

Kink and Anti-Kink Wave Solutions of the Forced Schamel KdV Equation and its Stability Analysis

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(Received on April 29, 2026; Revised on July 5, 2026 & August 3, 2026 & August 16, 2026; Accepted on August 25, 2026)

Abstract

In this paper, we solve the Schamel Korteweg-de Vries (Schamel KdV) equation with the help of the Kudryashov expansion technique. The Schamel KdV equation is important in various physical contexts, particularly in wave phenomena across different media. It is relevant in non-linear optics, plasma physics, solid-state physics, and fluid dynamics. For example, it models long waves on water surfaces, including solitary waves and solitons, and describes wave propagation in shallow water. The Kudryashov expansion technique is a simple and efficient technique. The obtained solutions are of the form of exponential functions, and two-dimensional, contour, and three-dimensional plots are also shown. The bifurcation analysis for the Schamel KdV equation is derived, and the stability of the model is also discussed.

Keywords- The Kudryashov expansion method, The Schamel KdV equation, Lyapunov exponent, Stability analysis.

1. Introduction

The exact solutions of the nonlinear partial differential equations (NPDEs) are studied to understand the inner mechanism of the physical models. There are several analytical methods recently applied by many

researchers; such methods are as follows: F-expansion method (Seadawy et al., 2021), Lie symmetry technique (Kumar et al., 2021), the (G'/G) -expansion method (Mohanty et al., 2022, 2023), extended generalized (G'/G) -expansion approach (Mohanty, 2024), Kudryashov method (Mohanty et al., 2024; Sahoo et al., 2025a), the Hirota bilinear method (Hao and Gao, 2025), $\tan(\phi/2)$ method (Chen et al., 2024), finite difference method (Zhang et al., 2025).

1.1 Background and Model Description

In plasmas, the Schamel KdV equation helps analyse ion acoustic waves and non-linear wave dynamics. It also aids in studying intense laser pulses in non-linear media, particularly optical solitons, which are stable wave packets that maintain their shape during transmission. The Schamel KdV equation supports a range of solutions, including solitons, solitary waves, and periodic waves, with solitons being notable for their localization, stability, and ability to interact without changing shape or speed.

The Schamel KdV equation represents a nonlinear partial differential equation that extends upon the classical Korteweg-de Vries (KdV) equation by adding a term that accounts for the influence of non-Maxwellian particle distributions, especially relevant in the field of plasma physics. This equation was proposed to characterise solitary waves, or solitons, in plasma environments, where particle behaviour deviates from the Maxwell-Boltzmann velocity distribution because of trapping phenomena. The Schamel KdV equation enhances this by incorporating a nonlinear term that addresses the deviation from thermal equilibrium within the plasma. The general form of the Schamel KdV equation is:

$$\frac{\partial u}{\partial t} + \left(Au^{\frac{1}{2}} + Bu \right) \frac{\partial u}{\partial x} + C \frac{\partial^3 u}{\partial x^3} = 0 \quad (1)$$

Here, $u(x, t)$ denotes the function of space and time. Here A , B and C are the nonlinear and the dispersion parameters. Note that if we take $A = 0$ in Equation (1), then the converted Equation (1) is called the non-linear KdV equation. The KdV equation is a prominent equation in the fields of fluid dynamics and plasma physics, describing the behavior of shallow water waves and ion-acoustic waves within plasmas. It can be expressed as:

$$\frac{\partial u}{\partial t} + Bu \frac{\partial u}{\partial x} + C \frac{\partial^3 u}{\partial x^3} = 0 \quad (2)$$

If we take $B = 0$ in Equation (1), then the Schamel KdV equation takes place in the form of the Schamel equation, whose formation is given by

$$\frac{\partial u}{\partial t} + Au^{\frac{1}{2}} \frac{\partial u}{\partial x} + C \frac{\partial^3 u}{\partial x^3} = 0 \quad (3)$$

The Schamel KdV equation possesses a stronger non-linearity as compared to the classical KdV equation. To understand the dynamical phenomena of this model equation, like the transition between different wave patterns, the onset of instabilities, and the emergence of solitons, bifurcation analysis plays a vital role. It is especially useful for studying how stable wave structures evolve into more complex behaviours, such as periodic or chaotic waves, under varying conditions (Chen and Chen, 2021; Chadha and Tomar, 2023; Fabiani et al., 2021; Hosseini et al., 2023). In nonlinear wave theory, bifurcation analysis has been instrumental in investigating travelling waves, pattern formation, and instabilities in systems like the KdV equation and the nonlinear Schrödinger equation. For more details, see (Chadha et al., 2023; Islam and

Basal, 2023; Rahman et al., 2024) in which the authors utilised bifurcation theory within the context of dynamical systems to explore various nonlinear differential equations.

Depending on the underlying mathematical structure of the governing equation, the Kudryashov expansion approach has been successfully applied to a variety of nonlinear evolution equations, producing several classes of precise solutions. Bulut et al. (2014), For instance, discovered soliton and hyperbolic function solutions by using the Kudryashov expansion approach to the time-fractional KdV problem. Eslami and Mirzazadeh (2014) proposed a one-soliton solution to the time-dependent fifth-order KdV equation using the same method. Using the Kudryashov expansion method, (Hosseini et al., 2022) examined the geophysical KdV equation, obtaining solitary-wave solutions and the conservation laws that go along with them. Similarly, dark, brilliant, singular, and combination bright-singular soliton solutions have been obtained by using the Kudryashov expansion approach to the (2+1)-dimensional combined KdV–mKdV equation. Baines (2022) provided a thorough discussion of how stratification and bottom topography contribute to the creation of driven waves. His monograph describes how forced KdV-type models arise in geophysical and environmental fluid mechanics and provides the theoretical basis for understanding topographically driven internal waves. For forced nonlinear wave events in stratified flows, the book is still a classic reference.

More recently, Ermakov and Stepanyants (2019) investigated the dynamics of the forced KdV equation with moving disturbances. They investigated the development of antecedent solitons, resonant events, trapped solitary waves, and the long-term evolution of nonlinear wave fields. The study showed that forcing significantly improves the solution dynamics when compared to the conventional unforced KdV equation. Furthermore, analytical techniques for the forced KdV equation have been developed. Lee (2018) used symmetry-based analytical techniques to create precise travelling-wave solutions and investigated the effect of force on nonlinear wave propagation. These results demonstrate that forcing modifies the wave profiles as well as the viable solution structures. Flamarion and Ribeiro Jr. (2021) utilised computational techniques to examine forced KdV models arising in stratified fluid flows from a numerical standpoint. Their models offer substantial insights into wave formation in practical environments by clarifying intricate connections among dispersion, nonlinear effects, and external forces. Mohanty et al. (2024) studied the force KdV equation with time-dependent force term $f(t) = t$ and solved by using the (G'/G) expansion method, the Kudrayashov expansion method and the $(1/G')$ expansion method and obtained solitary wave solutions. Sahoo et al. (2025b) solved the Schamel KdV equation without a force term by using the extended direct algebraic method and obtained kink and anti-kink type solutions. Sahoo et al. (2026) studied the Schamel KdV equation with cubic force term $(u-a)(u-b)(u-c)$, but did not solve it analytically, and studied the bifurcation analysis of it.

The forced Schamel–KdV equation, which combines both the Schamel-type fractional-power nonlinearity and an external forcing factor, has a significantly different nonlinear structure than the previous research. These extra nonlinear characteristics dramatically change the structure of the solution and make the analytical process more difficult. Consequently, instead of the bright, dark, singular, or solitary-wave solutions described in the previously cited literature, the Kudryashov expansion approach produces kink and anti-kink wave solutions. These kink-type solutions are essentially distinct from localised pulse type solitons and constitute transition waves that link two separate asymptotic equilibrium states. Therefore, the uniqueness of the present study does not only reside in applying an established analytical approach to another nonlinear problem. Rather, importance resides in proving that the Kudryashov expansion approach may generate new classes of precise solutions for the forced Schamel–KdV equation, especially kink and anti-kink waves originating from the combined effects of the Schamel nonlinearity and the forcing element. To the best of our knowledge, no prior reports of such accurate kink and anti-kink

solutions for the forced Schamel–KdV equation have been made using the Kudryashov expansion approach.

1.2 Research Gap in Current Study

The application of this forcing term lies in the field of seismology, such as for modelling seismic wave propagation and the response of geological structures to seismic events. By incorporating stepwise functions into the governing equations, we can effectively represent the sudden application of forces that simulate fault slip and other seismic sources. Additionally, the use of stepwise forcing terms aids in the calibration of models against observed seismic data, enhancing the accuracy of hazard assessments and the development of engineering solutions for earthquake-resistant structures. For instance, the work by Aki and Richards (2002) demonstrates how such models can be utilised to better understand the complex interactions between seismic waves and heterogeneous materials, ultimately improving earthquake preparedness and response strategies. See also (Peterson, 1997; Peterson and Salupere, 1997). The study of wave propagation in different media using a perturbed KdV equation (Engelbrecht and Khamidullin, 1988; Engelbrecht, 1991; Engelbrecht and Peipman, 1992) with these forcing terms.

The literature review reveals that the Schamel KdV equation, coupled with a force term $(u - a)(u - b)$, has yet to be investigated through the lens of the Kudryashov expansion method. No analytical approaches have examined the precise solutions for the Schamel KdV model incorporating a quadratic force term. None of the authors analysed the bifurcation of the Schamel KdV model with the quadratic force term.

1.3 Aims and Objectives

The first goal is implementing the Kudryashov expansion method to look into the precise solutions for the Schamel KdV equation with the quadratic force term. The Schamel KdV equation's graphical representations show the characteristics of the derived solutions. Using the Lyapunov exponent, we also conducted a stability analysis for the Schamel KdV model and its time series analysis.

1.4 Arrangement of the Study

The rest of this article is structured as follows: Section 2 gives the detailed procedures of the Kudryashov method. Section 3 informs about the precise solutions for the Schamel KdV equation with quadratic force term. In Section 4, we present the graphs and elaborate its importance to our work. Section 5 derives the dynamical system of the Schamel KdV equation with a stepwise forcing term. We have examined the linear stability analysis of both dynamical systems, that is, with and without the forcing term. Furthermore, using phase portraits, time-series graphs, and the Lyapunov exponent, we used the bifurcation tools in Section 5 to examine the impact of parameters. We provide a quick conclusion in section 6.

2. Procedures of the Kudryashov Method

Here, the NPDEs is of the form

$$S(w, w_t, w_x, \dots, w_{xt}, w_{tt}, w_{xx}, \dots) = 0 \quad (4)$$

where, w is a function of (x, t) . By, using $\eta = x - kt$ in Equation (4), it converts

$$S(w, w_\eta, w_{\eta\eta}, \dots) = 0 \quad (5)$$

where, $w = w(\eta)$. Considering the solutions for the Equation (5) is of the form

$$w(\eta) = \sum_{j=0}^n a_j (G)^j \quad (6)$$

where, $a_j, j = 0, 1, 2, \dots, n$, are constants and $G = G(\eta)$ satisfies

$$G' = G^2 - G \quad (7)$$

The solution of Equation (7) is

$$G = \frac{1}{1 + le^\eta} \quad (8)$$

where, l is a constant.

Step 1. Consider the degree of the assumed solutions is n i.e. $\deg(w) = n$, where $\deg$ = degree of the polynomial function w . Now $\deg(w') = n+1$ with help of Equation (7), similarly $\deg(w'') = n+2$, $\deg(w''') = nm$, for any positive integer m and so on. The highest order derivate term and the highest order nonlinear terms of Equation (5) are compared to determine the value of n and n should have a positive integer value.

Step 2. With the evaluated value of n from step-1, Equation (5) becomes a polynomial of G . Then, the coefficients of the same power of G are equal to zero. A system of equations exists, then solving the system of equations for $a_0, a_1, \dots, a_n$.

Step 3. Now, by utilizing the value of Equation (8) and $a_0, a_1, \dots, a_n$, we acquire the needed solutions of Equation (4).

3. Solution of the Schamel KdV Equation with Force Term

Then we can rewrite the SKdV equation is with the quadratic force terms (Engelbrecht and Khamidullin, 1988; Engelbrecht and Peipman, 1992),

$$\frac{\partial u}{\partial t} + \left(Au^{\frac{1}{2}} + Bu \right) \frac{\partial u}{\partial x} + C \frac{\partial^3 u}{\partial x^3} = (u-a)(u-b) \quad (9)$$

where, A, B and C are arbitrary coefficients, a, b are constants, and $u = u(x, t)$.

With help of the transformation $\eta = x - kt$, where k is constant, the Equation (9) becomes

$$-k \frac{du}{d\eta} + \left(Au^{1/2} + Bu \right) \frac{du}{d\eta} + C \frac{d^3 u}{d\eta^3} = u^2 - (a+b)u + ab \quad (10)$$

where, $u = u(\eta)$. Let us consider $u^{1/2}(\eta) = v(\eta)$ into the Equation (10), we get the following equation:

$$-2kv \frac{dv}{d\eta} + 2Av^2 \frac{dv}{d\eta} + 2Bv^3 \frac{dv}{d\eta} + 6C \frac{dv}{d\eta} \frac{d^2 v}{d\eta^2} + 2Cv \frac{d^3 v}{d\eta^3} = v^4 - (a+b)v^2 + ab \quad (11)$$

From Equation (11), it is noted that $\deg\left(v \frac{d^3 v}{d\eta^3}\right) = 2n + 3$ and $\deg\left(v^3 \frac{dv}{d\eta}\right) = 4n + 1$, now comparing $2n + 3 = 4n + 1$, we have $n = 1$. With the value of $n = 1$, Equation (6) gives the value $v(\eta) = a_0 + a_1 G$ (12)

Now the required derivatives are given below:

$$\begin{aligned}\frac{dv}{d\eta} &= a_1 G^2 - a_1 G, \\ \frac{d^2 v}{d\eta^2} &= 2a_1 G^3 - 3a_1 G^2 + a_1 G, \\ \frac{d^3 v}{d\eta^3} &= 6a_1 G^4 - 12a_1 G^3 + 7a_1 G^2 - a_1 G.\end{aligned}\quad (13)$$

Now, substituting the values of Equations (12), and (13) in Equation (11), it becomes a polynomial of $G(\eta)$ as follows:

$$\begin{aligned}& 4Aa_0 a_1^2 G^3 - 4Aa_0 a_1^2 G^2 + 12Ca_0 a_1 G^4 - 24Ca_0 a_1 G^3 + 14Ca_0 a_1 G^2 - 2Ca_0 a_1 G + 2aa_0 a_1 G \\ & + 2ba_0 a_1 G + ba_1^2 G^2 + aa_1^2 G^2 - 2ka_0 a_1 G^2 + 2ka_0 a_1 G + 2Ba_0^3 a_1 G^2 - 2Ba_0^3 a_1 G + 6Ba_0^2 a_1^2 G^3 \\ & - 6Ba_0^2 a_1^2 G^2 + 6Ba_0 a_1^3 G^4 - 6Ba_0 a_1^3 G^3 + 2Aa_0^2 a_1 G^2 - 2Aa_0^2 a_1 G - a_0^4 + ab - a_1^4 G^4 + ba_0^2 \\ & + aa_0^2 + 2Ba_1^4 G^5 - 2Ba_1^4 G^4 + 2Aa_1^3 G^4 - 2Aa_1^3 G^3 + 24Ca_1^2 G^5 - 54Ca_1^2 G^4 + 38Ca_1^2 G^3 \\ & - 8Ca_1^2 G^2 - 4a_0^3 a_1 G - 6a_0^2 a_1^2 G^2 - 4a_0 a_1^3 G^3 - 2ka_1^2 G^3 + 2ka_1^2 G^2 = 0\end{aligned}\quad (14)$$

Now, equating the coefficients of the same power of G^i , $i = 0, 1, 2, 3, 4, 5$ in Equation (14), the obtain system of algebraic equations are as follows:

$$\begin{aligned}G^5 : & 2Ba_1^4 + 24Ca_1^2 = 0, \\ G^4 : & 6Ba_0 a_1^3 - 2Ba_1^4 + 2Aa_1^3 - a_1^4 + 12Ca_0 a_1 - 54Ca_1^2 = 0, \\ G^3 : & 6Ba_0^2 a_1^2 - 6Ba_0 a_1^3 + 4Aa_0 a_1^2 - 2Aa_1^3 - 4a_0 a_1^3 - 24Ca_0 a_1 + 38Ca_1^2 - 2ka_1^2 = 0, \\ G^2 : & 2Ba_0^3 a_1 - 6Ba_0^2 a_1^2 + 2Aa_0^2 a_1 - 4Aa_0 a_1^2 - 6a_0^2 a_1^2 \\ & + 14Ca_0 a_1 - 8Ca_1^2 + aa_1^2 + ba_1^2 - 2ka_0 a_1 + 2ka_1^2 = 0, \\ G^1 : & -2Ba_0^3 a_1 - 2Aa_0^2 a_1 - 4a_0^3 a_1 - 2Ca_0 a_1 + 2aa_0 a_1 + 2ba_0 a_1 + 2ka_0 a_1 = 0, \\ G^0 : & -a_0^4 + aa_0^2 + ba_0^2 + ab = 0.\end{aligned}\quad (15)$$

The particular values for a_0, a_1, k, a, b, c are evaluated by solving algebraic Equations (15). The results are given in the following sets

$$\begin{aligned}\text{Set -1: } C &= -\frac{A^2 B}{48}, a = \frac{(B^2 + \sqrt{2B^4 + 32} + 4)A^2}{16B^2}, k = \frac{A^2(B^2 - 18)}{24B}, \\ b &= -\frac{(B - 2)^2 A^2 (4B + \sqrt{2B^4 + 32})}{16B^2 (2B^2 - 4B + \sqrt{2B^4 + 32} + 8)}, a_0 = \frac{A(B - 2)}{4B}, a_1 = \frac{-A}{2}.\end{aligned}\quad (16)$$

$$\begin{aligned} \text{Set - 2: } C &= -\frac{4A^2B}{75B^2 - 60B + 12}, a = 0, b = \frac{16A^2}{25B^2 - 20B + 4}, \\ k &= -\frac{16A^2B + 24A^2}{75B^2 - 60B + 12}, a_0 = 0, a_1 = -\frac{4A}{5B - 2}. \end{aligned} \quad (17)$$

$$\begin{aligned} \text{Set - 3: } C &= -\frac{4A^2B}{75B^2 - 60B + 12}, a = \frac{16A^2}{25B^2 + 20B + 4}, b = 0, \\ k &= -\frac{16A^2B - 24A^2}{75B^2 + 60B + 12}, a_0 = -\frac{4A}{5B + 2}, a_1 = \frac{4A}{5B + 2}. \end{aligned} \quad (18)$$

$$\text{Set - 4: } C = -\frac{A^2B}{3}, a = 0, b = A^2, k = \frac{2}{3}A^2B, a_0 = -A, a_1 = 2A \quad (19)$$

Now, by substituting Equation (16), $\eta = x - kt$ and $u = v^2$ into Equation (12), then Equation (12) becomes

$$u(x, t) = \left[\frac{A(B-2)}{4B} - \frac{A}{2} \left(\frac{1}{1 + le^{\frac{A^2(B^2-18)}{24B}t}} \right) \right]^2 \quad (20)$$

Again, substituting Equation (17), $\eta = x - kt$ and $u = v^2$, then Equation (12) becomes

$$u(x, t) = \left[-\frac{4A}{5B-2} \left(\frac{1}{1 + le^{\frac{16A^2B+24A^2}{75B^2-60B+12}t}} \right) \right]^2 \quad (21)$$

Similarly, by using Equation (18), $\eta = x - kt$ and $u = v^2$ in Equation (12), it becomes

$$u(x, t) = \left[-\frac{4A}{5B+2} + \frac{4A}{5B+2} \left(\frac{1}{1 + le^{\frac{16A^2B-24A^2}{75B^2+60B+12}t}} \right) \right]^2 \quad (22)$$

Further, similarly by using Equation (19), $\eta = x - kt$ and $u = v^2$ in Equation (12), it becomes

$$u(x, t) = \left[-A + 2A \left(\frac{1}{1 + le^{\frac{x-\frac{2}{3}A^2Bt}}}{1 + le^{\frac{x-\frac{2}{3}A^2Bt}}} \right) \right]^2 \quad (23)$$

4. Graphical Representations

The section that uses the solutions shows the 3-D plots, contour plots, and 2-D plots. The plots show wave structures that are kink-type, anti-kink-type, and solitary-type. The solutions (20) and (21) exhibit anti-kink wave structures characterized by varying amplitudes. The solution (22) has a kink-type wave structure, while the solution (23) has a solitary wave structure.

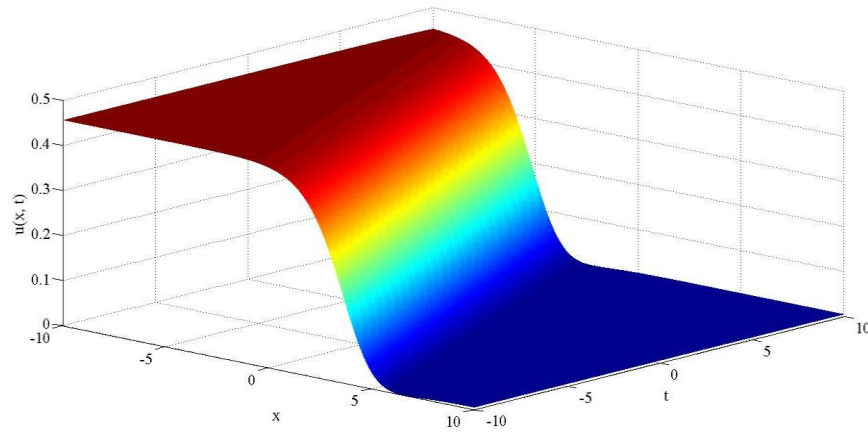


Figure 1. Anti kink wave structure for solution (20) with the particular values of the parameters are $A = 1.5$, $B = 2.5$, $l = 1$ and in the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.

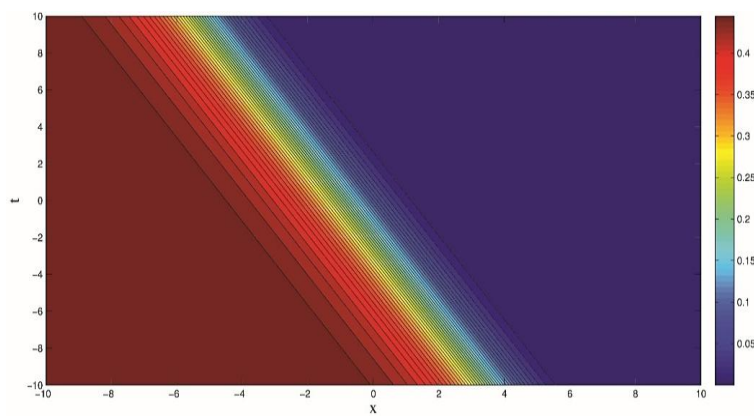


Figure 2. Contour plot for solution (20), where the free parameters are as follows $A = 1.5$, $B = 2.5$, $l = 1$ and in the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.

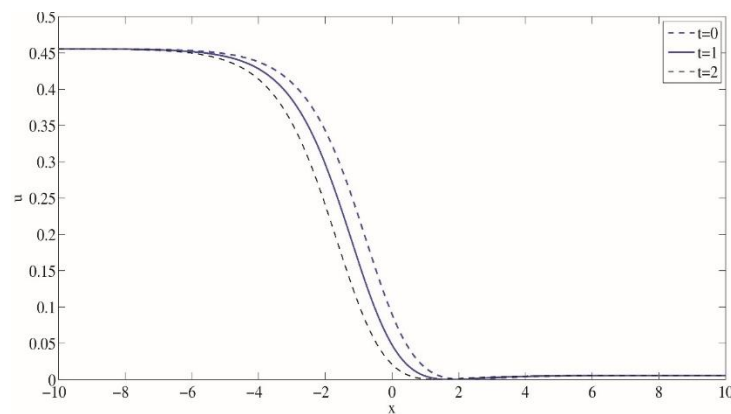


Figure 3. Two-dimensional plot for solution (20) with the particular value of the parameter $A = 1.5$, $B = 2.5$, $l = 1$ and in the interval $-10 \leq x \leq 10$ with $t = 0$, $t = 1$ and $t = 2$.

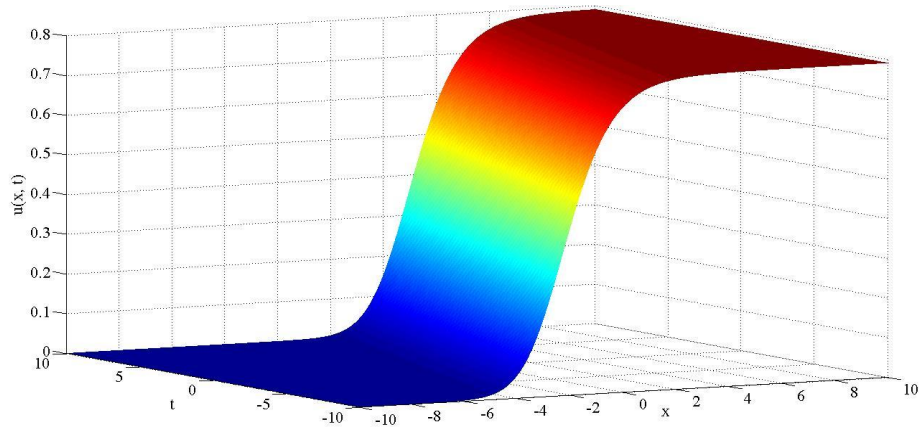


Figure 4. Kink wave structure for solution (22), where $A = 1.5$, $B = 0.5$, $L = 1$, and within the interval ranges $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.

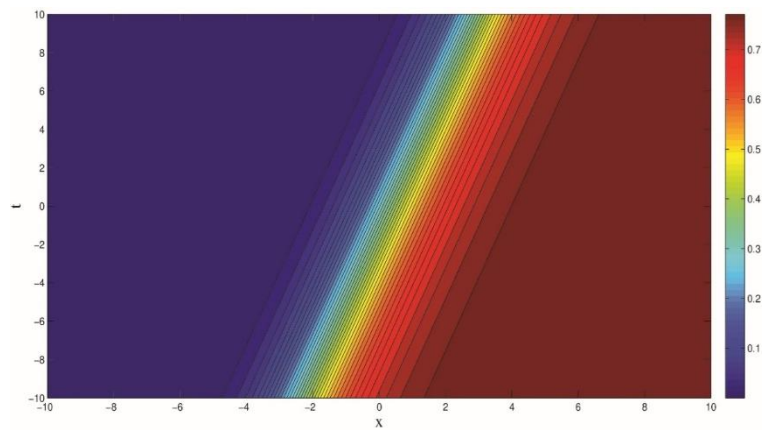


Figure 5. Contour plot for solution (22), where $A = 1.5$, $B = 0.5$, $l = 1$ and in the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.

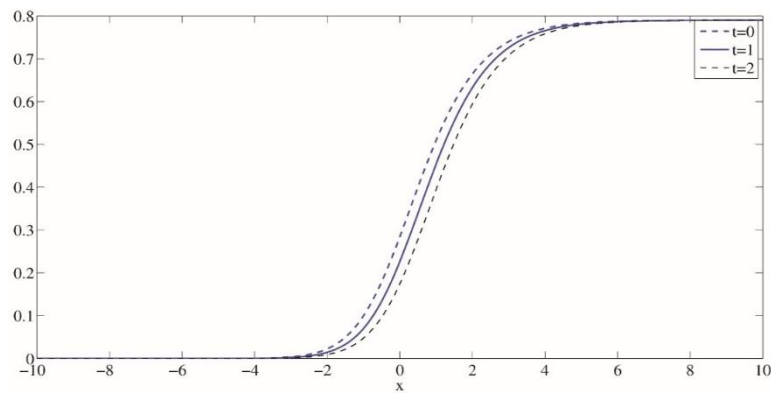


Figure 6. Two-dimensional plot for solution (22), where $A = 1.5$, $B = 0.5$, $l = 1$, and in the interval $-10 \leq x \leq 10$ with particular values of $t = 0$, $t = 1$, $t = 2$.

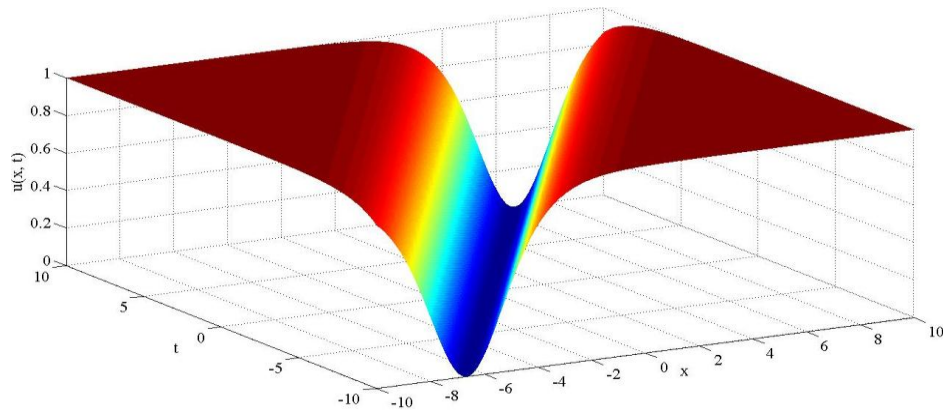


Figure 7. Solitary wave structure for solution (23), where $A = 1$, $B = 1$, $l = 1$ and in the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.

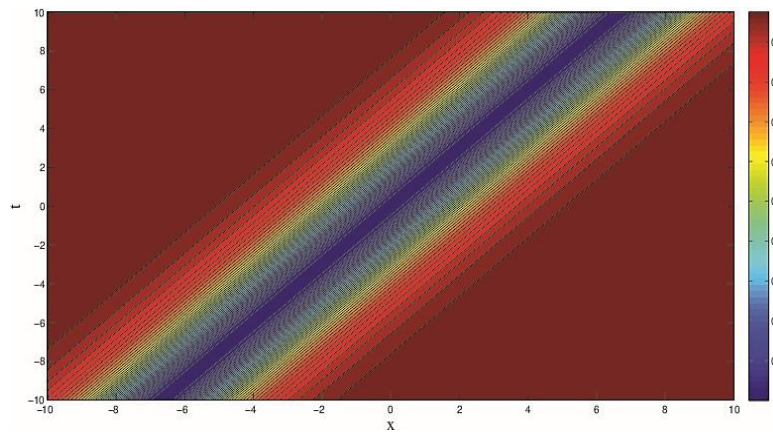


Figure 8. Contour plot for solution (23), where $A = 1$, $B = 1$, $l = 1$ and in the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.

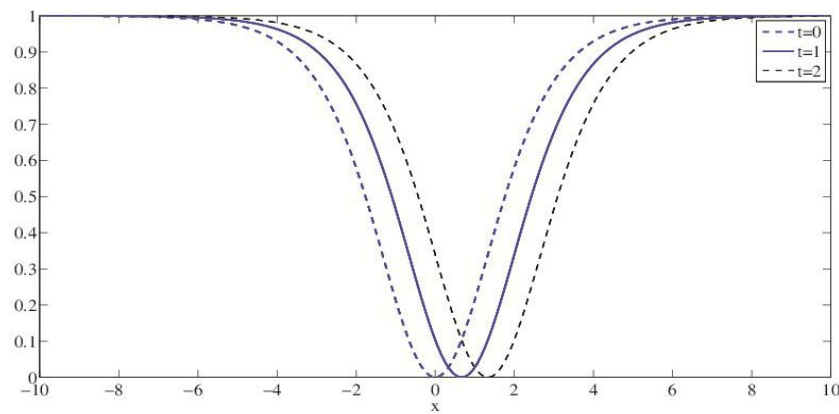


Figure 9. Two-dimensional plot for solution (23), where $A = 1$, $B = 1$, $l = 1$ and in the interval $-10 \leq x \leq 10$, and with $t = 0$, $t = 1$, $t = 2$.

The three-dimensional surface, contour, and two-dimensional profiles of the S-KdVF equation are shown according to the solution (20) for the parameter set $A = 1.5$, $B = 2.5$, and $l = 1$. **Figure 1** depicts anti-kink wave behavior within the ranges $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. **Figure 2** shows the contour plot in the domain $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$, whereas **Figure 3** the two-dimensional profiles at $t = 0$, $t = 1$, and $t = 2$ clearly reveal anti-kink wave patterns inside the interval $-10 \leq x \leq 10$. Solution (21) also produces the anti-kink type wave structure by providing the particular values of the free parameters and looks similar to **Figure 1** with different amplitude.

The three-dimensional surface, contour, and two-dimensional profiles of the S-KdVF equation are given based on the solution (22) for the parameter set $A = 1.5$, $B = 0.5$, and $l = 1$. **Figure 4** displays kink wave forms between the limitations $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. **Figure 5** illustrates a contour map within the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. **Figure 6** depicts the two-dimensional plot for time instances $t = 0$, $t = 1$, and $t = 2$, which forms an anti-kink type wave over the period $-10 \leq x \leq 10$.

The three-dimensional surface, contour, and two-dimensional profiles of the S-KdVF equation are shown according to the solution (23) for the parameter set $A = 1$, $B = 1$, and $l = 1$. **Figure 7** gives the solitary wave structures within the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. **Figure 8** presents the contour plot within the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. **Figure 9** shows the 2-D plot for the S-KdVF equation with $t = 0$, $t = 1$, and $t = 2$ gives the solitary wave within the interval $-10 \leq x \leq 10$.

5. Stability Analysis

In the section, we analyze the influence of the external forcing term on the dynamical behavior of the Schamel KdV equation. To illustrate this influence, we have employed a step-wise forcing term defined as $f(u) = (u - a)(u - b)$. The function $f(u) = 0$ has two roots: a and b . The forcing parameters are a and b . The forcing terms in its current formulation can act as an attenuator or amplifier for solitary waves, depending on the parameters.

Consider the Schamel KdV equation given by

$$\frac{\partial S}{\partial t} + \left(AS^{\frac{1}{2}} + BS \right) \frac{\partial S}{\partial x} + C \frac{\partial^3 S}{\partial x^3} = 0 \quad (24)$$

where, A , B and C are arbitrary coefficients and $S = S(x, t)$. By using the transformation $\eta = x - kt$, Equation (24) becomes

$$-k \frac{dS}{d\eta} + \left(AS^{1/2} + BS \right) \frac{dS}{d\eta} + C \frac{d^3 S}{d\eta^3} = 0 \quad (25)$$

After integrating Equation (25) with respect to η , we have

$$-kS + \frac{2}{3} AS^{3/2} + \frac{BS^2}{2} + C \frac{d^2 S}{d\eta^2} + c_1 = 0 \quad (26)$$

Let us consider $S^{1/2}(\eta) = u(\eta)$ into the Equation (26), we get the following equation

$$-ku^2 + \frac{2}{3} Au^3 + \frac{Bu^4}{2} + 2C \left(\frac{du}{d\eta} \right)^2 + 2Cu \frac{d^2 u}{d\eta^2} + c_1 = 0 \quad (27)$$

where, c_1 is the integration constant. Corresponding dynamical system will be

$$\frac{du}{d\eta} = v; \quad \frac{dv}{d\eta} = \frac{1}{2Cu} \left(ku^2 - \frac{2}{3}Au^3 - \frac{B}{2}u^4 - 2Cv^2 - c_1 \right) \quad (28)$$

Consider the forcing term $f(u) = u(u-a)(u-b)$, we get the following dynamical system

$$\frac{du}{d\eta} = v; \quad \frac{dv}{d\eta} = \frac{1}{2Cu} \left(ku^2 - \frac{2}{3}Au^3 - \frac{B}{2}u^4 - 2Cv^2 - c_1 \right) + E(u-a)(u-b) \quad (29)$$

After solving the dynamical system (28), we get the set solution as the equilibrium point of the system, say E_1 . With the help of the forcing term, the corresponding equilibrium points of the dynamical system (29) will be E_2 . Owing to the highly nonlinear nature of the equilibrium equations, deriving explicit closed-form expressions for the equilibrium points leads to lengthy and complicated algebraic expressions. Therefore, for the parameter values considered throughout this study, the equilibrium points are determined numerically given in **Table 1**.

Table 1. Real roots of the equilibrium equations and the corresponding equilibrium points of the dynamical systems associated with Equations (28) and (29), evaluated for the parameter set $A = 2, B = -0.8, C = 0.5, k = -0.7, c_1 = 0.7, E = 5, a = 1, \text{ and } b = 3$.

System	Roots (u^*)	Equilibrium points (u^*, v^*)
Equation (28)	$u_1 \approx -0.6866,$ $u_2 \approx 3.7477,$ $u_{3,4} \approx 0.1361 \pm 0.5738i.$	$E_1 = (-0.6866, 0)$ $E_2 = (3.7477, 0)$
Equation (29)	$u_1 \approx -0.1995,$ $u_2 \approx 0.2394,$ $u_3 \approx 0.8710,$ $u_4 \approx 3.0408.$	$E_1 = (-0.1995, 0)$ $E_2 = (0.2394, 0)$ $E_3 = (0.8710, 0)$ $E_4 = (3.0408, 0)$

Table 2. Jacobian matrices, characteristic polynomials, and eigenvalues corresponding to the dynamical systems described by Equations (28) and (29).

Quantity	Equation (28)	Equation (29)
Jacobian	$\begin{bmatrix} 0 & 1 \\ A_{27} & -\frac{2v}{u} \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ A_{27} + E(2u - a - b) & -\frac{2v}{u} \end{bmatrix}$
Characteristic polynomial at equilibrium	$\lambda^2 - A_{27} = 0$	$\lambda^2 - (A_{27} + E(2u_e - a - b)) = 0$
Eigenvalues	$\lambda = \pm \sqrt{A_{27}}$	$\lambda = \pm \sqrt{A_{27} + E(2u_e - a - b)}$
where, $A_{27} = \frac{ku_e^2 - \frac{4}{3}Au_e^3 - \frac{B}{2}Bu_e^4 + c_1}{2Cu_e^2}$.		

Table 3. Classification of equilibrium points based on the eigenvalues of the Jacobian matrix for the local stability analysis of the dynamical systems given by Equations (28) and (29).

Eigenvalue condition	Nature of equilibrium point	Stability
$\lambda_1 < 0, \lambda_2 < 0$	Stable node	Asymptotically stable
$\lambda_1, \lambda_2 < 0$ (opposite signs)	Saddle point	Unstable
$\lambda_{1,2} = \alpha \pm i\beta, \alpha < 0, \beta \neq 0$	Spiral sink (stable focus)	Asymptotically stable
$\lambda_{1,2} = \alpha \pm i\beta, \alpha > 0, \beta \neq 0$	Spiral source (unstable focus)	Unstable
$\lambda_{1,2} = \pm i\beta, \beta \neq 0$	Center (Hopf bifurcation threshold)	Neutrally stable

The explicit expressions of the Jacobian matrices, the corresponding characteristic polynomials, and the associated eigenvalues for the dynamical systems governed by Equations (28) and (29) are presented in **Table 2**. Based on the eigenvalues of the Jacobian matrices evaluated at the equilibrium points, the local

stability of each equilibrium is classified, and the corresponding stability characteristics for both dynamical systems are summarized in **Table 3**.

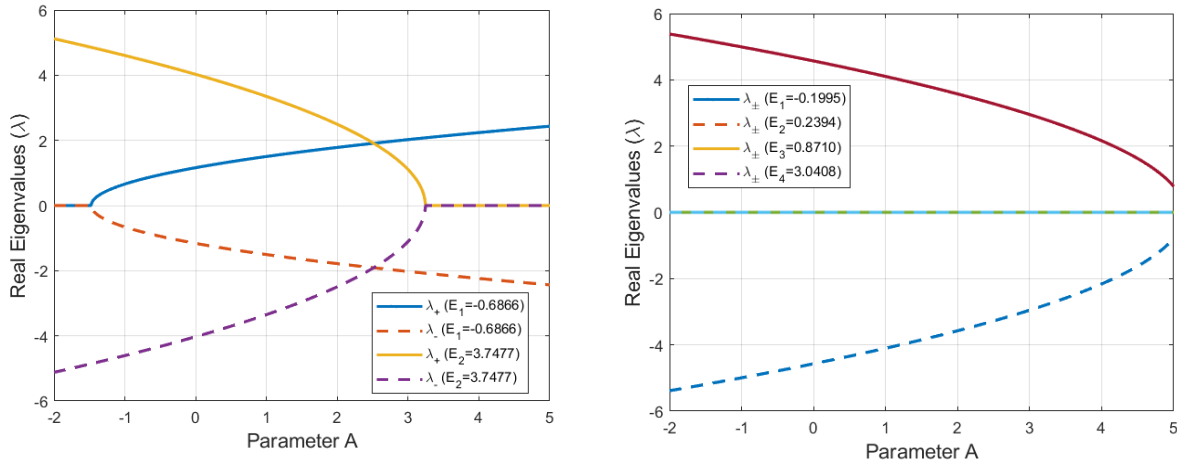


Figure 10. Variation of the real parts of the eigenvalues with respect to the parameter A . The first figure represents eigenvalue spectrum of the unforced dynamical system given by Equation (28) for the equilibrium points $E_1 = (-0.6866, 0)$ and $E_2 = (3.7477, 0)$. The second figure represents the eigenvalue spectrum of the forced dynamical system given by Equation (29) for the equilibrium points $E_1 = (-0.1995, 0)$, $E_2 = (0.2394, 0)$, $E_3 = (0.8710, 0)$, and $E_4 = (3.0408, 0)$, illustrating their local stability characteristics.

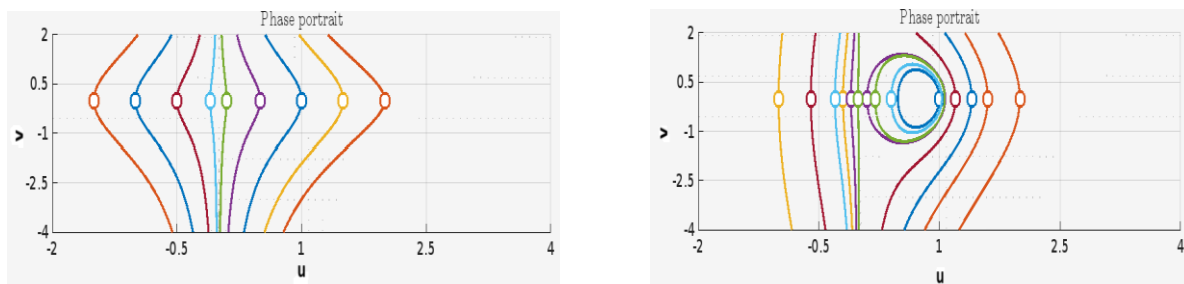


Figure 11. Phase portraits with respect to the model with and without forcing term. The value of the other parameters is $B = -0.8$, $k = -0.7$, $C = 0.5$, $A = 2$, $c_1 = 0.7$.

Figure 10 illustrate the variation of the real parts of the eigenvalues with the parameter A for the unforced and forced dynamical systems, respectively. For Equation (28), the equilibrium point $E_1 = (-0.6866, 0)$ remains an unstable saddle point since its eigenvalues have opposite signs throughout the considered range of A . In contrast, the equilibrium point $E_2 = (3.7477, 0)$ undergoes a stability transition near $A \approx 3.3$, where both eigenvalues approach zero, indicating the onset of a bifurcation. Beyond this critical value, the eigenvalues become purely imaginary, corresponding to center-type behavior in the linearized system. For Equation (29), no eigenvalue crosses the zero axis over the investigated parameter range. The equilibrium points E_1 and E_4 remain unstable saddle points, whereas E_2 and E_3 remain non-hyperbolic with zero real eigenvalues. These results show that the forcing term suppresses the bifurcation observed in the unforced system and significantly modifies the local stability of the equilibrium points.

With respect to these equilibrium points, the corresponding behavior of both dynamical systems is shown in **Figure 11**. The **Figure 11** shows the unstable behavior of the trajectories moving in the neighborhood of the equilibrium points of the dynamical system (28). The instability in the system can be viewed with the help of the opposite directions of the direction lines along the trajectories shown in the figure. But when we introduce the forcing term into the dynamical system (28), this converts the system into a stable one. This forcing term generates a stable point in the system around which we may get the periodic solutions of the Equation (24). This region of the stable solution can be visible in the given second figure between the range value of $u \in (2.5, 6)$.

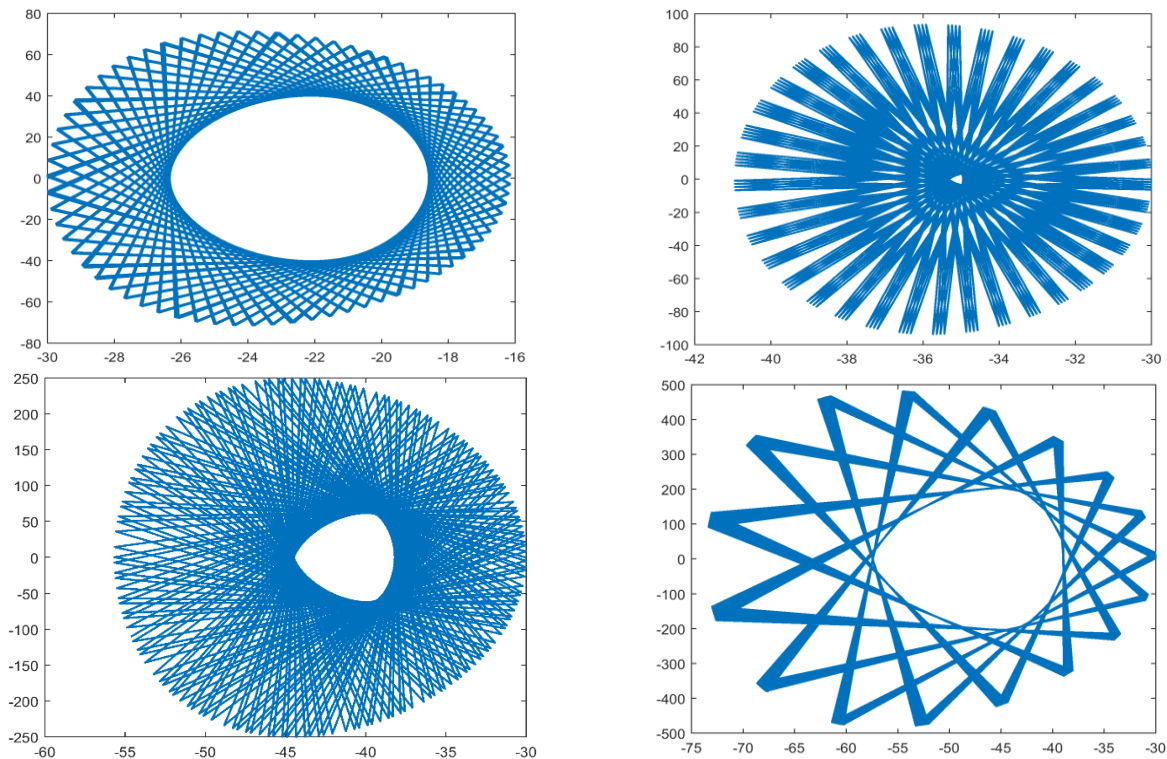


Figure 12. Phase portrait of the dynamical system Equation (28) for the range of the parameter $A \in [1, 2.33]$. The corresponding values of the parameters are $A = [1, 1.4, 1.8, 2.333]$, $B = -0.8$, $k = -0.7$, $c_1 = 0.7$, $C = 0.5$. The time interval is $0:6 \pi / 100:10 \pi$.

Figure 12 shows the quasi-periodic cyclic behavior with respect to the exact equilibrium point which is calculated by the evaluated equilibrium point from dynamical system (28). These figures represent the range of the parameter $A \in [1, 2.33]$ with respect to the four represented values of this parameter. The break point with respect to the range of the parameters is $A = 2.33$ at which the system becomes completely unstable. When we introduce the step-wise forcing term into the dynamical system then system does not show any representative behaviour of the system with respect to the same range of the parameter A shown in **Figure 13**. Therefore, we have perturbed the values of this parameter to get the desired behavior of the system.

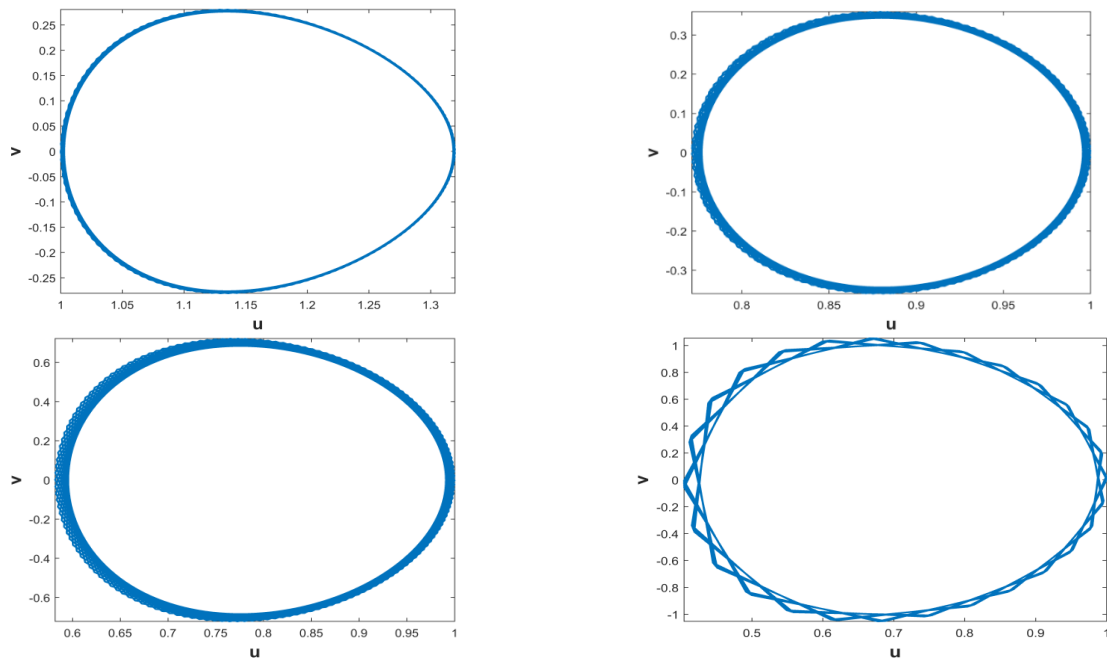


Figure 13. Phase portrait of the dynamical system Equation (29) after introducing the forcing term $f(u) = (u - a)(u - b)$ for the range of the parameter $A \in [-2.5, 3]$. The corresponding values of the parameters are $A = [-2.5, 0.001, 1.6, 3]$, $B = -0.8$, $k = 0.7$, $c_1 = 0.7$, $C = 0.5$, $E = 5$, $a = 1$ and $b = 3$. The time interval is $0: 6\pi/100: 10\pi$.

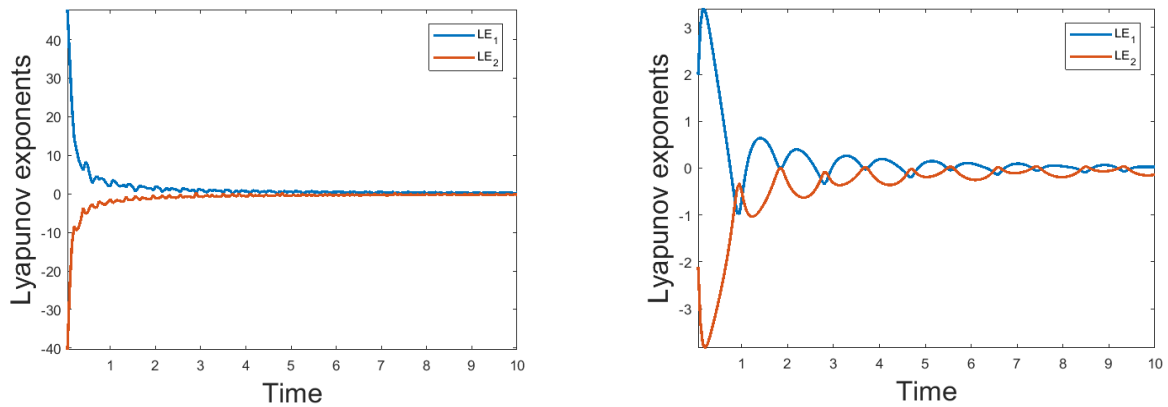


Figure 14. Lyapunov Exponent plot with respect to both the dynamical system Equations (28) and (29). The corresponding values of the parameters for the first figure are $B = -0.05$, $k = 0.7$, $c_1 = 0.7$, $C = 0.5$, $A = 0.2$ and for the second figure where the effect of the forcing term is clearly visible, are $A = 2$, $B = -0.8$, $k = 0.7$, $c_1 = 0.7$, $C = 0.5$, $E = 5$, $a = 1$ and $b = 3$.

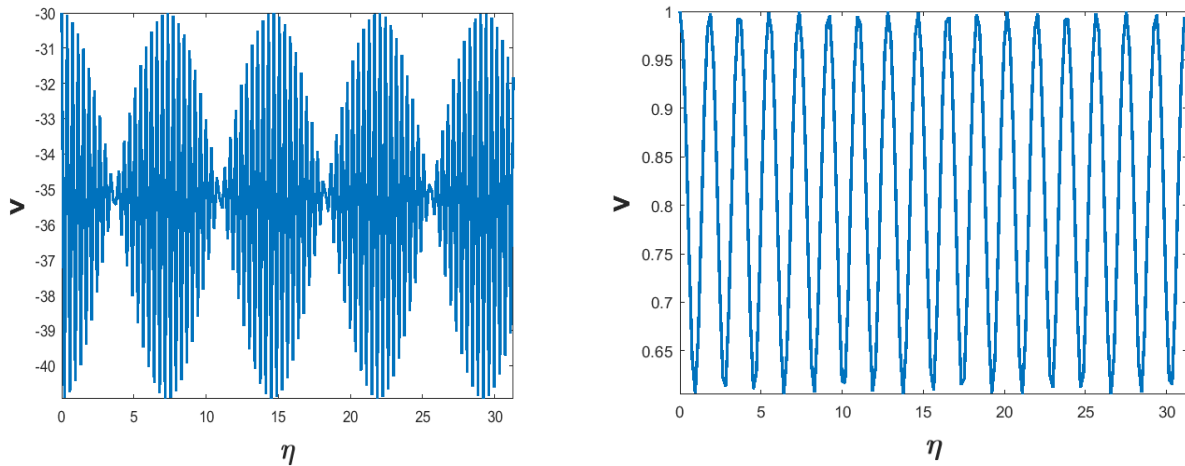


Figure 15. Time series plot with respect to both the dynamical system Equations (28) and (29). The corresponding values of the parameters are $A = 1.4$, $B = -0.8$, $k = -0.7$, $c_1 = 0.7$, $C = 0.5$, for the first figure and for the second figure the values are $A = 2$, $B = -0.8$, $k = 0.7$, $c_1 = 0.7$, $C = 0.5$, $E = 5$, $a = 1$ and $b = 3$.

To quantify the stability of the nonlinear dynamical system, the largest Lyapunov exponent (LLE) was computed by integrating both the original nonlinear systems given by Equations (28) and (29) and its associated variational equations simultaneously, shown in **Figure 14**. For a nonlinear system expressed as $\dot{x} = f(x)$, the evolution of a small perturbation vector δ is governed by the variational equation $\dot{\delta} = Df(x)\delta$, where $Df(x)$ denotes the Jacobian matrix evaluated along the system trajectory. The numerical computation of the largest Lyapunov exponent was carried out using the well-established algorithm proposed by Benettin et al. (1976). During the integration process, the perturbation vector was periodically normalized using the Gram–Schmidt orthogonalization procedure to prevent numerical overflow and to preserve its direction (Benettin et al., 1980; Golub and Van Loan, 2013). The largest Lyapunov exponent was then estimated from the long-term average growth rate of the perturbation according to

$$\Lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=1}^N \ln \left(\frac{||\delta_k||}{||\delta_0||} \right),$$

where, δ_0 and δ_k represent the initial and periodically normalized perturbation vectors, respectively. This procedure measures the average exponential rate at which nearby trajectories either converge or diverge in phase space. A negative value of the largest Lyapunov exponent ($\Lambda < 0$) indicates that perturbations decay with time, confirming that the equilibrium is asymptotically stable. A zero Lyapunov exponent ($\Lambda = 0$) corresponds to periodic or quasi-periodic motion, whereas a positive exponent ($\Lambda > 0$) signifies exponential divergence of nearby trajectories, which is a characteristic feature of chaotic dynamics. The **Figure 14** shows the corresponding chaos in the first figure for the dynamical system (28) and smooth behavior is shown in second figure with respect to the dynamical system (29). The corresponding strong quasi-periodic and weak quasi-periodic behavior of the systems are shown with the help of the time series plots in **Figure 15**.

6. Conclusion

We have effectively resolved the Schamel KdV equation with force terms utilising the Kudryashov expansion technique. The Kudryashov expansion method is determined to be an effective and uncomplicated technique for nonlinear partial differential equations involving wave transformations. The obtained solutions are exponential function forms. We have derived the surface plots, contour plots and two-dimensional plots for the solutions for the SKdV equation and also successfully obtained the kink and anti-kink type wave structures. Further, the obtained solutions are new for the SKdV equation with a force term. We have verified the obtained solutions by using the back substitution method and with the help of Maple. Every analytical method possesses inherent limitations, mainly when the balancing method is ineffective. The coefficients of the highest-order term in the initial approximation of the solution are zero. The resulting solutions include exponential functions and their contour plots, two-dimensional plots with predetermined periods, and three-dimensional surface plots. Further, the stability analysis of the model is well explained through the time series analysis and Lyapunov exponent. There may be significant uses for the derived solutions and techniques in the physical sciences, engineering, and applied mathematics.

Conflicts of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgments

This research did not receive any specific grant from funding agencies in the public, commercial, or non-profit sectors. The authors would like to thank the editor and the anonymous reviewers for their comments that helped improve the quality of this work.

AI Discourse

The author(s) confirm that generative AI tools were not used during the preparation of this article.

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