

An Economic Production Quantity (EPQ) Model for Multi-Product Manufacturing with Consideration Towards Complementary Price-Dependent Demand and the Possibility of Product Recovery

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Abstract

In production industries such as stainless steel manufacturing, a shared resource (also referred to as a constrained resource) can produce multiple complementary products that are consumed together. The production process yields defective outputs that can be recovered through reannealing and cutting coils into sheets on the constrained resource, while the customers' demand for each product is impacted by the price of the complementary product. This study develops an Economic Production Quantity (EPQ) model for two mutually complementary products manufactured on a shared resource with imperfect production. Product Type A's demand can only be satisfied through production from new feedstock, while Product Type B's demand can be satisfied through remanufacturing of non-conforming products and production from new feedstock. The model incorporates complementary price-dependent demands and remanufacturing to optimise lot sizes by minimising total cost. The total cost function is fundamentally driven by baseline elements (setup and holding costs) along with recovery operational metrics (remanufacturing and disposal costs). The objectives are to formulate a joint lot-sizing problem and provide insight for cost efficiency and sustainability. To achieve this, the study analyses two distinct operational scenarios based on remanufacturing capacity requirement: firstly, where remanufacturing rate exceeds demand rate, and secondly, where demand rate exceeds remanufacturing rate. Optimality condition was proven, and sensitivity analysis was done to identify which parameters have the greatest impacts on system performance. The model uniquely integrates in a single model: (i) complementary cross-price effect, (ii) imperfect production with remanufacturing on a shared resource, and (iii) joint lot-sizing of initial and remanufactured products. The model performance was compared to two other base models, and the possible extent of recoverable cost of imperfect quality was shown. Guidance for managerial decision was provided from the model's insights.

Keywords- Economic production quantity, Remanufacturing, Shared resource, Complementary demand.

1. Introduction

Capital-intensive production industries such as steel, stainless steel, or aluminium manufacturing plants, rely on shared (or constrained) resources (e.g., hot-rolling mills, annealing furnaces, cold mills, or cutting lines) for multi-product manufacturing. These products are often complementary to one another; examples include coils and sheets of different product grades, 3CR12 and 3CR12L chemistry or CR (Cold rolled) and NO1 finish products. Technical definition of these products is considered unimportant as these are only examples of general cases. A constrained resource may be manufacturing these complementary products, and some of the products manufactured may be imperfect, necessitating the remanufacturing of the defective products on the same resource. Without any loss of generality, the model presented here considers the case of two products, hereafter product type A and product type B, where A is a prime product and B is a product with less strict requirements. Consequently, when defective product types A or B are made, they are remanufactured into product type B. This remanufacturing process itself is not perfect, so, the remanufactured B itself may contain defective items again, and such items are simply discarded. Jobs on shared resources are usually scheduled as runs. This means if the resource is to produce a number of items (product type A, product type B, and the remanufactured B in this case, without loss of generality), the resource operates in repeated cycles, making type A first, shifting to type B after a changeover, and then to remanufactured B after another changeover. Once the remanufactured B is completed, the cycle run starts again.

This is the case of typical South African stainless-steel (or aluminium) manufacturers, where many resources are shared, and many products are complementary. A typical furnace has defect rate between 12% and 20% for each product made. Causes of such defects are diverse, including surface defects, mechanical property issues, inclusions, or dimensional defects. This results in returning the non-conforming quality products to the shared resource for remanufacturing into another product type. For example, coils with surface defects can be cut into sheets on the slitting lines, or CR products can be remanufactured into NO1 surface quality. The remanufacturing process competes directly with primary production on the shared resource with capacity constraints.

If the plant does not carefully plan for these defective products, there will be a significant increase in cost allocated to work-in-progress and lost time on the shared resource. This problem belongs to the economic lot scheduling (ELSP) problem, and an integrated model can help with joint lot-sizing, reduce unnecessary melting, and decrease carbon emissions through waste reduction. This study develops a mathematical model that addresses such problems by answering the following research questions:

- (i) What are the optimal production lot sizes for two complementary products manufactured on a shared resource with imperfect production and remanufacturing?
- (ii) How do complementary price-dependent demands and remanufacturing rates affect the total cost and inventory behaviour under light and heavy remanufacturing capacity requirement scenarios?
- (iii) What managerial insights can be derived from sensitivity analysis to improve cost efficiency and sustainability in stainless steel manufacturing?

While there have been many models that considered various combinations of the elements described earlier like multi-product scheduling, imperfect production with remanufacturing and price-dependent demand, none has all these features in such integrated manner as required in processes like steel making. For example, Laan and Teunter (2006) presents a remanufacturing models that treats demand as independent, while Mahlangu et al. (2023) excludes the impact of shared resources. Models that focus on shared resource scheduling omit the cross-price effect between complementary products (van Zyl and Adetunji, 2022). Most price-dependent demand models exclude remanufacturing entirely (Mokhtari, 2018; Taleizadeh et al.,

2019). This limits the applicability of all these models in the environment described. By integrating these elements simultaneously, the proposed model serves some real-life situations exemplified by the steel industry described. A detailed comparison of prior studies is provided in **Table 1** (Section 2.5), highlighting the contribution of this paper in the context of ongoing research.

Today's need for environmental sustainability in energy intensive industries (like steel making) provides further motivation for the incorporation of remanufacturing. Remanufacturing significantly reduces material waste, raw material consumption, energy usage and resource utilisation, aligning with heavy manufacturing industry policies.

This research develops an EPQ model for two complementary products, product Type A and product Type B, manufactured on a shared resource, with manufacturing imperfection, and the reuse of imperfect outputs in the remanufacture of the complementary price-dependent product type B. The model minimises the total costs (setup, holding, remanufacturing and disposal), with respect to the products' lot sizes. Two operational scenarios are analysed: where the remanufacturing rate exceeds the demand rate (light capacity requirement) and where demand rate exceeds the remanufacturing rate (heavy capacity requirement). Proof of optimality was provided and numerical examples were used to illustrate the application of the model, with sensitivity analysis done to provide insight for managerial decision-making.

The remainder of the paper is structured as follows. Section 2 presents the literature review. Section 3 describes the system and process flow. Section 4 introduces the notation and model assumptions. Section 5 develops the mathematical model. Section 6 presents the numerical results. Section 7 provides further discussion and managerial insights. Section 8 provides limitations and future developments, and Section 9 concludes the study.

2. Literature Review

2.1 Multi-Product Manufacturing

Multi-product manufacturing on a shared resource is a fundamental area of this study's model. A shared resource manufactures multiple products from feedstock using the same manufacturing process. van Zyl and Adetunji (2022) use these concepts to simplify multi-machine dynamics by focusing on the shared resource's impact rather than the downstream process.

Pirayesh and Poormoaid (2015) investigated a multi-item EPQ model on a shared resource with limited machine capacity. The model is developed from key aspects of Choi and Noble (2000) and incorporates material handling and flow into an EPQ model. Pasandideh et al. (2010) developed a non-linear integer model to solve the limited storage space problem. Taleizadeh et al. (2010) studied the effect of failure rates and rework on the single-machine lot sizing problem, later extending the model to consider multiple batches in Taleizadeh et al. (2011), thereby improving scheduling and planning.

Adetunji et al. (2020) concluded that shared ordering cost leads to valuable cost savings due to distributing fixed costs amongst a larger product group. Large setup costs can be distributed across the larger manufacturing pool, with a smaller setup cost associated with the different process steps. Mahlangu et al. (2023) incorporate joint ordering for complementary products with shared cost, a concept that supports multi-product cost optimisation.

2.2 Economic Lot Scheduling and Joint Manufacturing

The Economic Lot Scheduling Problem (ELSP) and joint manufacturing ensure the elimination of unnecessary high cost through proper scheduling. Elmaghraby (1978) provides a comprehensive review of

ELSP. Beck and Glock (2020) adapted Bomberger's (Bomberger, 1966) assumptions surrounding shared resources, manufacturing one product type at a time, keeping parameters known, with setup being separate from production sequencing and the holding cost dependent on inventory levels.

Nilsson et al. (2007) investigated two strategies associated with joint replenishment problem (JRP); direct grouping and indirect grouping strategies. Brander and Segerstedt (2009) modified the approach by incorporating the untraditional costs of production to find a balance between replenishment and the associated holding cost. Sangari et al. (2024) furthered this literature through extending the common-cycle policy to a finite planning horizon, including sequence-dependent setups, with demand uncertainty addressed through fuzzy set theory. Muckstadt and Isaac (1981) model remanufacturing with a Poisson arrival queuing system. Laan and Salomon (1997) extended their model by incorporating the concept of possible disposals if inventory levels reach a certain level, while Mahadevan et al. (2003) schedule the remanufacturing process only after inspection. Laan and Teunter (2006) incorporated a joint manufacturing model with an Economic Order Quantity model to find a solution for a push-and-pull remanufacturing model while considering two phases: where remanufacturing units are less than the demand, requiring additional manufacturing; and where remanufacturing exceeds demand, reducing first-time manufacturing.

While some of these reviewed studies rely on stochastic queuing framework to manage return uncertainties, our model utilises a deterministic framework where the production and remanufacturing rates are relatively steady on a shared resource. This deterministic approach simplifies the solution procedure by handling the relatively steady cycle stock separately as the deterministic component since the stochastic component can be handled with an appropriate safety stock model.

2.3 EPQ Models with Remanufacturing Processes

Imperfect manufacturing can increase production costs and lost sales. Remanufacturing products can, in many instances, be used to fulfil demand in a cost-saving way without using additional raw materials, however, it might affect the organisation's capacity plan and availability. Rosenblatt and Lee (1986) investigated the impact of imperfect quality on the EPQ model and the effects on production cost. Chiu (2003) expanded on this EPQ model by considering remanufacturing and disposal of non-conforming products to reduce total cost.

An important aspect of the remanufacturing process is when to schedule remanufacturing. Jamal et al. (2004) developed an EPQ model for two separate scenarios: in-cycle remanufacturing (within the same batch), and post-N-cycle remanufacturing (after N cycles). In the second scenario, the remanufacturable products are accumulated with additional holding costs. Laan and Teunter (2006) considered remanufacturing policies of customer returns on a shared resource, discussing the corresponding scheduling and constraint complexities.

Taleizadeh et al. (2010) introduced a stochastic defect rate (uniform and normal distribution) to the rework model, questioning the first-time completion viability. Taleizadeh et al. (2012) further developed an economic production quantity model that considers the rework process on a shared resource to optimise production quantities and the cycle times to minimise cost, emphasising the importance of using non-conforming finished goods within a remanufacturing process to reduce raw material costs. Taleizadeh et al. (2013) altered the model to consider the impact of capacity constraints and incorporates random defect rates and scrap rates. Inspection cost was also included in the production cost. van Zyl and Adetunji (2022) analysed the gap between remanufacturing and yield defects to satisfy both primary and secondary demand (lower-grade products).

Kohlmann and Sahling (2024) investigated a flexible planning approach for integrated lot sizing and rework planning with random proportion of non-conforming products, in turn replacing the assumption of a portion of deterministic defects, while incorporating in-line reworks on a shared resource. Rudert and Buscher (2022) developed heuristics for the single-item dynamic lot-sizing problem with rework of internal returns, shifting the focus from external customer returns to internal returns caused by imperfect production.

2.4 Price Dependent Demand

Demand for a product is dependent, not only on its selling price, but also on the selling prices of complementary or substitute products. Complementary products are usually purchased together. A change in the complementary product's price can alter the main product's demand through the cross-price effect; if one product's price decreases, it could result in the demand for both products increasing.

Karaöz et al. (2011) developed a model that accounts for the effects of time, complementary and substitute products on the economic quantity and total cost. This is one of the first inventory pricing models that considers the impact of the complementary products' selling price. Mokhtari (2018) incorporated substitutional and complementary products into the traditional inventory model. Taleizadeh et al. (2019) modelled the impact of complementary and substitutional products on average profit, exploring four scenarios, two for complementary products and two for substitutional products, where they looked at both deteriorating and non-deteriorating products. The model also introduced a degree of complementarity variability within the demand function.

Mahlangu et al. (2023) developed an EOQ model for deteriorating mutually complementary products with price and time-dependent demand, optimising the cycle time and cost. Durga and Chandrasekaran (2021) examined the effects of substitutional and complementary products on an EPQ model with quadratic demand functions.

2.5 Identified Research Gap

Existing studies fail to address the simultaneous integration of (i) complementary cross pricing, (ii) imperfect production with remanufacturing on a shared resource, and (iii) joint lot-sizing of initial and remanufactured products. Although there is extensive research done on each individual aspect, there is no literature that integrates the concepts simultaneously. To the best of the authors' knowledge, this is the first model to address this complex and integrated problem; providing original solution to problems that exist in certain industries like steel manufacturing with shared resources, fairly steady defect rates and price dependent demand. It develops an EPQ model for two complementary products, product Type A and product Type B manufactured on a shared resource with an integrated remanufacturing cycle, and analyses two operational scenarios, namely, bottleneck and non-bottleneck conditions. The research gap is illustrated in **Table 1**.

Table 1. Gap analysis of related literature studies.

Study	Multi-product / Shared resource	Remanufacturing / Imperfect production	Price-dependent demand / Complementary products	Joint manufacturing and lot scheduling	Gap addressed
van Zyl and Adetunji (2022)	✓	✓		✓	Addresses remanufacturing on a shared resource but does not include complementary price-dependent demand, resulting in suboptimal lot sizes when prices are interdependent.
Pasandideh et al. (2010), Pirayesh and Poormoaiied (2015)	✓				Focuses on multi-product production but excludes remanufacturing and price dependent demand. Defects and cross-price effects aren't accounted for.
Taleizadeh et al. (2012)	✓				Considers imperfect production but does not integrate remanufacturing on shared resources.
Mahlangu et al. (2023)	✓		✓		Incorporates complementary demand and joint ordering, but excludes remanufacturing.
Laan and Teunter (2006)	✓	✓		✓	Considers push-and-pull remanufacturing on a shared resource. Excludes cross-price effect between complementary products.
Mokhtari (2018)			✓		Addresses complementary demand, but excludes remanufacturing on a shared resource.
Taleizadeh et al. (2019)	✓		✓		Models complementary demand across different scenarios, but excludes remanufacturing on a shared resource.
Adetunji et al. (2020)	✓			✓	Cost saving through shared ordering, but excludes remanufacturing and cost-dependent demand.
Rudert and Buscher (2022)		✓		✓	Remanufacturing planning for internal returns on a shared resource, but excludes price dependent demand.
Kohlmann and Sahling (2024)		✓		✓	Remanufacturing planning with random defect portions, but excludes price dependent demand and joint lot sizing of initial remanufacturing products.
Sangari et al. (2024)	✓			✓	Extends common-cycle scheduling with sequence dependent setups, but excludes remanufacturing and price dependent demand
This study	✓	✓	✓	✓	Integrates all areas into one model

3. System Description

The process flow shown in **Figure 1** illustrates the system of the two complementary products, product Type A and Type B (a practical example would be CR Coil and No1 Sheet in steel making) manufactured on a shared resource. The flow diagram is split into sections, namely, input, process and output. The initial feedstock (produced from raw material) flows into a shared resource, producing Type A and Type B products. Inspections determine if these products conform to downstream quality expectations, with non-conforming products remanufactured into Type B products at the shared resource. Unsuccessfully remanufactured products are scrapped.

The sequence of processes on the shared resource is as follows:

- (i) Upstream feedstock enters the shared resource.
- (ii) The shared resource sequentially manufactures Product Type A and Product Type B at their respective production rates.
- (iii) Inspection identifies conforming and non-conforming products.
- (iv) All conforming products are transferred to finished goods to meet demand.
- (v) Non-conforming products of both Type A and Type B are transferred to the remanufacturable stockpile.
- (vi) After the manufacturing cycle, the line starts with the remanufacturing process for product type B.
- (vii) Successfully remanufactured products are transferred to finished goods for Type B (dashed arrow to solid Stock Type B).
- (viii) Unrecoverable items are scrapped.

The diagram shows that Type A can only be manufactured from upstream feedstock, while Type B can be satisfied by both upstream feedstock and remanufacturable stock, much like a stainless steel finishing line.

The system consists of four stock points:

- **Feedstock inventory** - the raw material received from upstream processes. It is assumed to be available when a new production cycle starts.
- **Remanufacturable stock** - the non-conforming products identified during the production of both Type A and Type B, which is accumulated during the initial manufacturing and depleted during remanufacturing.
- **Finished Type A inventory** - accumulated during the production of Type A product and depleted at a constant demand rate for Type A. Because Type A can only be manufactured from the initial feedstock, Product Type A will follow a saw-tooth inventory pattern.
- **Finished Type B inventory** - fed by both original production and successful remanufacturing, resulting in two accumulation phases.

Type A inventory increases linearly during its manufacturing phase, at the end of which the inventory decreases at a constant demand rate, resulting in the saw-tooth pattern. Type B inventory increases linearly during its manufacturing phase, drops during remanufacturing changeover, and increases again during the remanufacturing phase, resulting in a two-phase build-up. The remanufacturable stockpile grows as defects are accumulated from the two manufacturing phases, and gets used up during the remanufacturing phase of Type B. Scrap inventory grows at a constant rate from unsuccessfully remanufactured type B. Once the remanufacturing run is complete, the inventory level for Type B decreases at a constant demand rate. The full cycle then starts again.

A practical example of the system described is stainless steel manufacturing. The production run switches between CR coils (Type A) and No1 sheets (Type B), both made from upstream hot-rolled coils. Additionally, a portion of No1 sheets can be recovered, or non-conforming CR coils can be cut into No1 sheets (Type B), but No1 sheets won't be able to satisfy CR coil (Type A) demands. Between 12% and 20% of coils have surface defects like scratches, inclusions or skin lamination. This results in the material not being suitable for Type A demand. A proportion of these defective coils can be successfully recovered by cutting them into sheets (Type B). Unrecovered non-conforming stocks are scrapped. Customers usually order both CR coils and No1 sheets simultaneously for their projects, resulting in these products being mutually complementary to one another.

The possibility of remanufacturing defects from Type B into Type A is not considered because of Type A's stricter quality standards. Non-conforming products from Type B cannot satisfy the higher surface, mechanical, and dimensional requirements of Type A. In the stainless-steel industry, CR coils (Type A) demand superior quality compared to No1 sheets (Type B), making the flow strictly one-directional: hence non-conforming material from both Type A and Type B can only be remanufactured into Type B.

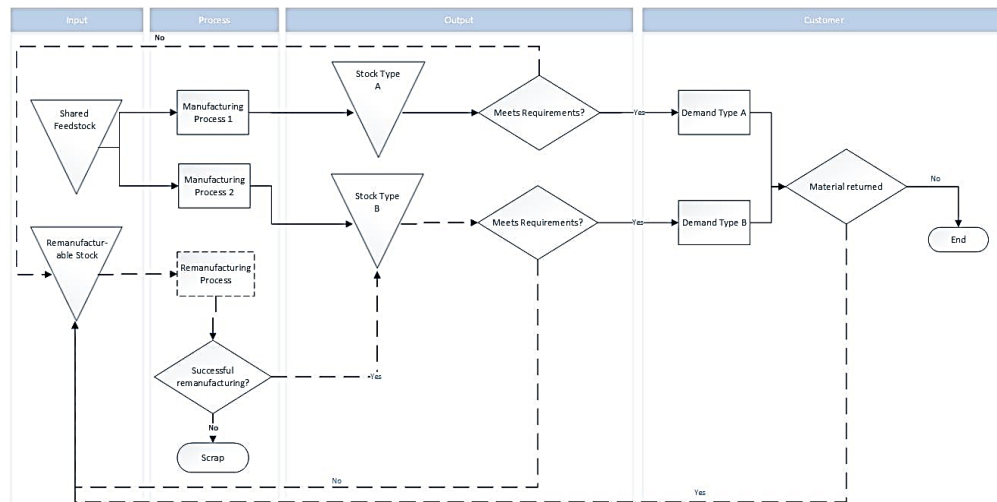


Figure 1. Flow diagram for multi-products through a shared resource and the respective processes to produce Type A and Type B products.

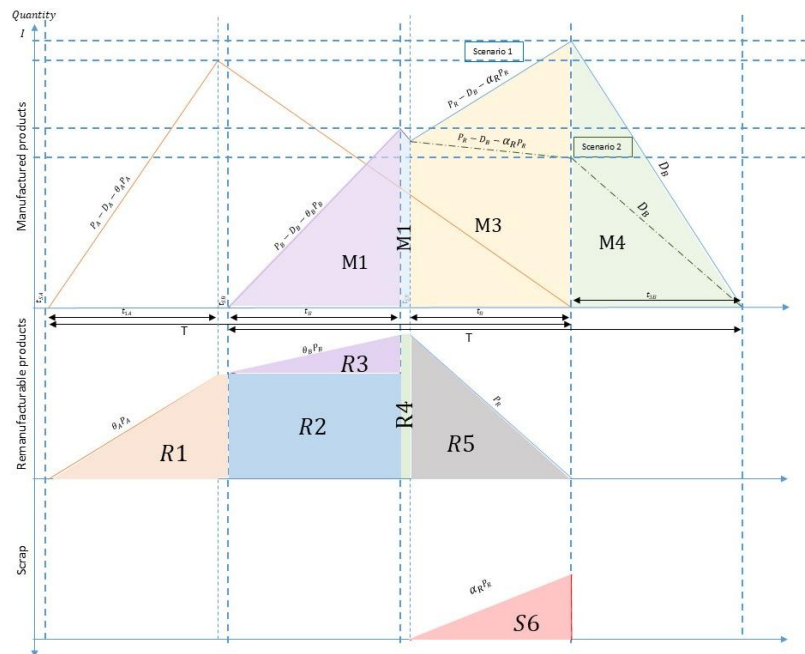


Figure 2. Quantity versus time graph manufacturing, remanufacturing and remanufacturable stocks for Type A and Type B products.

Two operational scenarios are examined:

- **Scenario 1:** Remanufacturing rate exceeds the rate of demand – where salvaging defective items requires relatively smaller reprocessing capacity, hence, many defective products can be remanufactured per unit time;
- **Scenario 2:** Rate of demand exceeds the remanufacturing rate - salvaging defective items requires large reprocessing capacity, hence, fewer defective products can be remanufactured per unit time.

4. Notation

The decision variables used in the model are listed in **Table 2**, and the parameters used in the model are listed in **Table 3**.

Table 2. Decision variables.

Variables	Description	Units
T	Cycle time of Process	time
Q_A, Q_B, Q_R	Lot size of manufactured Product Type A and Type B, and remanufactured product Type B respectively	Units (tons)

Table 3. Parameters.

Parameter	Description	Units
k_A, k_B, k_R	Setup (or change over) cost for manufacturing of product Type A, Type B, and remanufacturing of Type B respectively	Money/setup
P_A, P_B, P_R	Production rate of product Type A, Type B, and remanufactured B on a shared (constrained) resource, respectively	Units/time
t_{SA}, t_{SB}, t_{SR}	Setup time for the remanufacturing process of product Type A, Type B, and remanufactured B on a shared (constrained) resource, respectively	time
θ_A, θ_B	Proportion of product Type A and Type B, respectively, that requires remanufacturing	fraction
α_R	Proportion of unsuccessful remanufactured products that is scrapped	fraction
h_A, h_B, h_R, h_s	Holding cost rate of manufactured product Type A and Type B, remanufactured and scrap products, respectively	Money/units/time
d_R	Unit disposal cost of unsuccessfully remanufactured Type B	Money/units
r_A, r_B	Unit remanufacturing cost of product Type A and Type B, respectively	Money/units
p_A, p_B	Selling price of product Type A and Type B, respectively	Money/units
β_i, γ_i	Scaling parameters for demand ($i \in \{1, 2\}$), where $\beta_1, \gamma_2 > 0$ and $\beta_2, \gamma_1 \geq 0$	Money/units
D_A, D_B	Demand for product Type A and Type B, respectively	units/time
t_A, t_B, t_R	Production run time for product Type A, Type B, and remanufactured B respectively	time
$I_A, I_{B,i}$	Inventory levels for product Type A, and for Type B at different points of the production cycle ($i \in \{1, 2, 3\}$)	units
$I_{R,i}$	Inventory levels of non-conforming products waiting to be remanufactured at different points of the production cycle ($i \in \{1, 2\}$)	units
I_s	Inventory levels of unrecoverable scrap	units
T	Common cycle time for product Type A and product Type B	time

Lot sizes Q_A, Q_B, Q_R are interdependent on one another because they are all produced on a shared resource and the defects from Q_A and Q_B feed into the remanufacturing flow for Q_R . Consequently, only one of these variables would be the decision variable, and the other two become dependent on that. It is, therefore, important to choose one of them judiciously.

5. Assumptions

The assumptions made in this model were either observed within the steel manufacturing industry or adopted from some standard simplification logic identified within the literature to ensure model tractability without compromising the model's relevance within the industry context.

5.1 Manufacturing

- Manufacturing, or remanufacturing, happens on a shared resource, one process at a time, in a fixed sequence: manufacturing of product Type A, followed by product Type B, then remanufacturing of non-conforming products into product Type B thereafter.
- Production rates for product Type A (P_A), product Type B (P_B), and the remanufacturing rate (P_R) are known and constant. The model assumes stable conditions and results in predictable cycle times.
- Demand for product Type A can only be met by production from new feedstock.
- The quality of remanufactured products is acceptable for Type B demand standards, hence, demand for product Type B can be met by both production from new feedstock and remanufacturing of non-conforming products ($D_B T = Q_B \theta_B + Q_R \alpha_R$).
- All feedstock items meet production quality requirements.
- The defect rate θ and scrap rate α are fixed and known variables (see **Table 4** in Section 7.1 for baseline values).

5.2 Time

- There is a common cycle time for product Type A and Type B, and the cycle time for Type B includes its normal manufacturing and remanufacturing times.

5.3 Demand

- Demands for Product Type A and Product Type B are fully satisfied and arrive at constant rates. The model does not allow shortages.
- Demands for Product Type A and Type B are mutually complementary and price-dependent, and the dependency is well understood and represented by the demand function.
- Products are non-perishable and will not lose value due to time spent in storage, consistent with the durability of stainless steel.

5.4 Others

- Tax, inflation and shipping costs are not considered.
- Cost parameters are known and constant.
- There is no limitation on storage capacity for the different stock items.
- There are no customer returns; all customers are satisfied with the products they purchase.

Stochastic variations in demand, defect rates, production rate and setup times (amongst others) are not considered. Small variations in demand and other model parameters can be independently managed through a relevant safety stock model, which is not part of the problem considered in this research (see Zipkin, 2000, p.74 for different levels of variations in inventory systems and how they should be handled for modelling purposes). The defect rate θ and scrap rate α are considered fixed and known variables derived from historical data (calibrated base parameters are provided in **Table 4**, Section 7.1) and taken as 12%–20% based on steel industry expectations.

5.5 Justification of Methodology

The EPQ framework with analytical optimisation is used for this study because the system, generally, can be reasonably represented as deterministic with known and constant parameters. This allows for the derivation of the expressions for lot sizes and cycle times, enabling a global optimality search through Fermat's theorem, second-derivative test, boundary analysis and grid search. The conditions for optimality were presented, and it was shown that this is readily achieved under the operating conditions.

6. Model Formulation

The total cost of manufacturing in a cycle (1) consists of four cost components:

$$\text{Total cost} = \text{setup cost} + \text{holding cost} + \text{rework (remanufacturing) cost} + \text{disposal cost} \quad (1)$$

$$\text{manufacturing setup cost} = k_A + k_B + k_R \quad (2)$$

$$\text{rework (remanufacturing) cost} = r_A \theta_A Q_A + r_B \theta_B Q_B \quad (3)$$

$$\text{disposal cost} = d_R \alpha_R Q_R \quad (4)$$

The holding cost is derived from the inventory behaviour shown in **Figure 2**. The derivations formulate average inventory levels for product Type A (saw-tooth pattern), Type B (two-phase build-up), remanufacturable stock (triangular accumulation) and scrap inventory as presented in section 6.4.

6.1 Production Time Equations

The rate of manufacturing and remanufacturing can be used to determine the time of each process:

$$t_A = \frac{Q_A}{P_A} \quad (5)$$

$$t_B = \frac{Q_B}{P_B} \quad (6)$$

$$t_R = \frac{Q_R}{P_R} \quad (7)$$

The solution is feasible if the sum of all production and setup times is not greater than the cycle time, hence we have

$$t_{SA} + t_A + t_{SB} + t_B + t_{SR} + t_R \leq T \quad (8)$$

6.2 Demand Equations

The demand for Product Type A can be satisfied through the manufactured Type A produced minus the remanufacturing portion:

$$Q_A(1 - \theta_A) = D_A T \quad (9)$$

The demand for Product Type B can be satisfied through manufactured Type B products less the defective portion, plus the remanufactured portion less the scrap portion:

$$Q_B(1 - \theta_B) + Q_R(1 - \alpha_R) = D_B T \quad (10)$$

$$Q_R = \theta_A Q_A + \theta_B Q_B \quad (11)$$

Demand rates D_A and D_B incorporate the complementary cross-price effect through fixed selling prices, where a price reduction in one product raises demand for both. The functions are represented as $D_A = \alpha_1 - \beta_1 p_A - \gamma_2 p_B$ and $D_B = \alpha_2 - \beta_2 p_B - \gamma_1 p_A$, where the positive cross-price coefficients γ_2 and γ_1 reflect that a price reduction in one of the complementary products results in a lower overall project cost, raising the demand for both complementary products. However, selling prices p_A and p_B are treated as jointly fixed parameters, resulting in a price-dependent but constant demand function for each set of fixed prices for the two products. This is reasonable because only the price effect on demand (captured in the demand functions), and not the dynamic pricing structure, is being modelled in this problem. Hence, fixing the prices suffices for the modelling purpose.

We now proceed to derive the cost rate per unit time (annual), starting with the setup cost.

6.3 Setup Cost

There is a single manufacturing cycle for product types A and B, and a remanufacturing cycle for type B in each cycle run. Hence, the total set up cost in a cycle is Equation (2). To obtain the annual setup cost, we divide this by the cycle time. Since the cycle time is common for product types A and B, the cycle time can be obtained from either. It is apparent from the inventory structure that product type A will be a better choice for simplicity, hence T can be obtained from Equation (9) as $Q_A(1 - \theta_A)/D_A$, and we have the annual

$$\text{set up cost as} \quad \frac{D_A(K_A + K_B + K_R)}{Q_A(1 - \theta_A)} \quad (12)$$

6.4 Holding Cost

The annual holding cost for product type A can be calculated by multiplying the holding cost, h_A , with the area of the corresponding triangle in **Figure 2**, whose base is T and the height (the maximum stock level, I_A) is the rate of stock growth ($P_A - D_A - P_A\theta_A$), multiplied by the manufacturing time for product type A, Equation (5) divided by the cycle time, T . Substituting all these, we have

$$\frac{Q_A}{2}((1 - \theta_A) - \rho_A)h_A \quad (13)$$

where, ρ_A is the ratio of the demand rate, D_A , to the production rate, P_A .

The annual holding cost for B is more complicated; we find the area of the shape of the stock positions for B, which is the shaded regions in the manufactured products of **Figure 2** (topmost), and multiply by the unit holding cost, h_B . To calculate this area, we divide the shape into the constituent triangles and trapezia (labelled areas in the shape). For triangle M1, the area is $\frac{1}{2}t_B I_{B1}$, where I_{B1} is the height and is the product of $P_B - D_B - P_B\theta_B$ and t_B . The base is also t_B , which is Equation (6). Substituting these into the area and multiplying by the unit holding cost, h_B , the aggregate holding cost is

$$\frac{Q_B h_B}{2P_B} [Q_B[(1 - \theta_B) - \rho_B]] \quad (14)$$

where, ρ_B is the ratio of the demand rate, D_B , to the production rate, P_B . In similar manner, the aggregate holding cost for M2 is obtained from $\frac{1}{2}t_{SR}(I_{B1} + I_{B2})h_B$ as

$$t_{SB} h_B [Q_B[(1 - \theta_B) - \rho_B]] \quad (15)$$

The aggregate holding cost for M3 is obtained from $\frac{1}{2}(t_R)(I_{B3} + I_{B2})h_B$ as

$$\begin{aligned} & \frac{Q_A^2 \theta_A^2 h_B}{2P_R} [(1 - \alpha_R) - \rho_R] + \frac{Q_A \theta_A h_B}{2P_A} [Q_B[(1 - \theta_B) - \rho_B] + Q_B \theta_B [(1 - \alpha_R) - \rho_R] - \frac{D_B t_{SR}}{2}] \\ & + \frac{Q_B \theta_B h_B}{P_R} [Q_B[(1 - \theta_B) - \rho_B] + \frac{Q_B \theta_B}{2} [(1 - \alpha_R) - \rho_R] - \frac{D_B t_{SR}}{2}] \end{aligned} \quad (16)$$

where, ρ_R is the ratio of the demand rate, D_B , to the production rate, P_R ; and Q_R is given by Equation (11).

Similarly, the aggregate holding cost for M4 is obtained from $\frac{1}{2}t_{3B}(I_{B3})h_B$ as

$$\frac{h_B}{2D_B} [Q_B[(1 - \theta_B) - \rho_B] + \frac{(Q_A \theta_A + Q_B \theta_B)}{P_R} [(1 - \alpha_R) - \rho_R] - D_B t_{SR}]^2 \quad (17)$$

To find the holding cost for the remanufacturable and scrap items, we use the area under **Figure 3**, decomposing it in a similar manner to that of product type B, and multiplying each with the appropriate unit holding cost.

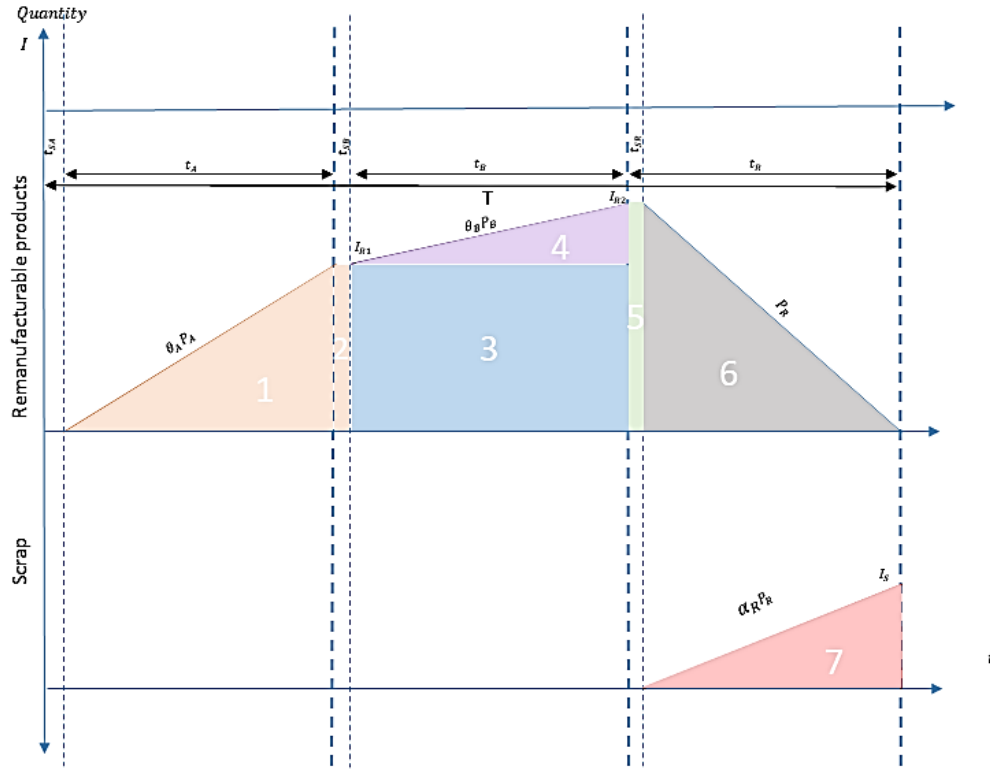


Figure 3. Area of inventory for remanufacturable products and scrap.

For R1, the area is $\frac{1}{2}I_{R1}t_Ah_R$, where I_{R1} is $\theta_A Q_A$, and t_A is given by Equation (5), hence, the aggregate holding cost is

$$\frac{\theta Q_A^2 h_R}{2P_A} \quad (18)$$

The aggregate holding cost under R2 is $I_{R1}t_{SB}h_R$, hence, we have

$$Q_A \theta_A t_{SR} h_R \quad (19)$$

The aggregate holding cost under R3 is $I_{R1}t_Bh_R$, where t_B is given in Equation (6), hence, we have

$$\frac{Q_A Q_B \theta_B h_R}{2P_B} \quad (20)$$

The aggregate holding cost under R4 is $\frac{1}{2}(I_{R2} - I_{R1})t_Bh_R$, hence,

$$\frac{\theta_B Q_B^2 h_R}{2P_B} \quad (21)$$

The aggregate holding cost under R5 is $I_{R2} t_{SR} h_R$, where I_{R2} is $\theta_A Q_A + \theta_B Q_B$, hence,

$$(\theta_A Q_A + \theta_B Q_B) t_{SR} h_R \quad (22)$$

The aggregate holding cost under R6 is $\frac{1}{2} (I_{R2}) t_R h_R$, with t_R given in Equation (7), hence,

$$\left(Q_A^2 \left(\frac{\theta_A}{P_R} \right)^2 + Q_A \left(\frac{\theta_A \theta_B Q_B}{P_R} \right) + Q_B^2 \left(\frac{\theta_B}{P_R} \right)^2 \right) h_R \quad (23)$$

The aggregate scrap holding cost is found from R7 as $\frac{1}{2} (I_S) t_R h_S$, where I_S is obtained from

$$(\theta_A Q_A + \theta_B Q_B) \alpha_R, \text{ and we have} \quad (24)$$

$$\frac{\alpha_R h_S}{2 P_R} [Q_A^2 \theta_A^2 + Q_B^2 \theta_B^2 + Q_A (2 \theta_A \theta_B Q_B)]$$

To find the annual average holding cost for product type B and the scrap items, we add all the relevant aggregate holding costs and divide by the cycle time, $T = Q_A (1 - \theta_A) / D_A$. Since the cycle time is defined in terms of product type A, we collect all terms of the holding costs in terms of the powers of Q_A . Hence, collecting terms in Q_A^2 and dividing by T , we have

$$\frac{Q_A D_A}{2(1-\theta_A)} \left[\theta_A h_R \left[\frac{1}{P_A} + \frac{\theta_A}{P_R} \left(\frac{2}{P_R} + \alpha_R \right) \right] + \theta_A^2 h_B [(1 - \alpha_R) - \rho_R] \left[\frac{[(1-\alpha_R)-\rho_R]}{D_B} + \frac{1}{P_R} \right] \right] \quad (25)$$

Collecting terms in Q_A and dividing by T , we have

$$\frac{D_A}{(1-\theta_A)} \left[\theta_A t_{SR} \left[2h_R - \frac{\rho_R h_B}{2} \right] + \theta_A Q_B \left[\frac{h_R}{P_B} + \frac{\alpha_R \theta_A h_S}{P_R} \right] + h_B [(1 - \theta_B) - \rho_B] \left[\frac{1}{P_R} + \frac{2[(1-\alpha_R)-\rho_R]}{D_B} \right] + \right. \\ \left. h_B \theta_A \theta_B Q_B [(1 - \alpha_R) - \rho_R] \left[\frac{1}{P_R} + \frac{2[(1-\alpha_R)-\rho_R]}{D_B} \right] \right] \quad (26)$$

Collecting terms without Q_A and dividing by T , we have

$$\frac{D_A}{Q_A(1-\theta_A)} \left[\left[h_B \left[[(1 - \theta_B) - \rho_B] \left[\frac{1}{2D_B} + \frac{\theta_B}{P_R} \right] + \frac{\theta_B}{2} [(1 - \alpha_R) - \rho_R] \left[\frac{1}{D_B} + \frac{\theta_B}{P_R} \right] + \frac{\theta_B h_R}{P_R} \left[\frac{1}{2} + \frac{\theta}{P_R} \right] + \frac{\alpha_R \theta_B^2 h_S}{2 P_R} \right] + \right. \right. \\ \left. \left. Q_B \left[h_B \left[[(1 - \theta_B) - \rho_B] \left[\frac{\theta_B}{2 P_B} + t_{SR} \right] + \theta_B t_{SR} \left[h_R + \frac{\theta_B D_B}{2 P_R} \right] \right] \right] + \frac{t_{SR} h_B}{2} [D_B t_{SR} + [(1 - \theta_B) - \rho_B] + [(1 - \right. \right. \\ \left. \left. \alpha_R) - \rho_R] \right] \right] \right] \quad (27)$$

6.5 Remanufacturing Cost

The annual cost of remanufacturing can be calculated from the total number remanufactured items in a cycle multiplied by the unit manufacturing cost, divided by the cycle time, which is

$$\frac{r_A \theta_A D_A}{(1-\theta_A)} + \frac{r_B \theta_B Q_B D_A}{Q_A (1-\theta_A)} \quad (28)$$

6.6 Disposal Cost

The annual disposal cost is calculated in a similar manner as

$$\frac{d_R \alpha_R \theta_A D_A}{(1-\theta_A)} + \frac{d_R \alpha_R \theta_B Q_B D_A}{Q_A (1-\theta_A)} \quad (29)$$

6.7 Total Cost

The total annual cost rate function is obtained by adding the setup cost (12) to the holding costs (13), (25), (26) and (27), the remanufacturing cost (28) and the disposal cost (29). Hence, we have

$$\begin{aligned} & \frac{Q_A D_A}{2(1-\theta_A)} \left[\theta_A h_R \left[\frac{1}{P_A} + \frac{\theta_A}{P_R} \left(\frac{2}{P_R} + \alpha_R \right) \right] + \theta_A^2 h_B [(1-\alpha_R) - \rho_R] \left[\frac{[(1-\alpha_R)-\rho_R]}{D_B} + \frac{1}{P_R} \right] + \frac{D_A}{(1-\theta_A)} \left[\theta_A t_{SR} [2h_R - \right. \right. \\ & \left. \left. \frac{\rho_R h_B}{2}] + \theta_A Q_B \left[\left[\frac{h_R}{P_B} + \frac{\alpha_R \theta_A h_S}{P_R} \right] + h_B [(1-\theta_B) - \rho_B] \right] \left[\frac{1}{P_R} + \frac{2[(1-\alpha_R)-\rho_R]}{D_B} \right] + h_B \theta_A \theta_B Q_B [(1-\alpha_R) - \right. \right. \\ & \left. \left. \rho_R] \left[\frac{1}{P_R} + \frac{2[(1-\alpha_R)-\rho_R]}{D_B} \right] + \theta_A (r_A + d_R \alpha_R) \right] + \frac{D_A}{Q_A (1-\theta_A)} \left[\left[h_B \left[[(1-\theta_B) - \rho_B] \left[\frac{1}{2D_B} + \frac{\theta_B}{P_R} \right] + \frac{\theta_B}{2} [(1-\right. \right. \right. \right. \\ & \left. \left. \left. \alpha_R) - \rho_R] \left[\frac{1}{D_B} + \frac{\theta_B}{P_R} \right] \right] + \frac{\theta_B h_R}{P_R} \left[\frac{1}{2} + \frac{\theta}{P_R} \right] + \frac{\alpha_R \theta_B^2 h_S}{2P_R} \right] + Q_B \left[h_B \left[[(1-\theta_B) - \rho_B] \left[\frac{\theta_B}{2P_B} + t_{SR} \right] + \theta_B t_{SR} [h_R + \right. \right. \right. \\ & \left. \left. \left. \frac{\theta_B D_B}{2P_R} \right] \right] + \frac{t_{SR} h_B}{2} [D_B t_{SR} + [(1-\theta_B) - \rho_B] + [(1-\alpha_R) - \rho_R]] + \theta_B Q_B (r_B + d_R \alpha_R) \right] \end{aligned} \quad (30)$$

6.8 Optimisation

Since the cycle time has been defined in terms of Q_A , this was selected as our decision variable. The optimum value of Q_A was found by differentiating the total cost rate function with respect to Q_A and setting this to zero, yielding

$$Q_A^* = \sqrt{\frac{2 \left[h_B \left[[(1-\theta_B) - \rho_B] \left[\frac{1}{2D_B} + \frac{\theta_B}{P_R} \right] + \frac{\theta_B}{2} [(1-\alpha_R) - \rho_R] \left[\frac{1}{D_B} + \frac{\theta_B}{P_R} \right] + \frac{\theta_B h_R}{P_R} \left[\frac{1}{2} + \frac{\theta}{P_R} \right] + \frac{\alpha_R \theta_B^2 h_S}{2P_R} \right] + Q_B \left[h_B \left[[(1-\theta_B) - \rho_B] \left[\frac{\theta_B}{2P_B} + t_{SR} \right] + \theta_B t_{SR} [h_R + \frac{\theta_B D_B}{2P_R}] \right] + \frac{t_{SR} h_B}{2} [D_B t_{SR} + [(1-\theta_B) - \rho_B] + [(1-\alpha_R) - \rho_R]] + \theta_B Q_B (r_B + d_R \alpha_R) \right]}{\theta_A h_R \left[\frac{1}{P_A} + \frac{\theta_A}{P_R} \left(\frac{2}{P_R} + \alpha_R \right) \right] + \theta_A^2 h_B [(1-\alpha_R) - \rho_R] \left[\frac{[(1-\alpha_R)-\rho_R]}{D_B} + \frac{1}{P_R} \right]} \quad (31)$$

6.9 Proof of Optimality

To prove the optimality of the equation, it suffices to check that the second derivative of the cost function is positive (semi)definite for the function of Q_A to be a minimum. This expression is shown in Equation (32).

$$\begin{aligned} & \frac{2D_A}{Q_A^3 (1-\theta_A)} \left[\left[h_B \left[[(1-\theta_B) - \rho_B] \left[\frac{1}{2D_B} + \frac{\theta_B}{P_R} \right] + \frac{\theta_B}{2} [(1-\alpha_R) - \rho_R] \left[\frac{1}{D_B} + \frac{\theta_B}{P_R} \right] + \frac{\theta_B h_R}{P_R} \left[\frac{1}{2} + \frac{\theta}{P_R} \right] + \frac{\alpha_R \theta_B^2 h_S}{2P_R} \right] + \right. \right. \\ & \left. \left. Q_B \left[h_B \left[[(1-\theta_B) - \rho_B] \left[\frac{\theta_B}{2P_B} + t_{SR} \right] + \theta_B t_{SR} [h_R + \frac{\theta_B D_B}{2P_R}] \right] + \frac{t_{SR} h_B}{2} [D_B t_{SR} + [(1-\theta_B) - \rho_B] + [(1-\right. \right. \right. \right. \\ & \left. \left. \left. \alpha_R) - \rho_R] \right] + \theta_B Q_B (r_B + d_R \alpha_R) \right] \end{aligned} \quad (32)$$

Since all the parameters involved are positive, it is apparent that this expression is positive for as long as $(1 - \theta_A)$, $[(1 - \theta_B) - \rho_B]$ and $[(1 - \alpha_R) - \rho_R]$ are positive. This is not difficult to achieve because the proportions of defective items manufactured are not expected to be close to 1, else, the process is incapable of producing good products. Also, it is expected that for the manufacturing of type A, type B and remanufacturing of type B products in a single production run, the corresponding ρ values cannot be close to 1, else, the process will be infeasible because there is no way the system can complete the three products with their necessary changeover if any of the manufacturing or remanufacturing processes alone has taken all the capacity available. Since all other terms are positive, one can expect that the overall expression has to be positive definite.

6.10 Batch Size for Product Type B

From the Q_A derived, we can derive Q_B by making T the subject from both Equations (9) and (10), equating the two expressions, substituting Equation (11) for Q_R , and simplifying. This leads to

$$Q_B = \frac{Q_A \left[(1 - \theta_A) \frac{D_B}{D_A} - \theta_A (1 - \alpha_R) \right]}{(1 - \theta_B \alpha_R)} \quad (33)$$

7. Solution Algorithm

Solution to the problem can be found by exploiting the structure of the variables to design an algorithm. It is obvious that the control variables, Q_A , Q_B and Q_R are not independent. It can also be seen that Q_B is contained in Equation (31), hence, the solution needs to be found iteratively for Q_A in terms of Q_B . This was solved using marginal analysis, by gradually incrementing the value of Q_B till when the turning point was found. Because they are all jointly determined through the relationship of remanufacturing lot size Equation (11), knowing one of them helps us to know the others, and to know the common cycle time T , hence, this relationship is explored for optimisation purpose. The algorithm is presented as follows:

Step 1: Use Equation (31) to determine the optimal value of Q_A by iterating over the values of Q_B

Step 2: Use Equation (9) to find T

Step 3: Use Equations (33) and (11) to find Q_B and Q_R

Step 4: Use Equations (5), (6), (7) and (8) to check the validity of T

If (8) is true, keep T from step 2, and keep Q_B and Q_R from step 3

Else, use (8) to reset T

Recalculate Q_A from the new T using Equation (9)

Use Equations (33) and (11) to recalculate Q_B and Q_R

Step 5: Use Equation (30) to find the optimal cost

7.1 Numerical Examples

This section demonstrates the use of the equations and the solution algorithm in solving a problem. In the first part, the parameters to be used in the numerical example are presented, followed by the solution of the sample problem. Two scenarios were presented in this illustrative example: the first is where the remanufacturing rate is greater than the demand rate ($P_R > D_B$) and the second where the remanufacturing rate is less than the demand rate ($P_R < D_B$). The section concludes with sensitivity analysis.

7.2 Parameter Values

The base parameters used are shown in **Table 4**. Some values were true values based on the industrial process (e.g. the defect rates), while others (e.g. financial and management information like, cost, price and demand) are partly scrambled because of confidentiality restrictions. The prices of the products are assumed to have included the cross-price effect of the complementary demand, with the price of product Type A set

at R55000/ton and type B at R45000/ton. The base value of each parameter represents the original value before variations for sensitivity analyses.

Table 4. Base parameter values for numerical examples.

Parameter	Description	Base value	Units
P_A	Production rate of Type A	125000	tons/year
P_B	Production rate of Type B	160000	tons/year
P_R	Remanufacturing rate	190000 (Scenario 1), 20000 (Scenario 2)	tons/year
D_A	Demand for Type A	50000	tons/year
D_B	Demand for Type B	30000	tons/year
θ_A	Defect rate of Type A	0.12	Fraction
θ_B	Defect rate of Type B	0.15	Fraction
α_R	Scrap rate of remanufacturing	0.2	Fraction
h_A	Holding cost of Type A	11000	R/tons/year
h_B	Holding cost of Type B	9000	R/tons/year
h_R	Holding cost of remanufacturable items	7000	R/tons/year
h_s	Holding cost of scrap	1500	R/tons/year
k_A	Setup cost for product type A	120000	R/setup
k_B	Setup cost for product type B	90000	R/setup
r_A	Remanufacturing cost of Type A defects	2750	R/tons
r_B	Remanufacturing cost of Type B defects	2250	R/tons
d_R	Disposal cost for scrap	1000	R/tons
t_{SA}	Set up time for product type A	0.003	year
t_{SB}	Set up time for product type B	0.002	year
t_{SR}	Set up time for product type remanufacturing	0.003	year

7.2.1 Scenario 1:

$$P_R > D_B$$

Using base parameters with $P_R = 190000 > D_B = 30000$, the optimal values are calculated to be ($Q_A^* = 1716.41$ tons), ($Q_B^* = 764.42$ tons), ($Q_R^* = 320.63$ tons), ($T^* = 0.0302$ years), with $TC = R43545036/\text{year}$.

7.2.2 Scenario 2:

$$P_R < D_B$$

Using base parameters with $P_R = 20000 < D_B = 30000$, the optimal values are calculated to be ($Q_A^* = 1637.93$ tons), ($Q_B^* = 729.47$ tons), ($Q_R^* = 305.97$ tons), ($T^* = 0.0288$ years), with $TC = R44146334/\text{year}$.

The graph of the total cost (TC) function against Q_A and the value of the derivative of the function at the optimal point for scenario 1 is shown in **Figure 4**. We see that the algorithm finds the minimum cost, and the derivative at the point is zero. It is interesting to note that the cost difference between the two scenarios is only R601298 per year, which is about 1.34 percent increase from scenario 1.

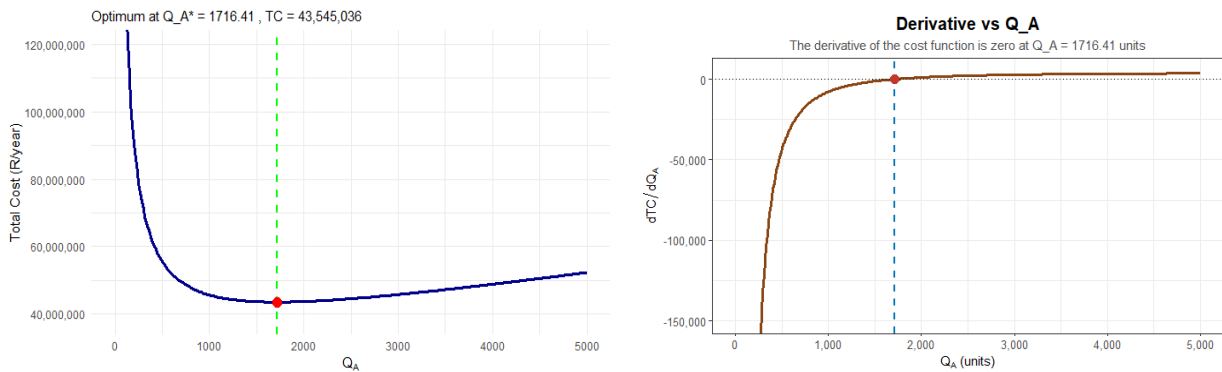


Figure 4. Graph of the total cost function and its derivative against Q_A for scenario 1.

7.3 Sensitivity Analysis

Sensitivity analysis was done on the model output to evaluate its robustness and identify the parameters' influence on total cost and optimal production quantities. Each parameter was varied between 50% below and above the base value while keeping other parameters constant and the change on TC, Q_A , Q_B and Q_R divided by the base value was recorded. The result is shown in **Tables 5 to 12**, where the top five parameters for each analysis are shown, and **Figures 5 and 6** where all the parameters are plotted.

Scenario 1: Top Five Most Influential Parameters (Percentage Changes).

Table 5. Most influential for total cost.

Parameter	ΔTC (%)
θ_A	-24.11 to 24.69
D_A	-24.11 to 20.07
r_A	-21.53 to 21.53
D_B	-16.34 to 15.63
θ_B	-10.68 to 11.06

Table 6. Most influential for Q_B .

Parameter	ΔQ_B (%)
D_B	-56.63 to 49.3
P_A	37.73 to -7.06
D_A	29.47 to -8.45
h_A	21.63 to -13.09
K_A	-15.39 to 13.32

Table 7. Most influential for Q_A .

Parameter	ΔQ_A (%)
D_A	41.74 to 54.49
P_A	37.73 to -7.06
h_A	21.63 to -13.09
K_A	-15.39 to 13.32
k_B	-11.29 to 10.14

Table 8. Most influential for QR.

Parameter	ΔQR (%)
θ_A	-31.93 to 38.32
D_A	-16.27 to 31.98
P_A	37.73 to -7.06
h_A	21.63 to -13.09
θ_B	-18.85 to 19.63

Scenario 2: Top Five Most Influential Parameters (Percentage Changes).

Table 9. Most influential for total cost.

Parameter	ΔTC (%)
θ_A	-20.65 to 23.37
D_A	-22.87 to 19.03
r_A	-21.24 to 21.24
D_B	-16.95 to 16.11
θ_B	-11.36 to 11.5

Table 10. Most influential for QB.

Parameter	ΔQB (%)
D_B	-55.44 to 46.08
D_A	35.67 to -6.31
P_A	33.1 to -6.55
h_A	19.11 to -12.13
K_A	-15.39 to 13.32

Table 11. Most influential for QA.

Parameter	ΔQA (%)
D_A	-38.95 to 58.1
P_A	33.1 to -6.55
h_A	19.11 to -12.13
D_B	14.58 to -9.33
θ_A	-10.2 to 13.9

Table 12. Most influential for QR.

Parameter	ΔQR (%)
θ_A	-31.93 to 38.32
D_A	-16.27 to 31.98
P_A	37.73 to -7.06
h_A	19.11 to -12.13
k_A	-18.85 to 19.63

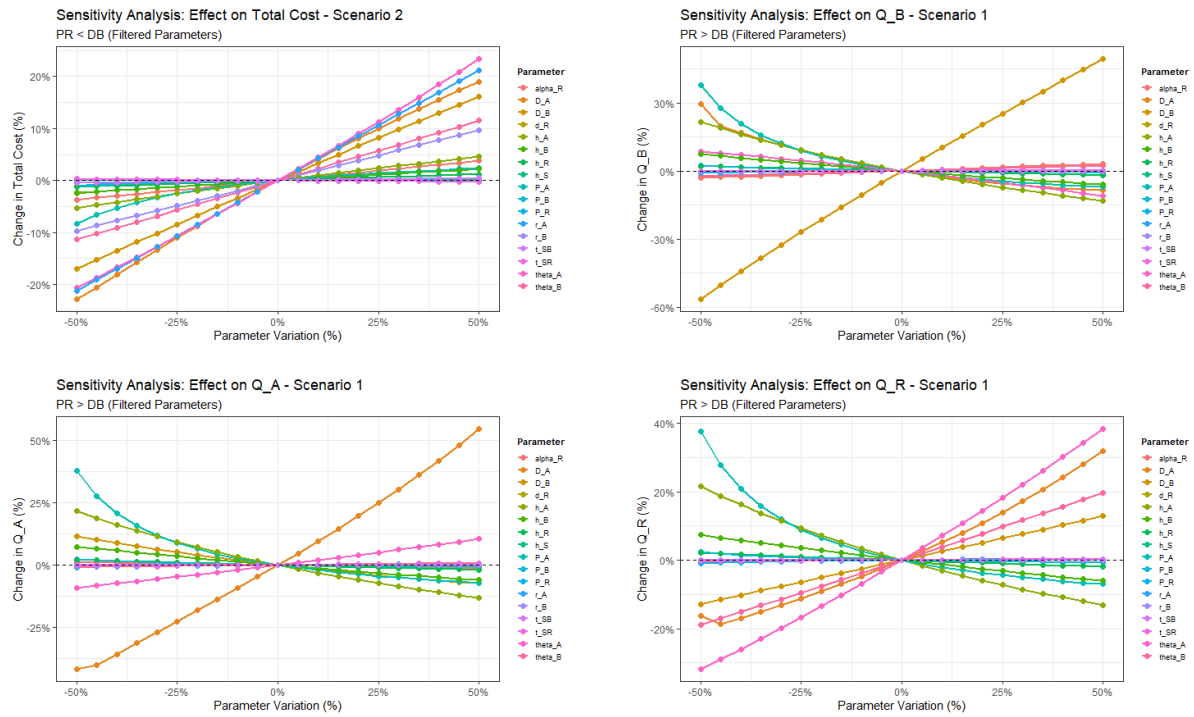


Figure 5. Sensitivity analysis (scenario 1).

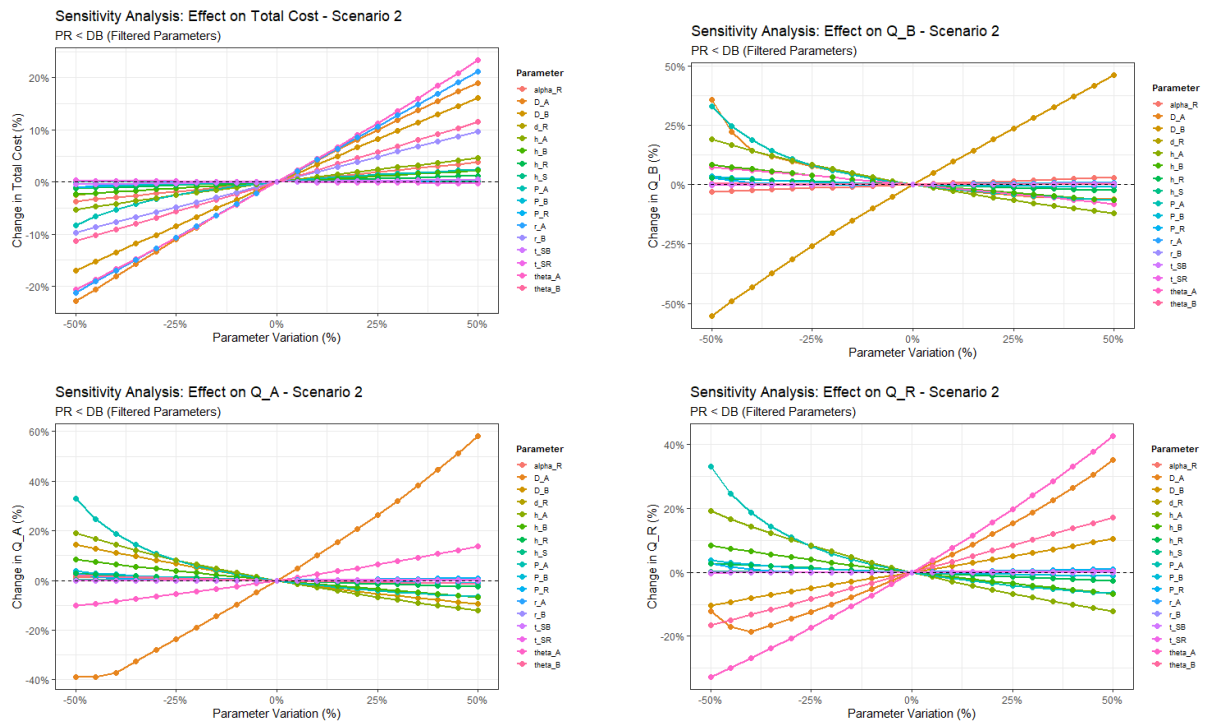


Figure 6. Sensitivity analysis (scenario 2).

Only 12 of the 20 parameters are found to be important in that they showed up in the top 5 parameters of at least one of the control variables and cost. It is interesting to note that the parameters related to product Type A seem to be more controlling than that of type B generally, even when considering the lot sizing decision for the manufacturing and remanufacturing product Type B, and this is repeated for both scenarios 1 and 2. In most cases where θ_A is not in the top 5, it is in the sixth position. It is also interesting to note that only three main types of parameters are most essential to managing the overall cost of this system: the demand rates, the defect rates, and the remanufacturing cost rates. It is particularly interesting that the rate of remanufacturing defect, α_R , does not come up anywhere among the important sensitivity parameters. It is also worth noting that whether the rate of remanufacturing is higher or lower than the demand rate, the same pattern is observable. In fact, the demand rate relative to the remanufacturing rate does not have much impact on the total cost of managing the system optimally. All these provide guidance for how best to manage the system, and will be discussed later.

7.4 Performance Comparison

To compare the performance of the model with some previous ones, we selected two previous models: the first is the classic economic lost scheduling problem (ELSP) for two perfect quality products (i.e. there was no manufacturing defect. To achieve this, we only need to set θ_A and θ_B to zero, which sets Q_R to zero, automatically); the second model is an imperfect ELSP without remanufacturing, where θ_A and θ_B are non-zero, but P_R was set to zero (also automatically setting Q_R to zero). This checks our model against two extreme case relaxations of our model.

Before we compute our values, however, we need to first reproduce our result without including the cost of remanufacturing (r_A and r_B) to properly compare similar cost structures. While extra remanufacturing cost was included in the model for completeness' sake, if it was not set to zero, it could skew the comparison, except if the unit manufacturing cost of production is also explicitly included. This, however, does not change the optimality decision since our cost rate is calculated per year, and only the demand (and not the order quantities) is relevant in comparing cost of two scenarios within a particular time interval (e.g. a year). Setting the remanufacturing cost to zero, we have:

7.4.1 Scenario 1: $P_R > D_B$

Using base parameters with $P_R = 190000 > D_B = 30000$, the optimal values are calculated to be ($Q_A^* = 1716.41$ tons), ($Q_B^* = 764.42$ tons), ($Q_R^* = 320.63$ tons), ($T^* = 0.0302$ years), with $TC = R16254736/\text{year}$.

7.4.2 Scenario 2: $P_R < D_B$

Using base parameters with $P_R = 20000 < D_B = 30000$, the optimal values are calculated to be ($Q_A^* = 1637.93$ tons), ($Q_B^* = 729.47$ tons), ($Q_R^* = 305.97$ tons), ($T^* = 0.0288$ years), with $TC = 16856034/\text{year}$.

We see that the optimal lot size with or without the inclusion of the disposal cost is the same, while the total cost for each scenario comes down by the same amount, R27290300. This is expected, as the annual cost depends on the demand rate only, and not the lot sizes, because the time interval is fixed, and hence, the remanufacturing cost is behaving as a fixed cost. This analysis makes us understand how significant remanufacturing cost could push up the overall system cost. We are now able to objectively compare the cost of the two extreme case relaxations for our model. For this analysis, it suffices to compare only one scenario, and we use scenario 1, $P_R > D_B$, without any loss of generality.

7.4.3 Extreme Case 1: Perfect Quality ELSP ($\theta_A = \theta_B = 0$)

Using base parameters with $P_R = 190000 > D_B = 30000$, the optimal values are calculated to be ($Q_A^* = 1386.48$ tons), ($Q_B^* = 831.39$ tons), ($Q_R^* = 0$ tons), ($T^* = 0.02773$ years), with $TC = R15014550/\text{year}$.

7.4.4 Extreme Case 2: Imperfect Quality ELSP without Remanufacturing ($P_R = 0$)

Using base parameters with $P_R = 190000 < D_B = 30000$, the optimal values are calculated to be ($Q_A^* = 1619.9$ tons), ($Q_B^* = 900.32$ tons), ($Q_R^* = 0$ tons), ($T^* = 0.02851$ years), with $TC = R23562095/\text{year}$.

We see that the total cost from our model is in between the two extreme cases. The difference between extreme case 2 and extreme case 1 (R8.55 million/year) is the total cost of manufacturing defects. However, by applying our model, the company only incurred R1.84 million/year additional cost over the perfect system, saving R6.71 million annually, meaning about 78.5% of the cost of imperfect manufacturing is recovered.

7.5.1 Managerial Insights and Strategic Recommendations

This section discusses the managerial insight that can be drawn from the model results and sensitivity analysis. Three main insights that are important for managing the overall system cost will be presented here: the implications of the remanufacturing capacity requirement, the importance of product type A's parameters, and the benefits of integrated remanufacturing planning.

Firstly, we observe from the total cost from the model scenarios that the cost difference between light and heavy remanufacturing capacity scenarios is only 1.34% (3.56% if remanufacturing cost is negligible), whereas having extra remanufacturing cost other than capacity utilisation increases the overall cost very quickly. This means management needs to be wary of other significant remanufacturing costs than utilising the extra capacity available after the regular manufacturing of product types A and B; i.e. the management should utilize the idle capacity as much as possible but carefully evaluate the benefit of allowing other remanufacturing costs before committing to remanufacturing, except when it is done for compliance purposes.

The second insight, which is from the sensitivity analysis, is that the parameters of product type A dominate the total cost structure. This is probably because item A is a premium product, and its cost factors relative to that of B are higher: 22%, 28.5% and 33% extra for the unit price, holding cost and set up cost respectively. It is particularly interesting that the defect rate of product type A, θ_A , is the most sensitive parameter in 50 percent of the cases evaluated for scenarios 1 and 2, while both θ_A and the demand rate for A, D_A , are in the top 2 positions about 75% of the time. It is also important to note that while the holding cost of product type A, h_A , features 60 percent of the times in the top 5 influential parameters, the holding cost of item B and the remanufacturable (stored defective) stocks did not feature at all, and moreover, the scrap rate for remanufactured products does not seem to have much significant effect on the overall cost. Stocks of product type A would, where necessary, be a more suitable candidate for JIT management than others. Therefore, management knows the product and parameters to focus on to minimise their operating cost.

The third lesson comes from the overall effectiveness of our model. It was shown that by integrating the manufacturing and remanufacturing cycles using our model, management can reclaim almost 80% of their cost of imperfect production. This can be significant in the steel manufacturing industry that trades in commodity products and basically operates on margins. This can be a main factor that determines which company survives or becomes insolvent in an environment where the profit margin is thin and operating

efficiency is key for survival.

8. Limitations and Future Work

All models are built with some assumptions, without which the model becomes mathematically intractable. Some of the key assumptions of these models and their implications are discussed here.

The model is deterministic and assumes key parameters (e.g. demand) a known with certainty. While this is not totally true, mathematical modelling of inventory systems has evolved very well to manage this level of uncertainty. For as long as the demand distribution is fairly stable and the coefficient of variation is reasonably low, most such systems can be readily modelled as deterministic using the expected values of the model parameters. All variations (demand, time, cost etc.) can be handled by an appropriate safety stock model, which is outside the scope of this work.

Also, once the spot values of the prices are known, it suffices to build the model. This is a reasonable assumption because the purpose of the model is not to study the pricing dynamics of the products, but the relationship of the prices to the demand function. This is the general approach in all models that do not directly study price dynamics in inventory modelling. This is because all such dynamic relationships are already captured in the price evolution of the two products until the stable states of the two prices relative to each other is established. Including this price movement in the model will require a more complicated model, which is outside the scope of this research.

Others include that the model is developed for a complementary system of two products without any loss of generality but can be readily extended to an $n > 2$ systems, and while the model was parameterized with guidance from industry data, some of the actual values were scrambled to some extent for confidentiality reasons. While the numerical analysis shows the opportunities presented by the model, a real validation of the proposed model is yet to be done in an active production facility. Future research may also focus on deploying the mathematical framework within a manufacturing execution system to evaluate its performance against dynamic interferences.

9. Conclusion

This study develops an economic production quantity model for a multi-product manufacturing system of two complementary products with imperfect production quality and the remanufacturing of the second product, on a shared manufacturing resource. The mathematical model determines the optimal lot sizes, joint cycle time, and annual manufacturing cost rate. The optimality condition was shown using Fermat's theory on stationary point and the second derivative test, boundary analysis, and grid search verification. Due to the level of complexity of the model, an optimisation algorithm and a numerical example were presented on the use of the model. Two scenarios were illustrated in the numerical analysis: where the remanufacturing rate is greater than the demand rate, and vice versa. It was found that the overall cost is dominated by the additional manufacturing cost other than the level of capacity demand for remanufacturing. Our model recovered almost 80% of the extra cost that resulted from manufacturing defect.

Sensitivity analysis provides further insight into production management where the model parameters of the first product (made only by manufacturing) dominate the overall cost structure. It was particularly observed that its defect rate is the most important parameter, followed by its demand rate.

The research contributes a valuable model for a unique lot sizing problem for which no prior model that has all its characteristics has been identified in literature.

Conflicts of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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During the preparation of this work the author(s) used generative AI in order to improve the language of the article. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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