$ expresses the superposition principle: adding any solution ψ of the linear equation to a given solution u yields another solution $u + \psi$. The absence of Galilean boosts and spatial scalings distinguishes the spherical case from its Euclidean counterpart.

Table 2. Lie brackets of finite-dimensional symmetry generators.

$[\cdot,\cdot]$	L_1	L_2	L_3	T	D
L_1	0	$-L_3$	L_2	0	0
L_2	L_3	0	$-L_1$	0	0
L_3	$-L_2$	L_1	0	0	0
T	0	0	0	0	0
D	0	0	0	0	0

If the source term is $f(u) = \alpha u + \mu$ on $\mathbb{S}^2$, then the geometric generators L_1, L_2, L_3 and T are symmetries for all (α, μ). The scaling $D = u\partial_u$ is admitted precisely when $\mu = 0$, i.e., in the homogeneous linear case. The constant vertical shift $U = \partial_u$ is admitted if and only if $\alpha = 0$, regardless of μ (i.e., for $u_t = \Delta_{\mathbb{S}^2} u + \mu$). This shift fails for $\alpha \neq 0$ because the symmetry requires that $f(u)$ be constant, meaning the reaction term must be independent of u (as explained in the next proposition). For the homogeneous case ($\mu = 0$), regardless of α, the infinite-dimensional superposition ideal $\{\psi\partial_u : \psi_t = \Delta_{\mathbb{S}^2}\psi + \alpha\psi\}$ is admitted, reflecting linearity.

Proposition 1. Let

$$F[u] := u_t - \Delta_{\mathbb{S}^2} u - f(u), \qquad (\alpha, \mu \in \mathbb{R}) \tag{25}$$

For the affine linear source $f(u) = \alpha u + \mu$ on $\mathbb{S}^2$, the rotational generators L_1, L_2, L_3 and the time translation $T = \partial_t$ are Lie point symmetries for all (α, μ). The scaling $D = u\partial_u$ is a symmetry if and only if $\mu = 0$.

Proof. We use the infinitesimal invariance criterion $X^{[2]}(F)|_{F=0} = 0$, with the parameters α and μ held fixed, where $F[u] := u_t - \Delta_{\mathbb{S}^2} u - f(u)$. For the rotational generators L_i ($i = 1,2,3$), note that each L_i is a Killing field on ($\mathbb{S}^2$,g); hence the Laplace–Beltrami operator is invariant under its flow and $[L_i, \Delta_{\mathbb{S}^2}] = 0$. Also $[L_i, \partial_t] = 0$ and $L_i(\mu) = 0$. Therefore,

$$L_i^{[2]} F = L_i(u_t) - L_i(\Delta_{\mathbb{S}^2} u) - \alpha L_i(u) - L_i(\mu) = (L_i u)_t - \Delta_{\mathbb{S}^2}(L_i u) - \alpha(L_i u) \tag{26}$$

Now, if u satisfies $F[u] = 0$, then $u_t - \Delta_{\mathbb{S}^2} u - \alpha u = \mu$. Applying L_i to this equation (and using the commutation properties and $L_i(\mu) = 0$) yields

$$(L_i u)_t - \Delta_{\mathbb{S}^2}(L_i u) - \alpha(L_i u) = 0 \tag{27}$$

Hence, $L_i^{[2]} F = 0$ when $F = 0$, so each L_i is a symmetry for all α and μ.

For the time translation $T = \partial_t$, the coefficients of F are autonomous in t, so $[T, \partial_t] = [T, \Delta_{\mathbb{S}^2}] = 0$ and $T(\mu) = 0$. Then,

$$T^{[2]} F = T(u_t) - T(\Delta_{\mathbb{S}^2} u) - \alpha T(u) - T(\mu) = (Tu)_t - T(\Delta_{\mathbb{S}^2} u) - \alpha T(u) \tag{28}$$

Again, if u satisfies $F[u] = 0$, then applying T to Equation (26) gives

$$T(u_t) - \Delta_{\mathbb{S}^2}(Tu) - \alpha(Tu) = 0 \tag{29}$$

So, $T^{[2]} F = 0$ when $F = 0$. Thus T is a symmetry for all α and μ.

For the scaling generator $D = u\partial_u$, the second prolongation satisfies

$$D^{[2]}(u_t) = u_t, \quad D^{[2]}(\Delta_{\mathbb{S}^2} u) = \Delta_{\mathbb{S}^2} u, \quad D^{[2]}(\alpha u) = \alpha u, \quad D^{[2]}(\mu) = 0 \tag{30}$$

Since, μ is a constant parameter. Hence

$$D^{[2]} F = D^{[2]}(u_t - \Delta_{\mathbb{S}^2} u - \alpha u - \mu) = u_t - \Delta_{\mathbb{S}^2} u - \alpha u = F + \mu \tag{31}$$

Therefore, $D^{[2]}(F)|_{F=0} = \mu$, and the invariance criterion implies that D is admitted if and only if $\mu = 0$.

These three items show that L_1, L_2, L_3 and T persist for all α and μ, while the scaling symmetry D is admitted precisely in the homogeneous case $\mu = 0$.

Since the finite-dimensional algebra is $\mathfrak{g}_{\text{fin}} \cong \mathfrak{so}(3) \oplus \langle \mathrm{T} \rangle \oplus \langle \mathrm{D} \rangle$, we classify one-dimensional subalgebras up to adjoint action. To construct an optimal system of one-dimensional subalgebras, we use the adjoint representation of the Lie group. The adjoint action is computed using the commutation relations given in **Table 2**.

Table 3. Infinitesimal adjoint action.

$\mathrm{Ad}\big(\exp(\varepsilon X_i)\big)X_j$	L_1	L_2	L_3	T	D
L_1	L_1	$L_2 - \varepsilon L_3$	$L_3 + \varepsilon L_2$	T	D
L_2	$L_1 + \varepsilon L_3$	L_2	$L_3 - \varepsilon L_1$	T	D
L_3	$L_1 - \varepsilon L_2$	$L_2 + \varepsilon L_1$	L_3	T	D
T	L_1	L_2	L_3	T	D
D	L_1	L_2	L_3	T	D

The adjoint representation is used to simplify general linear combinations of generators and to classify inequivalent one-dimensional subalgebras in the construction of the optimal system (Raza et al., 2022).

Proposition 2 (Optimal System of One-Dimensional Subalgebras). A minimal set of representatives for the one-dimensional subalgebras of the finite-dimensional Lie symmetry algebra $\mathfrak{g}_{\text{fin}} \cong \mathfrak{so}(3) \oplus \langle \mathrm{T} \rangle \oplus \langle \mathrm{D} \rangle$, classified up to the adjoint action and scaling, is given by the collection $\mathcal{O}_1 = \{L_3 + aT + bD,\ T + cD,\ D \mid a, b, c \in \mathbb{R}\}$.

Proof. Consider a general element of the five-dimensional Lie algebra $\mathfrak{g}_{\text{fin}}$ expressed as $X = a_1 L_1 + a_2 L_2 + a_3 L_3 + a_4 T + a_5 D$, where $a_i \in \mathbb{R}$. The classification follows from the adjoint representation $\mathrm{Ad}\big(\exp(\varepsilon X)\big)Y$, whose action on the basis elements is summarized in **Table 3.** Since T and D are central elements spanning the center of the algebra, they commute with all generators such that $[T, X] = 0$ and $[D, X] = 0$. Consequently, their adjoint action is trivial, and they remain invariant under any inner automorphism. Therefore, the classification of one–dimensional subalgebras is determined primarily by the $\mathfrak{so}(3)$ subalgebra.

The rotational generators $\{L_1, L_2, L_3\}$ span the $\mathfrak{so}(3)$ subalgebra, and the adjoint action of SO(3) on its own Lie algebra corresponds to a rigid rotation in the three-dimensional subspace. For any non–zero rotational part (a_1, a_2, a_3), there exists an element of the rotation group that rotates the vector to be parallel to the L_3 axis. Applying such a rotation and normalizing the coefficient to unity yields the canonical form $L_3 + aT + bD$. In the case where the rotational part vanishes, the generator reduces to a linear combination of the central elements $a_4 T + a_5 D$. If $a_4 \neq 0$, scaling yields the representative $T + cD$, whereas if $a_4 = 0$ and $a_5 \neq 0$, the generator is equivalent to D. Because D and $T + cD$ cannot be mapped into one another by any adjoint action for finite γ, they constitute distinct classes.

This optimal system organizes the possible symmetry reductions into three primary physical categories. Stationary and zonal solutions are generated when the coefficients a and b vanish, leading directly to the axisymmetric profiles and zonal harmonics discussed in the subsequent derivations. Alternatively, traveling waves correspond to the case where a is non–zero and b is zero, representing helical or spiral propagation patterns across the sphere. Finally, self-similar solutions are generated by those subalgebras involving the scaling operator D, which include both the pure scaling cases and the coupled time-scaling reductions. These categories provide a complete framework for analyzing the invariant solutions of the reaction–diffusion equation on the spherical manifold.

4. Symmetry Reductions and Exact Invariant Solutions of the Linear Reaction–diffusion Equation on the Sphere

This section presents a systematic Lie symmetry analysis of the linear reaction–diffusion equation on the two-dimensional sphere $\mathbb{S}^2$. The compact, curved geometry of $\mathbb{S}^2$ imposes strong constraints on admissible solutions, leading to quantization conditions and discrete spectra that contrast with the Euclidean case. The curved, compact nature of $\mathbb{S}^2$ leads to fundamental differences from the Euclidean case: solutions must be globally defined and smooth, imposing stringent regularity conditions that quantize admissible growth rates and spatial patterns.

The symmetry structure of Equation (5) reflects the geometric constraints of the sphere. While the linearity implies an infinite-dimensional solution symmetry algebra, we focus on the finite-dimensional geometric subalgebra generated by isometries of $\mathbb{S}^2$, time translations, and scaling.

We employ the method of symmetry reduction to systematically construct exact invariant solutions. We next consider selected two-dimensional subalgebras that lead to tractable and representative reductions. For each selected subalgebra, we determine the corresponding invariant variables, derive the reduced ordinary differential equations, obtain explicit closed-form solutions, and analyze the global regularity conditions required on $\mathbb{S}^2$. This procedure reveals how the intrinsic geometry and symmetry structure of the sphere dictate the form of the fundamental solutions, reproducing the spherical harmonic basis through purely group-theoretic arguments and providing clear geometric interpretations for the various harmonic types.

4.1 Reduction Using Subalgebra $\langle T, L_3\rangle = \langle \partial_t, \partial_y\rangle$

Using the invariants of $T = \partial_t$:

$$q = x,\ v = y,\ P(q,v) = u(x,y,t) \tag{32}$$

we obtain the time-independent PDE

$$P_{qq} + (\cot q)P_q + (\csc^2 q)\, P_{vv} + \alpha P = 0 \tag{33}$$

The second symmetry

$$L_3 = \frac{\partial}{\partial y} = 0\frac{\partial}{\partial q} + 1\frac{\partial}{\partial v} + 0\frac{\partial}{\partial P} \tag{34}$$

is inherited by Equation (33). The characteristic system

$$\frac{dq}{0} = \frac{dv}{1} = \frac{dP}{0} \tag{35}$$

gives the invariants

$$\xi(q,v) = q,\ \omega(\xi) = P(q,v) \tag{36}$$

Since we seek solutions invariant under $L_3 = \partial_y$, we impose $P_v = 0$, hence P depends only on $\xi = q$, and we write $P(q,v) = \omega(\xi)$. Then

$$P_v = 0,\ P_{vv} = 0,\ P_q = \omega'(\xi),\ P_{qq} = \omega''(\xi) \tag{37}$$

and Equation (33) becomes

$$\omega'' + \cot\xi\, \omega' + \alpha\omega = 0 \tag{38}$$

Introducing $z = \cos\xi$, Equation (38) transforms into the standard Legendre differential equation:

$$(1 - z^2)\omega_{zz} - 2z\omega_z + \alpha\omega = 0 \tag{39}$$

The general solution is expressed in terms of Legendre functions of the first and second kind
$$\omega(\xi) = AP_v(\cos\xi) + BQ_v(\cos\xi),\ v(v+1) = \alpha \tag{40}$$

Global smoothness on the compact manifold $\mathbb{S}^2$ requires $B = 0$ and $v = n \in \mathbb{N}$, yielding:
$$u(x, y, t) = AP_{\mathrm{n}}(\cos x),\ \alpha = n(n+1) \tag{41}$$

where, $A \in \mathbb{R}$ is an arbitrary amplitude constant, and P_n denotes the Legendre polynomial of degree n. The quantization condition $\alpha = n(n+1)$ arises from requiring solutions to be regular at the poles ($z = \pm 1$), reflecting the discrete spectrum of the spherical Laplacian. This reduction yields zonal harmonics– axisymmetric patterns where n determines the number of nodal latitude lines.

We now look for closed-form solutions of Equation (38). Testing the trial solution $\omega(\xi) = \cos\xi$, with $\omega' = -\sin\xi$ and $\omega'' = -\cos\xi$, and substitution into Equation (38) gives:
$$\omega'' + \cot\xi\,\omega' + \alpha\omega = -\cos\xi - \cot\xi\,(\sin\xi) + \alpha\cos\xi = (\alpha - 2)\cos\xi \tag{42}$$

Thus, $\cos\xi$ is a solution if and only if $\alpha = 2$. We denote this first independent solution as
$$\omega_1(\xi) = \cos\xi \tag{43}$$

To find a second independent solution, we employ the method of reduction of order
$$\omega_2(\xi) = \omega_1(\xi) \int \frac{e^{-\int \cot\xi d\xi}}{[\omega_1(\xi)]^2}\, d\xi \tag{44}$$

This yields the second solution:
$$\omega_2(\xi) = 1 + \cos\xi\, \ln(\csc\xi - \cot\xi) \tag{45}$$

Therefore, for $\alpha = 2$, the general solution for the radial part is:
$$\omega(\xi) = C_1\cos\xi + C_2[1 + \cos\xi\, \ln(\csc\xi - \cot\xi)] \tag{46}$$

Substituting back to the original coordinates, we obtain the exact invariant solution:
$$u(x, y, t) = C_1\cos x + C_2[1 + \cos x \ln(\csc x - \cot x)] \tag{47}$$

This solution is invariant under rotations about the vertical polar axis, so u depends only on the polar angle and not on the azimuth. It represents an axisymmetric temperature distribution constant along each circle of latitude.

The first term $\cos x$ is a regular spherical harmonic corresponding to a dipolar pattern (hot in one hemisphere, cold in the opposite). The second term $\cos x \ln(\csc x - \cot x)$ is the singular associated Legendre function which diverges at the poles. On the full sphere $\mathbb{S}^2$, only the regular part is physically admissible. The logarithmic term would cause the solution to blow up at the North and South poles. Geometrically, **Figure 2** illustrates this axisymmetric pattern that has a warm cap on one pole and a cool cap on the opposite pole, with temperature constant along each latitude circle. The horizontal axis represents the colatitude x, while the vertical axis represents the solution profile $u(x)$, which is independent of the azimuthal angle. The linear reaction term αu acts as a uniform source or sink that must be balanced by diffusion, which is possible only for the discrete eigenvalue $\alpha = 2$, reflecting the quantization condition on the sphere.

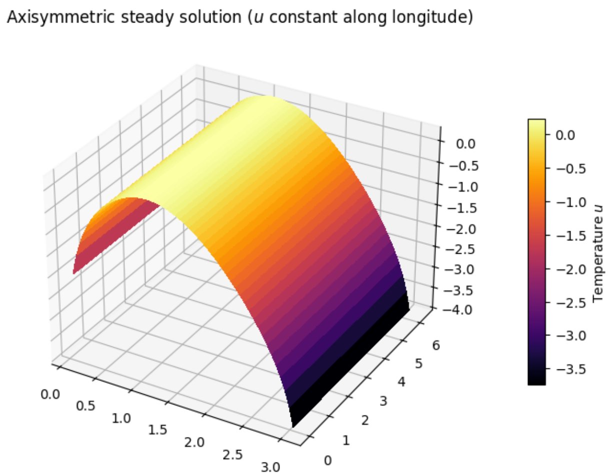

Figure 2. Axisymmetric stationary solution for the discrete eigenvalue $\alpha = 2$, showing a latitude-dependent profile constant in the azimuthal direction.

4.2 Reduction Using Rotational Subalgebras $\langle T, L_1\rangle$ and $\langle T, L_2\rangle$

We now consider two equivalent stationary reductions corresponding to the subalgebras $\langle T, L_1\rangle$ and $\langle T, L_2\rangle$. Both represent rotationally deformed steady-state patterns on the sphere, differing only by a rotation of the coordinate system about the polar axis.

Using $T = \partial_t$ first (stationary solutions), the characteristic systems for L_1 and L_2 are, respectively,

$$\frac{dx}{\cos y} = \frac{dy}{-\cot x \sin y} + \frac{du}{0}, \; \frac{dx}{\sin y} = \frac{dy}{\cos y \cot x} + \frac{du}{0} \tag{48}$$

These yield the invariants

$$\xi_1 = \sin x \sin y\,, \; \xi_2 = \sin x \cos y\,, \; \omega(\xi) = u \tag{49}$$

Substitution of either invariant into the original equation reduces it to the same Legendre-type ordinary differential equation,

$$(1-\xi^2)\omega''(\xi) - 2\xi\omega'(\xi) + \alpha\omega(\xi) = 0 \tag{50}$$

The general solution is

$$\omega(\xi) = AP_\nu(\xi) + BQ_\nu(\xi), \; \nu(\nu+1) = \alpha \tag{51}$$

where, P_ν and Q_ν are the Legendre functions of the first and second kind. Global smoothness on the sphere requires $B = 0$ and $\nu = n \in \mathbb{N}$, giving

$$u(x,y,t) = \begin{cases} AP_{\mathrm{n}}(\sin x \sin y), & \text{for } \langle T, L_1\rangle, \\ AP_{\mathrm{n}}(\sin x \cos y), & \text{for } \langle T, L_2\rangle, \end{cases} \qquad \alpha = n(n+1) \tag{52}$$

where, $A \in \mathbb{R}$ is an arbitrary amplitude constant

Both reductions generate rotational profiles corresponding to non-axisymmetric spherical harmonics, which are stationary temperature fields. In **Figure 3**, the horizontal and vertical axes correspond to the spherical coordinates (x, y), representing the colatitude and azimuthal angle, respectively, while the plotted surface represents the solution $u(x, y)$. The solution corresponding to $\langle T, L_2\rangle$ is symmetric about the x-axis, while that for $\langle T, L_1\rangle$ is symmetric about the y-axis. Since the two subalgebras are related by a rotation in the azimuthal angle, their associated solutions form a pair of rotationally equivalent families that, taken together, constitute a subset of the non-axisymmetric spherical harmonic modes.

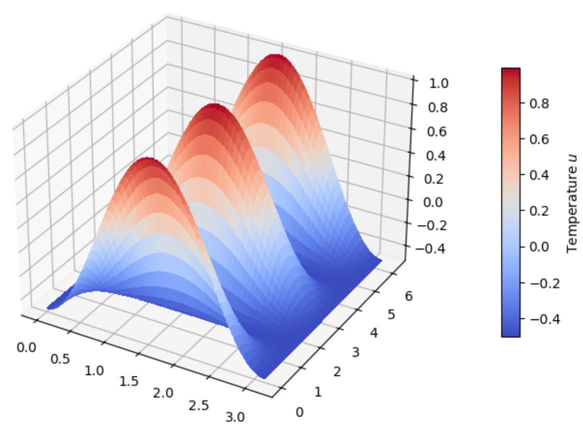

Figure 3. Representative non-axisymmetric stationary harmonic with $\alpha = n(n+1)$ along the equatorial x-axis.

4.3 Reduction Using Subalgebra $\langle \beta T + \gamma L_3, \delta T + \lambda D + \kappa L_3\rangle$ and Non-Stationary Solutions

Using the first symmetry $\beta T + \gamma L_3$ with invariants

$$q = x,\ v = \beta y - \gamma t,\ P(q, v) = u(x, y, t) \tag{53}$$

Equation (5) reduces to

$$-\gamma P_v = P_{qq} + (\cot q) P_q + \beta^2 (\csc^2 q)\, P_{vv} + \alpha P \tag{54}$$

Applying the second symmetry $\delta T + \lambda D + \kappa L_3$ yields the invariant representation

$$P(q, v) = \exp(\omega(\zeta) + mv),\ \zeta = q,\ m = \frac{\lambda}{\beta\kappa - \gamma\delta},\ \beta\kappa - \gamma\delta \neq 0 \tag{55}$$

Substitution of this form into Equation (54) gives

$$P_v = mP, \qquad P_{vv} = m^2 P, \qquad P_q = \omega' P, \qquad P_{qq} = (\omega'' + \omega'^2) P \tag{56}$$

and hence division by $P \neq 0$ yields the reduced ordinary differential equation

$$\omega'' + (\omega')^2 + \cot\zeta\, \omega' + \beta^2 m^2 \csc^2\zeta + \alpha + \gamma m = 0 \tag{57}$$

For the particular choice of parameters

$$\alpha = 1,\ \beta = \gamma = 1,\ \kappa = 0,\ \lambda = \delta \tag{58}$$

we have

$$m = \frac{\lambda}{\beta\kappa - \gamma\delta} = \frac{\lambda}{-\lambda} = -1 \tag{59}$$

and Equation (57) reduces to

$$\omega'' + (\omega')^2 + \cot\zeta\,\omega' + \csc^2\zeta = 0 \tag{60}$$

Setting $Y = \omega'$, we obtain

$$Y' + Y^2 + \cot\zeta\, Y + \csc^2\zeta = 0 \tag{61}$$

whose solution is

$$\omega(\zeta) = C_0 + \ln\left[\cos\left(\ln\left(\frac{1+\cos\zeta}{\sin\zeta}\right)\right)\right] \tag{62}$$

Consequently, the corresponding invariant solution of Equation (5) is

$$u(x, y, t) = Ce^{-(y-t)}\cos\left(\ln\left(\frac{1+\cos x}{\sin x}\right)\right) \tag{63}$$

Remark 4. Since $y \sim y + 2\pi$ on $\mathbb{S}^2$, the factor e^{-y} is not periodic; hence Equation (63) is a local invariant solution, not a globally single-valued solution on the sphere.

This mixed reduction provides non-stationary solutions exhibiting wave-like propagation in the azimuthal direction while maintaining specific spatial patterns. In **Figure 4**, the horizontal and vertical axes correspond to the spherical variables (x, y), while the third dimension represents the solution amplitude $u(x, y, t)$ at a fixed time. The exponential time dependence coupled with the complex angular structure represents interference patterns in the heat distribution. **Figure 4** shows a schematic of this helical solution: a hot region spirals around the sphere from the equator toward one pole, fading as it goes, followed by a cold region trailing behind it, and so on, with the spiral loops getting closer together near the pole.

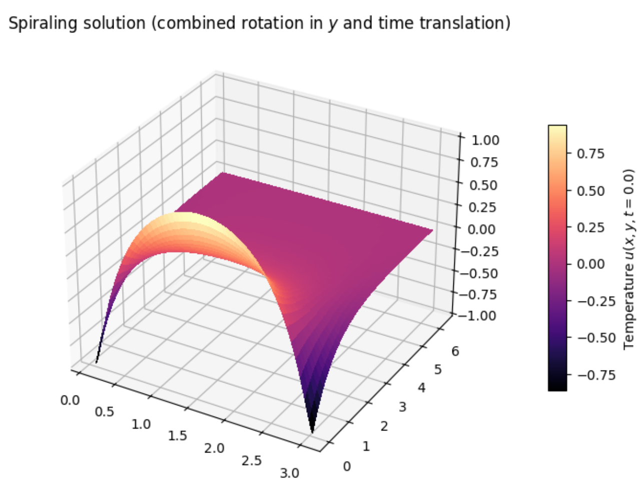

Figure 4. Local non-stationary invariant solution for $\alpha = 1$, showing azimuthal exponential variation at $t = 0$.

The factor $e^{-(y-t)}$ shows that along the characteristic lines $y - t =$ constant, the amplitude is constant. At fixed time t, the amplitude decays exponentially as y increases, whereas at fixed azimuth y it grows exponentially in time. Geometrically, the one-parameter symmetry underlying this solution is a screw motion, combining a simultaneous rotation in y, a time shift, and a scaling in u. Consequently, the temperature distribution is not periodic in the azimuthal direction but instead diminishes with each 2π revolution. The term $\cos(\ln((1 + \cos x)/\sin x))$ encodes an oscillatory dependence on the latitude x; it oscillates infinitely many times, producing an increasingly fine alternation of hot and cold bands near the poles.

5. Conclusion

We have presented a complete Lie symmetry analysis of the spherical reaction–diffusion equation $u_t = \Delta_{\mathbb{S}^2} u + \alpha u$. The classification of subalgebras up to $\mathrm{SO}(3)$-conjugacy yields curvature-adapted similarity variables and corresponding invariant solutions. Stationary reductions recover the spherical harmonic structure and impose the usual quantization conditions through global smoothness on the compact sphere, while time-dependent reductions show that the reaction term acts as a uniform spectral shift controlling temporal amplification or decay. A mixed reduction gives non-stationary azimuthally propagating patterns, and full rotational invariance isolates the spatially uniform mode.

The analysis provides a unified symmetry–based derivation of invariant solutions using the complete algebra $\mathfrak{so}(3) \oplus \langle \partial_t, u\partial_u \rangle$, offering a geometry-adapted route to the spectral decomposition of the Laplace–Beltrami operator and clarifying how curvature restricts admissible symmetry extensions compared with the Euclidean case. It thus complements spectral and heat–kernel methods with a streamlined, symmetry-first framework specific to $\mathbb{S}^2$. Beyond the specific equation studied here, the analysis demonstrates how geometric symmetry, compactness, and spectral structure interact in the symmetry reduction of parabolic equations on curved manifolds.

Several extensions suggest themselves, including nonlinear reaction terms, anisotropic or spatially varying diffusivities, and non-constant curvature manifolds. Higher-dimensional analogues on $\mathbb{S}^n$ and reaction–diffusion systems incorporating advection or boundary effects offer additional avenues for analytical and numerical investigation. The invariant structures identified here provide a natural foundation for studying pattern formation, spectral stability, and symmetry breaking in nonlinear parabolic flows on curved surfaces.

Conflicts of Interest
The author confirms that there is no conflict of interest to declare for this publication.

Acknowledgments
This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The author would like to thank the editor and anonymous reviewers for their comments that help improve the quality of this work.

AI Disclosure
The author declares that no assistance is taken from generative AI to write this article.

References

Bluman, G.W., & Cole, J.D. (1974). *Similarity methods for differential equations*. Springer, New York. ISBN: 978-0-387-90107-7. https://doi.org/10.1007/978-1-4612-6394-4

Bluman, G.W., & Kumei, S. (1989). *Symmetries and differential equations*. Springer, New York. ISBN: 978-1-4757-4307-4. https://doi.org/10.1007/978-1-4757-4307-4

Chavel, I. (1984). *Eigenvalues in Riemannian geometry*. Academic Press, London. https://doi.org/10.1016/s0079-8169%2808%29x6051-9

Clarkson, P.A., & Mansfield, E.L. (1994). Symmetry reductions and exact solutions of a class of nonlinear heat equations. *Physica D: Nonlinear Phenomena*, *70*(3), 250-288. https://doi.org/10.1016/0167-2789(94)90017-5

Dicke, A., & Veselić, I. (2024). Spherical Logvinenko–Sereda–Kovrijkine type inequality and null-controllability of the heat equation on the sphere. *Archiv der Mathematik*, *123*(5), 543-556. https://doi.org/10.1007/s00013-024-02051-4

Grigor'yan, A. (2009). *Heat kernel and analysis on manifolds*. American Mathematical Society, USA. https://doi.org/10.1090/amsip/047

Hydon, P.E. (2000). *Symmetry methods for differential equations: a beginner's guide*. Cambridge University Press. https://doi.org/10.1017/CBO9780511623967

Ibragimov, N.H. (1993). *CRC handbook of lie group analysis of differential equations, vol.* 1. CRC Press, Boca Raton. https://doi.org/10.1201/9781003419808

Ibragimov, N.H. (1999). *Elementary lie group analysis and ordinary differential equations*. John Wiley & Sons, New York.

Koval, S.D., & Popovych, R.O. (2023). Point and generalized symmetries of the heat equation revisited. *Journal of Mathematical Analysis and Applications*, *527*(2), 127430. https://doi.org/10.1016/j.jmaa.2023.127430

Miller, W. (1977). *Symmetry and separation of variables*. Addison–Wesley Publishing, United States. https://www.osti.gov/biblio/5260230

Olver, P.J. (1986). *Applications of lie groups to differential equations.* Springer, New York. https://doi.org/10.1007/978-1-4684-0274-2

Ovsiannikov, L.V. (1982). *Group analysis of differential equations*. Academic Press, Cambridge. ISBN: 978-0-12-531680-4. https://doi.org/10.1016/C2013-0-07470-1

Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations*. Springer, New York. ISBN: 978-0-387-90845-8. https://doi.org/10.1007/978-1-4612-5561-1

Raza, A., Mahomed, F.M., Zaman, F.D., & Kara, A.H. (2022). Optimal system and classification of invariant solutions of nonlinear class of wave equations and their conservation laws. *Journal of Mathematical Analysis and Applications*, *505*(1), 125615. https://doi.org/10.1016/j.jmaa.2021.125615

Sepanski, M.R. (2007). *Compact lie groups*. Springer, New York. https://doi.org/10.1007/978-0-387-49158-5

Stephani, H. (1990). *Differential equations: their solution using symmetries*. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511599941