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A Multi-Operator Framework of Triangular Interval Type-2 Fuzzy Prioritized Aggregation in Expert Evaluation

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Abstract
When many people and many factors are difficult due to uncertainty and ambiguous information, to make decisions in real situations. To address this, Fuzzy sets are used, of which interval type-2 fuzzy sets are particularly good at dealing with larger levels of uncertainty. In this paper, we set up a new concept called triangular interval type-2 fuzzy set. This is simple to work with and is still effective in presenting uncertainty. The main goal of this work is to develop new opportunities for combing information from different people using prioritized fuzzy aggregation operators. These include average, geometric and harmonic operators and they allow us to give importance not just to the criteria but also to the people making the decisions. It defines how these operators work, examine their properties and demonstrates how to apply them when making actual decisions. To prove the usefulness of our method. We apply it to a real-world example related to selecting the right decision in multi-criteria decision-making process. We also compare existing methods and show that these are better in terms of handling uncertainty and achieve more reliable results. In the future, this work can be extended to use fuzzy statements of the intuitionistic type and use them to apply a changing environment. On large scale, we can use these operators to make feasible decision with different priorities of experts and criteria in uncertain conditions.

Keywords- Fuzzy sets, Prioritized aggregation, Triangular interval type-2 fuzzy set, Multi-criteria decision-making.

Abbreviations

MAGDM	Multi-Attribute Group Decision-Making
T2FS	Type-2 Fuzzy Sets
IT2FS	Interval Type-2 Fuzzy Sets
TIT2FS	Triangular Interval Type-2 Fuzzy Sets
TIT2FPWAO	Triangular Interval Type-2 Fuzzy Prioritized Weighted Aggregation Operator
TIT2FPWGO	Triangular Interval Type-2 Fuzzy Prioritized Weighted Geometric Operator
TIT2FPWHO	Triangular Interval Type-2 Fuzzy Prioritized Weighted Harmonic Operator

1. Introduction

In real-world problems, many mathematical models lack complete or exact information. To handle such uncertainties, fuzzy environments provide a robust framework for achieving reliable results. Fuzzy sets, introduced by Zadeh (1965), and interval type-2 fuzzy sets (IT2FSs), introduced by Atanassov and Gargov (1989), have been widely applied across various disciplines. These theories have proven effective in numerous decision-making processes. Xu and Zhao (2016) offer a comprehensive review of existing fuzzy decision-making theories and methods.

In multiple attribute group decision-making (MAGDM), it is essential to aggregate information from various sources to make informed decisions. Gupta and Anupam (2017), Wan and Dong (2020) worked on intuitionistic fuzzy problems. Gong (2013), Zamri et al. (2013) worked on interval type-2 fuzzy sets. Several methods exist for combining such data, as discussed by Beliakov et al. (2007), Calvo et al. (2012), and Xu (2003). One significant method is the aggregation operator (AO) known as the ordered weighted averaging (OWA) operator, introduced by Yager (1988). This operator provides a parameterized family of aggregation operations, including maximum, minimum, and average functions. Since its introduction, the OWA operator has attracted considerable attention and has been applied in various fields (Fodor et al., 1995; Bordogna et al., 1997; Chen et al., 2004; Merigó, 2011).

To extend the concept of OWA, Xu and Da (2002) and Xu and Yager (2006) introduced the ordered weighted geometric (OWG) operator, which focuses on the geometric mean. Furthermore, type-2 fuzzy sets (T2FSs), proposed by Zadeh (1975) as an extension of type-1 fuzzy sets (T1FSs), offer a more powerful approach to modeling uncertainty. A T2FS includes a primary membership function, a secondary membership function, and a footprint of uncertainty (FOU). Türksen (2002) highlighted the three-dimensional structure of T2FSs and the importance of FOU in managing vague and imperfect information in real-life applications.

T2FSs have attracted significant research interest for both theoretical and practical developments (Karnik and Mendel, 2001; Mitchell, 2005; Mendel, 2007; Aisbett et al., 2011) pointed out that the high computational complexity of T2FSs limits their practical use. To address decision-making problems under T2FSs, various authors have proposed different techniques. Wu and Mendel (2007) introduced a weighted average operator, while Lee and Chen (2008) proposed ranking and arithmetic operations. Mendel et al. (2006) developed an optimal representation for handling the uncertainty and complexity of T2FSs, going beyond crisp interval limitations noted by Chen et al. (2013).

The concept of IT2FSs was refined by Chen and Lee (2010), who also proposed a TOPSIS-based method for MAGDM problems. Jain et al. (2022) worked on defuzzification techniques. Further advancements were made by Chen et al. (2012) and De et al. (2020), who defined ranking formulas for IT2FSs. Kumar et al. (2023), and Qin and Liu (2014) explored critical decision-making issues under IT2FS settings. The concept of triangular interval type-2 fuzzy sets (TIT2FSs) was elaborated by Chiao (2011), Hagras and Wagner (2012), Hu et al. (2013), Qin and Liu (2014), Wang et al. (2012), Wang and Fan (2003), and Xu (2003), described fuzzy OWA (FOWA) operators, and Zhao et al. (2013) introduced techniques for solving fuzzy prioritized operators (POs) with applications in MAGDM. Wang and Liu (2012) developed the fuzzy-induced ordered weighted averaging (FIOWA) operators. Yager (2008, 2009) presented fuzzy prioritized weighted average (FPWA) and fuzzy prioritized weighted harmonic average (FPWHA) operators. Pathak et al. (2024), Tadic and Komatina (2025).

Many group decision methods using fuzzy logic usually assume that all criteria and all experts are equally important. However, this is rare in real life. Different criteria can have different importance, and some expert opinions can have more weight than others. To better reflect these actual conditions, this paper presents new aggregation methods that allow for different levels of prioritization. These methods are based on the triangular interval type-2 fuzzy sets (TIT2FS) that is useful for dealing uncertainty and ambiguous information. We propose three new operators: TIT2F priorty weight average (TIT2FPWA), TIT2F weighted geometry (TIT2FPWG), and TIT2F weighted harmony (TIT2FPWH). These operators allow priority to be include in combination of information from various sources.

This paper is divided into several parts: First, we will explain the basic ideas and concepts related to the Type 2 Fuzzy Set and its triangular spacing format. Next, we'll hire new prioritized operators and explain how they work. These operators are then used to show how different criteria and experts can be used to solve key group decision problems. We'll also add detailed examples to show how the method actually works and compare it to existing methods. The final part of the paper summarizes the main results and proposes how this approach can be used in future research.

2. Preliminaries

This section, we explore the fundamental definition of concerning T2FSS, TIT2FSS, and numbers:

Definition 1. T2FS: A T2FS β in universe of discourse or defined, κ can be defined by a type-2 acceptance region or membership function $\mu_\beta(k,\xi)$ as following form:

$$\beta = \left\{\left((k,\xi),\mu_\beta(k,\xi)\right) | \forall k \in \kappa, \forall \xi \in J_k \subset [0,1]\right\} \tag{1}$$

where, J_k defines an interval in [0,1].

Moreover, the T2FS expressed as follows:

$$\beta = \int_{k\in\kappa}\int_{\xi\in J_k}\frac{\mu_\beta(k,\xi)}{(k,\xi)} = \int_{k\in\kappa}\frac{\left(\int_{\xi\in J_k}\frac{\mu_\beta(k,\xi)}{\xi}\right)}{k}.$$

where, J_k define as the primary membership function at κ, and $\int_{\xi\in J_k}\frac{\mu_\beta(k,\xi)}{\xi}$ shows as the second acceptance region or membership at κ. In the discrete cases, $\int$ is changed by $\sum$.

Definition 2. (Interval-valued type-2 fuzzy set) involves representing the membership grades of each point in the domain as a crisp set confined within the interval [0,1].

Definition 3. (Footprint of uncertainty: FOU) The primary Acceptance region or memberships of T2FSS introduce uncertainty, which is encapsulated by a boundary region referred to as the FOU. Analytically, the FOU is defined as the union of all primary acceptance regions or memberships, denoted as $FOU(\beta) = \cup_{k\in\kappa} J_k$.

Definition 4. Arithmetical operations on TIT2FNs: (Gong, 2015).

Let

$\beta_1 = \langle\beta_1^{\bar{U}},\beta_1^{L}\rangle = \langle\left(b_{11}^{\bar{U}},b_{12}^{\bar{U}},b_{13}^{\bar{U}};h_1^{\bar{U}}\right),\left(b_{11}^{\bar{L}},b_{12}^{\bar{L}},b_{13}^{\bar{L}};h_1^{\bar{L}}\right)\rangle,$

$\beta_2 = \langle\beta_2^{\bar{U}},\beta_2^{\bar{L}}\rangle = \langle\left(b_{21}^{\bar{U}},b_{22}^{\bar{U}},b_{23}^{\bar{U}};h_2^{\bar{U}}\right)\left(b_{21}^{\bar{L}},b_{22}^{\bar{L}},b_{23}^{\bar{L}};h_2^{\bar{L}}\right)\rangle.$

be two sets and k be a real positive number, then the arithmetic operations defined on set of TIT2FNs as follows:

- Addition:

$$\beta_1 \oplus \beta_2 = (\beta_1^{\bar{U}}, \beta_1^{\bar{L}}) \oplus \beta_2^{\bar{U}}, \beta_2^{\bar{L}})$$

$$= \langle \left(b_{11}^{\bar{U}} + b_{21}^{\bar{U}}, b_{12}^{\bar{U}} + b_{22}^{\bar{U}}, b_{13}^{\bar{U}} + b_{23}^{\bar{U}}; min\left(h_1^{\bar{U}}, h_2^{\bar{U}}\right)\right), \left(b_{11}^{\bar{L}} + b_{21}^{\bar{L}}, b_{12}^{\bar{L}} + b_{22}^{\bar{L}}, b_{13}^{\bar{L}} + b_{23}^{\bar{L}}; min\left(h_1^{\bar{L}}, h_2^{\bar{L}}\right)\right)\rangle$$

- Multiplication:

$$\beta_1 \otimes \beta_2 = (\beta_1^{\bar{U}}, \beta_1^{\bar{L}}) \otimes \beta_2^{\bar{U}}, \beta_2^{\bar{L}})$$

$$= \langle \left(b_{11}^{\bar{U}} \times b_{21}^{\bar{U}}, b_{12}^{\bar{U}} \times b_{22}^{\bar{U}}, b_{13}^{\bar{U}} \times b_{23}^{\bar{U}}; min\left(h_1^{\bar{U}}, h_2^{\bar{U}}\right)\right), \left(b_{11}^{\bar{L}} \times b_{21}^{\bar{L}}, b_{12}^{\bar{L}} \times b_{22}^{\bar{L}}, b_{13}^{\bar{L}} \times b_{23}^{\bar{L}}; min\left(h_1^{\bar{L}}, h_2^{\bar{L}}\right)\right)\rangle$$

- Scalar Multiplication:

$$k\beta_1 = k\left(\beta_1^{\bar{U}}, \beta_1^{\bar{L}}\right)$$

$$= \left(kb_{11}^{\bar{U}}, kb_{12}^{\bar{U}}, kb_{13}^{\bar{U}}; h_1^{\bar{U}}\right), \left(kb_{11}^{\bar{L}}, kb_{12}^{\bar{L}}, kb_{13}^{\bar{L}}; h_1^{\bar{L}}\right)$$

- Power:

$$\beta_1^k = \left(\beta_1^{\bar{U}}, \beta_1^{L}\right)^k$$

$$= \left(\left(b_{11}^{\bar{U}}\right)^k, \left(b_{12}^{\bar{U}}\right)^k, \left(b_{13}^{\bar{U}}\right)^k\right); h_1^{\bar{U}}), \left(\left(b_{11}^{\bar{L}}\right)^k, \left(b_{12}^{\bar{L}}\right)^k, \left(b_{13}^{\bar{L}}\right)^k; h_1^{\bar{L}}\right)$$

Definition 5. If γ is a TIT2FNs $\gamma = \langle a, b, c\rangle$ then score value of β is:

$$S(\gamma) = \frac{a \oplus b \oplus c}{3}, S(\gamma) \in [0,1] \tag{2}$$

Definition 6. (Prioritized weighted average (PWA) operator:) Let $R = \{R_1, R_2, \ldots, R_{\bar{n}}\}$ the collection of attributes is prioritized using a linear ordering $R_1 \succ R_2 \succ \cdots \succ R_{\bar{n}}$, where attribute R_j is considered to have a higher region of priority than R_k if $j < k$. Values $R_j(x)$ represents performance based any alternatives x under or below attribut*e* R_j, and lies within the interval [0,1].

$$PWA\left(R_1, R_2, \cdots, R_{\{\bar{n}\}}\right) = \sum_{\bar{j}}^{\bar{n}} \frac{\tau_{\bar{j}}}{\sum_{\bar{j}}^{\bar{n}} \tau_{\bar{j}} R_{\bar{j}}} \tag{3}$$

where,

$$\tau_{\bar{j}} = \prod_{k=1}^{\bar{j}-1} R_{\bar{j}},\ (\bar{j} = 2, \ldots, \bar{n}), \tau_1 = 1.$$

3. TIT2 Fuzzy Prioritized Weighted Average Operators

The PA operators are utilized in some different situations where the input or defined arguments consist of precise data sets.

Definition 7. Let $\beta_i = \langle \beta_{i1}^{U}, \beta_{i1}^{L}\rangle = \langle \left(b_{i1}^{\bar{U}}, b_{i2}^{\bar{U}}, b_{i3}^{\bar{U}}; h_i^{\bar{U}}\right), \left(b_{i1}^{\bar{L}}, b_{i2}^{\bar{L}}, b_{i3}^{\bar{L}}; h_i^{\bar{L}}\right)\rangle$ be collection of TIT2FNs, then we define the operator as follows:

$$\text{TIT2FPWA}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) = \oplus_{j=1}^{\bar{n}} \frac{\tau(\beta_j)}{\sum_{i=1}^{\bar{n}} \tau(\beta_j)} \beta_j \tag{4}$$

$$= \frac{\tau(\beta_1)}{\sum_{j=1}^{\bar{n}} \tau(\beta_j)} \beta_1 \oplus \frac{\tau(\beta_2)}{\sum_{j=1}^{\bar{n}} \tau(\beta_j)} \beta_2 \oplus \ldots \oplus \frac{\tau(\beta_j)}{\sum_{j=1}^{\bar{n}} \tau(\beta_j)} \beta_j$$

where, $\tau(\beta_j) = \prod_{k=1}^{j-1} S(\beta_k)(j = 2,3,\ldots,\bar{n}), \tau(\beta_1) = 1$ and $S(\beta_k)$ represented as the score value of β_k.

Remark 1. Priority levels of combined arguments decrease to same or equal level or situation, operator simplifies to TIT2F weighted average operator:
$$\text{TIT2FPWA}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) = (w_1\beta_1 \oplus w_2\beta_2 \oplus \cdots \oplus w_{\bar{n}}\beta_{\bar{n}}) \tag{5}$$

Utilizing the law of operations TIT2FPWA, we can establish a theorem as follows:

Theorem 1. Let $\beta_i = \langle \beta_{i1}^{\bar{U}}, \beta_{i1}^{\bar{L}} \rangle = \langle (b_{i1}^{\bar{U}}, b_{i2}^{\bar{U}}, b_{i3}^{\bar{U}}; h_i^{\bar{U}}), (b_{i1}^{\bar{L}}, b_{i2}^{\bar{L}}, b_{i3}^{\bar{L}}; h_i^{\bar{L}}) \rangle$ constitutes TIT2FNs, after aggregating value by TIT2FPWA operator is also TIT2FN, and

$$\begin{aligned}\text{TIT2FPWA}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) &= \oplus_{i=1}^{\bar{n}} \frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i} \beta_i. \\ &= \langle \left(\oplus_{i=1}^{\bar{n}} \frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i} b_{i1}^U, \oplus_{i=1}^{\bar{n}} \frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i} b_{i2}^U, \oplus_{i=1}^{\bar{n}} \frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i} b_{i3}^U; \min_{i=1,\cdots,\bar{n}} \{h_i^U\} \right) \\ &\left(\oplus_{i=1}^{\bar{n}} \frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i} b_{i1}^L, \oplus_{i=1}^{\bar{n}} \frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i} b_{i2}^L, \oplus_{i=1}^{\bar{n}} \frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i} b_{i3}^L; \min_{i=1,\cdots,\bar{n}} \{h_i^L\} \right) \rangle\end{aligned} \tag{6}$$

where,
$$\tau_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k) \tag{7}$$

is the estimated value of β_k.

Let $\beta_i = \langle \beta_i^U, \beta_i^L \rangle = \langle (b_{i1}^U, b_{i2}^U, b_{i3}^U; h_i^U), (b_{i1}^L, b_{i2}^L, b_{i3}^L; h_i^L) \rangle$ be a collection of TIT2FNs, $\tau_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k)$ is the expected value of $\beta_k = \langle \beta_k^U, \beta_k^L \rangle = \langle (b_{k1}^U, b_{k2}^U, b_{k3}^U; h_k^U),$ $(b_{k1}^L, b_{k2}^L, b_{k3}^L; h_k^L) \rangle$. Then, from the above operator, it has been analyzed that for a collection of TIT2FNs, the proposed operator fulfills several properties of idempotency, boundedness, and monotonicity. These properties are described as follow:

Property 1. (Idempotency:) If all $\beta_j = \beta_0$ for all j then
$$\text{TIT2FN}(\beta_1, \beta_2, \ldots, \beta_{\bar{n}}) = \beta_0 \tag{8}$$

Proof: Since for all j, we have $\beta_j = \beta_0 = \langle (a_{01}^U, a_{02}^U, a_{03}^U; h_0^U)(a_{01}^L, a_{02}^L, a_{03}^L; h_0^L) \rangle$ then by Theorem 1, we get

$$\begin{aligned}\text{TIT2FPWA}(\beta_1, \beta_2, \ldots, \beta_{\bar{n}}) = \quad & \langle \left(\oplus_{j=1}^{\bar{n}} \frac{\tau_0 a_{01}^U}{\sum_{j=1}^{\bar{n}} \tau_0}, \oplus_{j=1}^{\bar{n}} \frac{\tau_0 a_{02}^U}{\sum_{j=1}^{\bar{n}} \tau_0}, \oplus_{j=1}^{\bar{n}} \frac{\tau_0 a_{03}^U}{\sum_{j=1}^{\bar{n}} \tau_0} \right), \\ & \left(\oplus_{j=1}^{\bar{n}} \frac{\tau_0 a_{01}^L}{\sum_{j=1}^{\bar{n}} \tau_0}, \oplus_{j=1}^{\bar{n}} \frac{\tau_0 a_{02}^L}{\sum_{j=1}^{\bar{n}} \tau_0}, \oplus_{j=1}^{\bar{n}} \frac{\tau_0 a_{03}^L}{\sum_{j=1}^{\bar{n}} \tau_0} \right) \rangle \\ = \quad & \left\langle \left(\frac{\tau_0 a_{01}^U}{\tau_0}, \frac{\tau_0 a_{02}^U}{\tau_0}, \frac{\tau_0 a_{03}^U}{\tau_0} \right), \left(\frac{\tau_0 a_{01}^L}{\tau_0}, \frac{\tau_0 a_{02}^L}{\tau_0}, \frac{\tau_0 a_{03}^L}{\tau_0} \right) \right\rangle \\ = \quad & \beta_0.\end{aligned}$$

Property 2. (Boundedness:) Let $\beta^- = \left\langle \left[\min_i a_{i1}^U, \min_i a_{i2}^U, \min_i a_{i3}^U, \min_i a_{i1}^L, \min_i a_{i2}^L, \min_i a_{i3}^L \right] \right\rangle$ and $\beta^+ = \langle [\max_i a_{i1}^U, \max_i a_{i2}^U, \max_i a_{i3}^U, \max_i a_{i1}^L, \max_i a_{i1}^L, \max_i a_{i3}^L] \rangle$ then

$$\beta^- \leq \text{TIT2FPWA}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) \leq \beta^+ \tag{9}$$

Proof: Since we can rewrite Equation (4):
$\text{TIT2FPWA}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) = \oplus_{i=1}^{\bar{n}} \mathrm{w_i} \beta_i$.

where,
$w_i = \frac{\tau_i \beta_{i\cdot}}{\sum_{i=1}^{\bar{n}} \tau_i}$.

then we have $w_i \geq 0, (j = 1,2,\ldots,\bar{n})$ and $\sum_{i=1}^{\bar{n}} w_i = 1$.

Take a_i, Denote:
$\min_i \{a_{i1}^U | i = 1,2,\cdots,\bar{n}\} = \left(a_{i1}^U\right)_{\min}$
$\max_i \{a_{i1}^U | i = 1,2,\cdots,\bar{n}\} = \left(a_{i1}^U\right)_{\max}$

By equation the following holds:
$\left(a_{i1}^U\right)_{\min} = \oplus_{i=1}^{\bar{n}} w_i \left(a_{i1}^U\right)_{\min} \leq \oplus_{i=1}^{\bar{n}} w_i a_i \leq \oplus_{i=1}^{\bar{n}} w_i \left(a_{i1}^U\right)_{\max} = \left(a_{i1}^U\right)_{\max}$

Similarly, we have

$\left(a_{i1}^U\right)_{\min} \leq \oplus_{i=1}^{\bar{n}} w_i b_i \leq \left(a_{i1}^U\right)_{\min}, \left(a_{i1}^U\right)_{\max} \leq \oplus_{i=1}^{\bar{n}} w_i \left(a_{i2}^U\right) \leq \left(a_{i2}^U\right)_{\max},$

$\left(a_{i3}^U\right)_{\min} \leq \oplus_{i=1}^{\bar{n}} w_i \left(a_{i3}^U\right) \leq \left(a_{i3}^U\right)_{\max}, \left(a_{i2}^L\right)_{\min} \leq \oplus_{i=1}^{\bar{n}} w_i \left(a_{i2}^L\right) \leq \left(a_{i2}^L\right)_{\max},$

$\left(a_{i3}^L\right)_{\min} \leq \oplus_{i=1}^{\bar{n}} w_i \left(a_{i3}^L\right) \leq \left(a_{i3}^L\right)_{\max}.$

$$\min_i \{\beta_i \,\big|\, i = 1,2,\cdots,\bar{n}\} \leq \text{TIT2FPWA}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) \leq \max_i \{\beta_i \big|\, i = 1,2,\cdots,\bar{n}\}$$

Property 3. Let β_i and β_i', $(i = 1,2,\ldots,\bar{n})$ be the collections of two different TIT2FNs such that $\beta_i \leq \beta_i'$ for all i, then we have

$$TIT2FN(\beta_1, \beta_2, \ldots, \beta_{\bar{n}}) \leq TIT2FN(\beta_1', \beta_2', \ldots, \beta_{\bar{n}}') \tag{10}$$

4. TIT2F Prioritized Weighted Geometric (TIT2FPWG) Operator

Definition 8. The weighted geometric operator of TIT2FN:
Collection of TIT2FNs,
$\beta_i = \langle \beta_i^U, \beta_i^L \rangle = \langle \left(b_{i1}^U, b_{i2}^U, b_{i3}^U; h_i^U\right), \left(b_{i1}^L, b_{i2}^L, b_{i3}^L; h_i^L\right) \rangle, i = 1,2,\cdots,\bar{n}.$

TIT2FPWG operator is defined as follows:

$$\text{TIT2FPWG}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) = \otimes_{i=1}^{\bar{n}} \beta_i^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}} \tag{11}$$

$$= \langle (\otimes_{i=1}^{\bar{n}} (b_{i1}{}^{U})^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i2}{}^{U})^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i3}{}^{U})^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}; \min_{i=1,\cdots,\bar{n}} \{h_i^U\}),$$

$$(\otimes_{i=1}^{\bar{n}} (b_{i1}{}^{L})^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i2}{}^{L})^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i3}{}^{L})^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}; \min_{i=1,\cdots,\bar{n}} \{h_i^L\})\rangle$$

where, $\tau_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k)$ is achieved value of β_k.

Remark 2. When the priority levels of the combined arguments equalize, the TIT2FPWG operator transforms into the TIT2F Weighted Geometric operator.

$$TIT2FPWG(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) = \beta^{w_1} \otimes \beta^{w_2} \otimes \cdots \otimes \beta^{w_{\bar{n}}} \quad (12)$$

Utilizing the laws of operations TIT2FNs, we can establish a theorem as follows:

Theorem 2. Let $\beta_i = \langle \beta_i^U, \beta_i^L \rangle = \langle (b_{i1}^U, b_{i2}^U, b_{i3}^U; h_i^U), (b_{i1}^L, b_{i2}^L, b_{i3}^L; h_i^L) \rangle, i = 1,2,\cdots,\bar{n}$, which is aggregation of TIT2FNs, after aggregate by TIT2FPWG operator is also a TIT2FNs, and

$$\mathrm{TIT2FPWG}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) = \bigotimes_{i=1}^{\bar{n}} \beta_i^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}} \quad (13)$$

$$= \langle \left(\otimes_{i=1}^{\bar{n}} (b_{i1}^U)^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i2}^U)^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i3}^U)^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}; \min_{i=1,\cdots,\bar{n}} \{h_i^U\} \right),$$

$$\left(\otimes_{i=1}^{\bar{n}} (b_{i1}^L)^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i2}^L)^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}, \otimes_{i=1}^{\bar{n}} (b_{i3}^L)^{\frac{\tau_i}{\sum_{i=1}^{\bar{n}} \tau_i}}; \min_{i=1,\cdots,\bar{n}} \{h_i^L\} \right)\rangle.$$

where, $\tau_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k)$ is the extimated value of β_k.

Proof. We prove this theorem by mathematical induction on $\bar{n}$ same as Theorem 1.

Let $\beta_i = \langle \beta_i^U, \beta_i^L \rangle = \langle (b_{i1}^U, b_{i2}^U, b_{i3}^U; h_i^U), (b_{i1}^L, b_{i2}^L, b_{i3}^L; h_i^L) \rangle$ be a collection of TIT2FNs, $T_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k)$ is the defined or achieved results $\beta_k = \langle \beta_k^U, \beta_k^L \rangle = \langle (b_{k1}^U, b_{k2}^U, b_{k3}^U; h_k^U), (b_{k1}^L, b_{k2}^L, b_{k3}^L; h_k^L) \rangle$. Then, from the above operator, it has been analyzed that for a collection of TIT2FNs, the proposed operator fulfills several of the properties of idempotency, boundedness, and monotonicity. These properties are described as follows:

Property 4. (Idempotency) If all $\beta_j = \beta_0$ for all j then

$$\mathrm{TIT2FPWG}(\beta_1, \beta_2, \ldots, \beta_{\bar{n}}) = \beta_0 \quad (14)$$

Property 5. (Boundedness) Let $\beta^- = \left\langle \left[\min_i a_{i1}^U, \min_i a_{i2}^U, \min_i a_{i3}^U, \min_i a_{i1}^L, \min_i a_{i2}^L, \min_i a_{i3}^L \right] \right\rangle$ and $\beta^+ = \langle [\max_i a_{i1}^U, \max_i a_{i2}^U, \max_i a_{i3}^U, \max_i a_{i1}^L, \max_i a_{i1}^L, \max_i a_{i3}^L] \rangle$ then

$$\beta^- \leq \mathrm{TIT2FPWG}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) \leq \beta^+ \quad (15)$$

Property 6. (Monotonicity) Let β_i and β_i', $(i = 1,2,\ldots,\bar{n})$ is the collections of two different TIT2FNs such that $\beta_i \leq \beta_i'$ for all i, then we have

$$TIT2FN(\beta_1,\beta_2,\ldots,\beta_{\bar{n}}) \leq TIT2FN(\beta_1',\beta_2',\ldots,\beta_{\bar{n}}') \tag{16}$$

5. TIT2F Prioritized Weighted Harmonic Average Operator

On the basis of and harmonic average, the authors define TIT2FPWHA operator as follows:

Definition 9. The weighted harmonic operator of TIT2FN:

Collection of TIT2FNs,

$\beta_i = \langle \beta_i^U, \beta_i^L \rangle = \langle \left(b_{i1}^U, b_{i2}^U, b_{i3}^U; h_i^U\right), \left(b_{i1}^L, b_{i2}^L, b_{i3}^L; h_i^L\right) \rangle, i = 1,2,\cdots,\bar{n},$

The TIT2FPWHA operator is defined as follows:

$$TIT2FPWHA(\beta_1,\beta_2,\cdots,\beta_{\bar{n}}) = \Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{\beta_i}\Big)^{-1} = \frac{1}{\frac{\frac{\tau_1}{\sum_{i=1}^{\bar{n}}\tau_i}}{\beta_1} \oplus \frac{\frac{\tau_2}{\sum_{i=1}^{\bar{n}}\tau_i}}{\beta_2} \oplus \cdots \oplus \frac{\frac{\tau_{\bar{n}}}{\sum_{i=1}^{\bar{n}}\tau_i}}{\beta_{\bar{n}}}} \tag{17}$$

where, $\tau_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k)$ is the expected value of $\beta_k = \langle \beta_k^U, \beta_k^L \rangle = \langle \left(b_{k1}^U, b_{k2}^U, b_{k3}^U; h_k^U\right), \left(b_{k1}^L, b_{k2}^L, b_{k3}^L; h_k^L\right) \rangle$.

Remark 3. When the priority levels of the combined arguments equalize, the TIT2FPWHA operator transforms into TIT2F Weighted harmonic operator.

$$TIT2FPWHA(\beta_1,\beta_2,\cdots,\beta_{\bar{n}}) = \frac{1}{\frac{w_1}{\beta_1} \oplus \frac{w_2}{\beta_2} \oplus \cdots \oplus \frac{w_{\bar{n}}}{\beta_{\bar{n}}}} \tag{18}$$

Utilizing the law of operations TIT2FNs, we can establish a theorem as follows:

Theorem 3. Let $\beta_i = \langle \beta_i^U, \beta_i^L \rangle = \langle \left(b_{i1}^U, b_{i2}^U, b_{i3}^U; h_i^U\right), \left(b_{i1}^L, b_{i2}^L, b_{i3}^L; h_i^L\right) \rangle, i = 1,2,\cdots,\bar{n},$ which is a collection of TIT2FNs, after aggregating by TIT2FPWHA is TIT2FNs:

$$\begin{aligned} TIT2FPWHA(\beta_1,\beta_2,\cdots,\beta_{\bar{n}}) &= \Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{\beta_i}\Big)^{-1} \\ &= \Big\langle \Big[\Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{b_{i1}^U}\Big)^{-1}, \Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{b_{i2}^U}\Big)^{-1}, \left(\Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{b_{i3}^U}\Big)^{-1}; \min_{i=1,\cdots,\bar{n}}\{h_i^U\}\right), \\ &\quad \Big(\Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{b_{i1}^L}\Big)^{-1}, \left(\Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{b_{i2}^L}\Big)^{-1}, \left(\Big(\bigoplus_{i=1}^{\bar{n}} \frac{\frac{\tau_i}{\sum_{i=1}^{\bar{n}}\tau_i}}{b_{i3}^L}\Big)^{-1}; \min_{i=1,\cdots,\bar{n}}\{h_i^L\}\right)\right)\Big\rangle \end{aligned} \tag{19}$$

where, $\tau_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k)$ is the expected value of $\beta_k = \langle \beta_k^U, \beta_k^L \rangle = \langle \left(b_{k1}^U, b_{k2}^U, b_{k3}^U; h_k^U\right), \left(b_{k1}^L, b_{k2}^L, b_{k3}^L; h_k^L\right) \rangle$.

Proof. Similarly as Theorem 1

Let $\beta_i = \langle \beta_i^U, \beta_i^L \rangle = \langle \left(b_{i1}^U, b_{i2}^U, b_{i3}^U; h_i^U\right), \left(b_{i1}^L, b_{i2}^L, b_{i3}^L; h_i^L\right) \rangle$ be a collection of TIT2FNs, $\tau_i = \prod_{k=1}^{i-1} S(\beta_k), i = 1,2,\cdots,\bar{n}, \tau_1 = 1, S(\beta_k)$ is achieved results of $\beta_k = \langle \beta_k^U, \beta_k^L \rangle = \langle \left(b_{k1}^U, b_{k2}^U, b_{k3}^U; h_k^U\right),$

$\left(b_{k1}^{L}, b_{k2}^{L}, a_{k3}^{L}; h_{k}^{L}\right)\rangle$. Then, from the above operator, it has been analyzed that a collection of TIT2FNs, fulfills several essential properties, including idempotence, boundedness, and monotonicity. These properties are described as follows:

Property 7. (Idempotency) If all $\beta_j = \beta_0$ for all j then

$$\text{TIT2FPWHA}(\beta_1, \beta_2, \ldots, \beta_{\bar{n}}) = \beta_0 \tag{20}$$

Property 8. (Boundedness) Let $\beta^- = \left\langle\left[\min_i a_{i1}^U, \min_i a_{i2}^U, \min_i a_{i3}^U, \min_i a_{i1}^L, \min_i a_{i2}^L, \min_i a_{i3}^L\right]\right\rangle$ and $\beta^+ = \langle[\max_i a_{i1}^U, \max_i a_{i2}^U, \max_i a_{i3}^U, \max_i a_{i1}^L, \max_i a_{i1}^L, \max_i a_{i3}^L]\rangle$ then

$$\beta^- \le \text{TIT2FPWHA}(\beta_1, \beta_2, \cdots, \beta_{\bar{n}}) \le \beta^+ \tag{21}$$

Property 9. (Monotonicity) Let β_i and β_i', $(i = 1,2, \ldots, \bar{n})$ which is aggregation of two different TIT2FNs s.t. $\beta_i \le \beta_i'$ for all i, then we have

$$TIT2FN(\beta_1, \beta_2, \ldots, \beta_{\bar{n}}) \le TIT2FN(\beta_1', \beta_2', \ldots, \beta_{\bar{n}}') \tag{22}$$

7. MAGDM Based on Proposed Operator

Basis of this particular section, the authors represent a technique to tackle FMAGDM examples as well as problems by using an IT2F framework. Consider a set Y repenting criteria or alternatives and a set F comprising an attribute as well as, where $Y = \{y_1, y_2, \cdots, y_{\bar{n}}\}$ and $\bar{F} = \{\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_{\bar{m}}\}$. Consider l decision makers (D-Ms), denoted as $\widehat{D}_1, \widehat{D}_2, \cdots, \widehat{D}_l$. Let $Q^k = \left(\Gamma_{ij}^k\right)_{\bar{n}\times\bar{m}}$ which are defined as IT2F decision matrix where Γ_{ij}^k is an IT2IFS, which defined by the decision maker (DM) $\widehat{D}_k$ for alternatives x_i concerning the attribute $\bar{f}_j$. When it comes to options, you can break them down into two kinds: ones that bring benefits and ones that cost. Essentially, the attributes group $\bar{F}$ can be split into two parts $\bar{F}_1$ and $\bar{F}_2$. In $\bar{F}_1$, you have the good stuff, and in $\bar{F}_2$, you have the expensive stuff. They don't overlap ($\bar{F}_1 \cap \bar{F}_2 = \emptyset$), but together they make up all of $\bar{F}$ ($\bar{F}_1 \cup \bar{F}_2 = \bar{F}$). To make fair decisions, the scorecard Q^k needs to be made even unless everything in $\bar{F}_j (j = 1,2, \ldots, \bar{m})$ falls under the same category. This study uses a standard formula to even out the playing field with a balanced scorecard called Q^k.

$$\Gamma_{ij}^k = \begin{cases} \Gamma_{ij}^k; j \in \bar{F}_1 \\ \left(\Gamma_{ij}^k\right)^c; j \in \bar{F}_2 \end{cases} \tag{23}$$

The complement of $\left(\Gamma_{ij}^k\right)$denoted as $\left(\Gamma_{ij}^k\right)^c$. Hence, we established a normalized decision matrix Q^k.

Step 1: Compute the values of $\tau_{ij}^k (k = 1,2, \cdots, t)$ as follows:

$$\tau_{ij}^{(k)} = \prod_{i=1}^{j-1} S\left(\beta_k\right), j = 1,2, \cdots, \bar{n}, \tau_1 = 1 \tag{24}$$

Step 2: Aggregate TIT2FNs $p_{ij}^k (i = 1,2, \cdots, \bar{m}; j = 1,2, \cdots, \bar{n}$ for every alternative $P^k (k = 1,2, \cdots, t)$ by the proposed TIT2FPWA operator.

Step 3: Compute the values of $\tau_{ij} (i = 1,2, \cdots, \bar{m}; j = 2, \cdots, \bar{n})$

$$\tau_{ij} = \prod_{i=1}^{j-1} S\left(P^{(k)}\right), j = 1,2, \cdots, \bar{n}, \tau_1 = 1 \tag{25}$$

Step 4: Aggregate all TIT2FNs $p_{ij}'s(j = 2, \cdots, \bar{n})$ for each option by the TIT2FPWA or TIT2FPWG or TIT2FPWH operator.

Step 5: By using Equation (2) we get the score value for each $S(\beta_i), (i = 1,2, \cdots, \bar{n})$.

Step 6: Calculate rank the alternatives $\Gamma_i, (i = 1,2, \cdots, \bar{n})$ according to their score values, $S(\beta_i)(i = 1,2, \cdots, \bar{n})$ representing the overall fuzzy preference values p_i and then select the best alternatives.

8. Numerical Example

Basis of the section, an illustration affects Fuzzy MAGDM process major of the defined way. **Table 1** and **Table 2** shows the linguistic conditions “Totally Agree" (TA), " Agree"(A), "Moderate Agree" (MA), "Moderate"(M), "Moderate Disagree"(MD), "Disagree" (D), "Totally Disagree"(TD). Consider a production company facing the challenge of identifying the optimal supplier globally for one of its components used in assembly.

Three global suppliers, denoted as Y_1, Y_2 and Y_3 are under evaluation across four attributes:
$\bar{f}_1$**:** for product quality,
$\bar{f}_2$**:** for safety concerns,
$\bar{f}_3$**:** for supplier performance, and
$\bar{f}_4$**:** for supplier’s concept

Whose weight vector $w = (0.3, 0.15, 0.2, 0.35)$. An expert group was developed which consists of three experts D_1, D_2 and D_3 (whose weight vector is $k = (0.3, 0.45, 0.25)$) from each strategic decision area. The experts D_1, D_2 and D_3 apply the linguistic terms appear in **Table 3** to present the features of the global best suppliers Y_1, Y_2 and Y_3 for various attributes $f_i(i = 1,2,3,4)$, listed in **Table 3**.

Considering that the attributes are the benefit attributes except for the attribute $\bar{f}_2$ (risk factor), then based on the **Table 3**, the decision matrices $Q^k = \left(\Gamma_{ij}^k\right)_{3\times4}$ can be updated to the following normalized matrices respectively, listed in **Table 4**.

Based on **Table 3**, we utilize Definition 4 to aggregate all individual normalized interval type-2 fuzzy decision matrix $Q^k = \left(\Gamma_{ij}^k\right)_{3\times4}$ into a collective normalized interval type-2 fuzzy decision matrix $Q = \left(\Gamma_{ij}\right)_{3\times4}$ shown as follows:

$$Q = \begin{bmatrix} & \bar{f}_1 & \bar{f}_2 & \bar{f}_3 & \bar{f}_4 \\ y_1 & B_{11} & B_{12} & B_{13} & B_{14} \\ y_2 & B_{21} & B_{22} & B_{23} & B_{24} \\ y_3 & B_{31} & B_{32} & B_{33} & B_{34} \end{bmatrix}$$

Table 1. Linguistic terms for membership.

Linguistic terms	Interval type-2 fuzzy sets (PMF, SMF)
Totally Disagree (TD)	[(0.0,0.0,0.1;1.0,1.0); (0.0,0.0,0.05;0.9,0.9)]
Disagree(D)	[(0.0,0.1,0.3;1.0,1.0); (0.05,0.1,0.2;0.9,0.9)]
Moderate Disagree (MD)	[(0.1,0.3,0.5;1.0,1.0); (0.2,0.3,0.40;0.9,0.9)]
Moderate (M)	[(0.3,0.5,0.7;1.0,1.0); (0.40,0.5,0.6;0.9,0.9)]
Moderate Agree (MA)	[(0.5,0.7,0.9;1.0,1.0); (0.6,0.7,0.80;0.9,0.9)]
Agree (A)	[(0.7,0.9,1.0;1.0,1.0); (0.8,0.9,0.95;0.9,0.9)]
Totally Agree (TA)	[(0.9,1.0,1.0;1.0,1.0); (0.95,1.0,1.0;0.9,0.9)]

Table 2. Ranking values of alternatives of the three decision-makers.

Attributes	Alternatives	Decision Makers
Product Quality (f_1)	y_1	MA A MA
	y_2	A MA A
	y_3	TA A MA
Risk Factor (f_2)	y_1	M TA A
	y_2	MA A TA
	y_3	TA TAA
Services of Supplier (f_3)	y_1	TA A A
	y_2	A TA TA
	y_3	M MA MA
Supplier Profile (f_4)	y_1	TA A A
	y_2	A TA A
	y_3	A TA TA

$\Gamma_{11} = \langle (0.5900,0.7900,0.9450;1),(0.6900,0.7900,0.8675;0.9) \rangle$
$\Gamma_{12} = \langle (0.0000,0.0550,0.2100;1),(0.0275,0.0550,0.1325;0.9) \rangle$
$\Gamma_{13} = \langle (0.7600,0.9300,1.0000;1),(0.8450,0.9300,0.9650;0.9) \rangle$
$\Gamma_{14} = \langle (0.7600,0.9300,1.0000;1),(0.8450,0.9300,0.9650;0.9) \rangle$
$\Gamma_{21} = \langle (0.6100,0.8100,0.9550;1),(0.7100,0.8100,0.8825;0.9) \rangle$
$\Gamma_{22} = \langle (0.0300,0.1350,0.3100;1),(0.0825,0.1350,0.2225;0.9) \rangle$
$\Gamma_{23} = \langle (0.8400,0.9700,1.0000;1),(0.9050,0.9700,0.9850;0.9) \rangle$
$\Gamma_{24} = \langle (0.7900,0.9450,.9725;1),(0.8675,0.9450,1.0000;0.9) \rangle$
$\Gamma_{31} = \langle (0.7100,0.8800,0.9750;1),(0.7950,0.8800,0.9275;0.9) \rangle$
$\Gamma_{32} = \langle (0.0000,0.0250,0.1500;1),(0.0125,0.0250,0.0875;0.9) \rangle$
$\Gamma_{33} = \langle (0.4400,0.6400,0.8400;1),(0.5400,0.6400,0.7400;0.9) \rangle$
$\Gamma_{34} = \langle (0.8400,0.9700,1.0000;1),(0.9050,0.9700,0.9850;0.9) \rangle$

Table 3. Normalized values of alternatives of the three decision-makers.

Attributes	Alternatives	Decision Makers
Product Quality (f_1)	y_1	MA A MA
	y_2	A MA A
	y_3	TA A MA
Risk Factor (f_2)	y_1	M TD D
	y_2	MD D TD
	y_3	TD TD D
Services of Supplier (f_3)	y_1	TA A A
	y_2	A TA TA
	y_3	M MA MA
Supplier Profile (f_4)	y_1	TA A A
	y_2	A TA A
	y_3	A TA TA

- After using Equation (23) for normalization, then utilize Equations (6) and (7) to calculate the $\tau_{ij}^{(1)}, \tau_{ij}^{(2)}, \tau_{ij}^{(3)}$

$$\tau_{ij}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\tau_{ij}^{(2)} = \begin{bmatrix} 0.7750 & 0.1983 & 0.8967 & 0.8967 \\ 0.7917 & 0.1583 & 0.9367 & 0.9117 \\ 0.8550 & 0.0417 & 0.6400 & 0.9367 \end{bmatrix},$$

$$\tau_{ij}^{(3)} = \begin{bmatrix} 0.6064 & 0.0385 & 0.8190 & 0.8190 \\ 0.6340 & 0.0239 & 0.8930 & 0.8463 \\ 0.7417 & 0.0023 & 0.4096 & 0.8930 \end{bmatrix}.$$

- Utilize the informative decision in matrix P^k, and operator for aggregate all individual decision matrices.
- Calculating the values of $\tau_{ij}(i = 1,2,\dots,m, j = 2,\dots,n)$ as follows:

$$\tau_{ij} = \begin{bmatrix} 1.0000 & 0.2479 & 0.3458 & 0.0896 \\ 1.0000 & 0.2509 & 0.5145 & 0.1360 \\ 1.0000 & 0.2607 & 0.2879 & 0.0615 \end{bmatrix}$$

- To consolidate all triangular fuzzy preference values $p_{ij}(j = 1,2,\dots,n)$into a unified measure, employ the fuzzy prioritized weighted Aggregation operator (FPAWO), Calculate $p_i(i = 1,2,\dots,m)$.
  $p_1 = \langle\{\, 0.1548, 0.2432, 0.8568\}\rangle$
  $p_2 = \langle\{\, 0.1697, 0.2952, 1.0000\}\rangle$
  $p_3 = \langle\{\, 0.1545, 0.2209, 0.7884\}\rangle$
- Calculating the valuables TIT2FPWA expectations $S(p_i)$.
  S(p_1) = 0.4182, S(p_2) = 0.4908, S(p_3) = 0.3879.
- Calculating the valuables based on TIT2FPWG expectations $S(p_i)$.
  S(p_1) = 0.3604, S(p_2) = 0.3948, S(p_3) = 0.2993.
- Calculating the valuables TIT2FPWH expectations $S(p_i)$.
  S(p_1) = 0.5019, S(p_2) = 0.5689, S(p_3) = 0.4676.
- Rank all a different choosing $\Gamma_i(i = 1,2,3)$ in agreeance with the scoring numbers $S(p_i)(i = 1,2,3)$ of all-embracing fuzzy preference values p_i.Therefore, the ranking order of the four alternatives is $\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$ and found that Γ_2 is the most desirable one while Γ_3 is the least one.

9. Comparative Study

To compare the performance of the suggested methods with certain pre-existing methods, comparative studies have been conducted. These studies evaluate the existing operators based on averaging methods by Chen and Lee (2010), Gong (2013, 2015), Lee and Chen (2008), Hu et al. (2013), Wang (2012), & Zamri et al. (2013) and the outcomes connected to it have been display in **Table 4**.

Table 4. Comparative analysis.

Method	**Score values**	**Order of alternatives**
Gong (2015)	[0.2887, 0.4896, 0.2218]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
Chen (2010)	[0.6212, 0.8383, 0.5405]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
Hu (2013)	[0.3250, 0.4036, 0.2715]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
Gong (2013)	[0.3590, 0.4773, 0.1667]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
Zamri (2013)	[0.4997, 0.4999, 0.4995]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
Chen (2013)	[4.0059, 4.1068, 3.8871]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
Lee (2008)	[0.6100, 0.8700, 0.3100]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
Wang (2012)	[8.8892, 9.0788, 8.3035]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
	Proposed Method	
TIT2FPWA	[0.4182, 0.4908, 0.3879]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
TIT2FPWG	[0.3604, 0.3948, 0.2993]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$
TIT2FPWH	[0.5019, 0.5689, 0.4676]	$\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$

10. Discussion

The comparison showed in **Table 4** that the proposed TIT2FPWA, TIT2FPWG, and TIT2FPWH methods provided stable and reliable results when we compare with existing approaches. All methods has same ranking of alternatives: $\Gamma_2 \succ \Gamma_1 \succ \Gamma_3$, which confirms validity of the proposed models. The proposed methods generate more balanced and moderate results. This suggests that our approach handles uncertainty more effectively and reflects decision-makers preferences more realistically. Based on the comparative results, it is clear that the proposed methods perform equally well or better than existing ones. Therefore, they can be considered as a strong and useful alternative for solving complex mutli-criteria decision-making problems in uncertain environments.

11. Conclusion

This study compared the proposed methods—TIT2FPWA, TIT2FPWG, and TIT2FPWH—with several existing techniques used in multi-criteria decision-making (MCDM). All methods, including existing ones, give the same ranking results, ensuring consistency in decision outcomes. These proposed methods generate more balanced and realistic score values, which better capture the uncertainty and vagueness in multi-criteria decision making problems. These operators that allow us to give importance to each expert and factor using the weight vectors. Therefore, these methods can be considered as practical and effective alternatives to traditional aggregation techniques for solving complex multi-criteria group decision-making problems under uncertainty. In the future these contributions can expand further to develop smarter and powerful tools that help decision-makers in handling complicated situations with more ease and understanding.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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