

Optimization of a Fuzzy Inventory Model with Quadratic Demand, Time-Dependent Costs and Shortages Using the Convergent Gradient Adaptive Search Algorithm

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Abstract

Inventory decisions for deteriorating items become complex when demand varies over time and cost parameters are uncertain. This study develops a fuzzy inventory model with quadratic demand, time-dependent holding cost, time-dependent deterioration and fully backlogged shortages. The ordering, holding, shortage and deterioration costs are represented by hexagonal fuzzy numbers and converted into representative crisp values using the graded mean integration method. The resulting total average cost function is minimized using the proposed Convergent Gradient Adaptive Search Algorithm (CGASA). The novelty of CGASA lies in combining feasible initialization, adaptive learning-rate control, descent correction and local refinement within a deterministic gradient-based procedure. A numerical example based on a hospital pharmacy inventory setting is used to demonstrate the applicability of the model to temperature-sensitive medical products. The results indicate stable convergence and support replenishment decisions for healthcare and perishable inventory systems under cost uncertainty. Sensitivity analysis further shows the influence of ordering cost, holding cost, deterioration rate and shortage cost on the inventory policy.

Keywords- Quadratic demand, Total average cost, Hexagonal fuzzy numbers, Convergent gradient adaptive search algorithm, Exploration–exploitation.

1. Introduction

Inventory decisions play an important role in the long-term stability of business organizations, as they are influenced by changing demand, ordering cost, holding cost and product deterioration. Deterioration is common in products such as fruits, vegetables, grains, gasoline, alcohol and electronic components, which may lose their quality or value over time. Traditional deterministic inventory models may not adequately represent practical situations in which demand changes over time, products deteriorate and cost parameters are uncertain. To overcome these limitations, researchers increasingly use fuzzy logic and other mathematical methods to represent uncertainty more realistically.

Inventory models are used to determine the stock levels required to satisfy customer demand at minimum cost. The classical Economic Order Quantity (EOQ) model was formulated under the assumptions of constant demand and fixed cost parameters (Harris, 1913). Later, this framework was extended by incorporating a permissible delay in payments (Goyal, 1985). The exponential decay model became an important basis for inventory studies involving deteriorating items (Ghare and Schrader, 1963). However,

this model did not consider uncertainty or situations involving multiple decision factors. Fuzzy set theory was introduced to represent imprecise information (Zadeh, 1965). It was later extended to intuitionistic fuzzy sets for improved representation of uncertainty (Atanassov, 1999). The application of fuzzy logic to real-life decision problems has also been discussed in the literature (Dubois and Prade, 1980). The graded mean integration (GMI) method has been used to convert fuzzy quantities into representative crisp values for decision-making under uncertainty (Klir and Yuan, 1995). Neutrosophic hesitant fuzzy programming (NHFP) and the Pythagorean hesitant fuzzy programming approach (PHFPA) have also been applied to an inventory model with quadratic demand and time-varying costs (Das and Islam, 2024). Previous studies have shown that fuzzy numbers are useful for representing uncertainty in both single-objective and multi-objective inventory models (Chen et al., 2007; Taleizadeh et al., 2013; Theeba et al., 2024). As fuzzy inventory models become more complex, suitable optimization techniques are required to obtain stable numerical solutions. Gradient-based optimization methods are widely used to update decision variables in nonlinear optimization problems (Boyd and Vandenberghe, 2004; Nesterov, 2018). Several adaptive learning-rate and gradient-based methods have also been developed for nonlinear and unconstrained optimization problems (Kingma and Ba, 2014). Recent studies on the Gradient-Based Optimizer, its improved variants and neutrosophic optimization methods have further demonstrated the usefulness of adaptive search procedures (Altbawi et al., 2023; Daoud et al., 2023; Elsayed and Kolankeh, 2025). In addition, uncertainty-based decision-making methods have been applied to multi-objective transportation and supply chain problems (Giri and Roy, 2022; Giri and Roy, 2024).

Adaptive learning-rate methods regulate the step size during iterative optimization according to gradient behaviour. These methods are useful for nonlinear objective functions because a fixed learning rate may lead to slow convergence or oscillation. Several studies have examined adaptive learning-rate strategies and their convergence properties (Chakrabarti and Chopra, 2021; Jiang et al., 2024; Liu et al., 2019; Nawi et al., 2017). Methods such as AdaGrad, RMSProp and Adam have also demonstrated the importance of step-size adaptation in neural network training and related applications (Huang et al., 2020; Jin et al., 2024). However, these methods were mainly developed for stochastic and large-scale optimization and usually rely on accumulated gradients or moment estimates. They do not directly incorporate inventory feasibility, descent-check correction or problem-specific learning-rate bounds.

Most fuzzy inventory models are solved by first converting fuzzy quantities into crisp values and then applying classical or heuristic optimization methods. However, limited attention has been given to a deterministic adaptive gradient procedure specifically designed for a nonlinear total average cost (TAC) function involving quadratic demand, time-dependent deterioration, shortages and fuzzy cost parameters. To address this research gap, the present study proposes the Convergent Gradient Adaptive Search Algorithm (CGASA). Unlike Adam and RMSProp, CGASA does not rely on moving averages or accumulated moment-based gradient information. Instead, it combines feasible initialization based on the time at which the inventory level reaches zero, data-based learning-rate selection, descent-check correction, bounded step-size control, β -based local refinement and exploration–exploitation monitoring. These features provide a problem-oriented procedure for obtaining a stable replenishment decision. The proposed model is applicable to perishable inventory systems in which demand varies over time, deterioration increases with storage duration and shortages lead to additional costs.

The remainder of the paper is organized as follows. Section 2 presents the preliminary definitions. Section 3 describes the inventory management model. Section 4 develops the inventory formulation and mathematical model. Section 5 presents the proposed CGASA solution procedure and convergence analysis. Section 6 provides the computational example. Section 7 discusses the sensitivity analysis and computational performance. Finally, Section 8 presents the conclusion, managerial implications, limitations

and future scope.

2. Preliminaries

This section presents the basic definitions required for the development of the proposed fuzzy inventory model.

Definition 2.1 (Zadeh, 1965). A fuzzy number $\tilde{A}$ is a fuzzy set of the real line $\mathbb{R}$ with a membership function $\mu_{\tilde{A}}: \mathbb{R} \rightarrow [0,1]$ satisfying the following properties:

- $\exists x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$.
- Fuzzy convexity: $\forall x_1, y_1 \in \mathbb{R}, \lambda \in [0, 1], \mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)y_1) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(y_1)\}$.
- $\mu_{\tilde{A}}(x)$ is piecewise continuous on $\mathbb{R}$.
- $\mu_{\tilde{A}}(x)$ is upper semi-continuous on $\mathbb{R}$.

Definition 2.2 (Das and Islam, 2024). A hexagonal fuzzy number (HFN) is denoted by $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6, 1)$, where $\omega_1 \in [0,1]$, $a_i \in \mathbb{R}$ (real number) $i = 1, 2, \dots, 6$ and its membership function $\mu_{\tilde{A}_H}(x)$ is given as follows.

$$\mu_{\tilde{A}_H}(x) = \begin{cases} \omega_1 \left(\frac{x - a_1}{a_2 - a_1} \right), & a_1 \leq x \leq a_2 \\ \omega_1 + (1 - \omega_1) \left(\frac{x - a_2}{a_3 - a_2} \right), & a_2 \leq x \leq a_3 \\ 1, & a_3 \leq x \leq a_4 \\ 1 - (1 - \omega_1) \left(\frac{x - a_4}{a_5 - a_4} \right), & a_4 \leq x \leq a_5 \\ \omega_1 \left(\frac{a_6 - x}{a_6 - a_5} \right), & a_5 \leq x \leq a_6 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 (Das and Islam, 2024). Consider a hexagonal fuzzy number (HFN) $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6; \omega)$, where $0 < \omega \leq 1$. By the graded mean integration representation, the corresponding crisp value is given by

$$R(\tilde{A}_H) = \frac{\omega(2a_1 + 3a_2 + 4a_3 + 4a_4 + 3a_5 + 2a_6)}{18}.$$

Definition 2.4 (Fuzzy Arithmetic). Let $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6; \omega)$ and $\tilde{B}_H = (b_1, b_2, b_3, b_4, b_5, b_6; \omega)$ be two hexagonal fuzzy numbers with the same height ω . The arithmetic operations on their Graded Mean Integration (GMI) values are defined as follows:

- **Addition:** $R(\tilde{A}_H + \tilde{B}_H) = R(\tilde{A}_H) + R(\tilde{B}_H)$.
- **Subtraction:** $R(\tilde{A}_H - \tilde{B}_H) = R(\tilde{A}_H) - R(\tilde{B}_H)$.
- **Multiplication:** $R(\tilde{A}_H \cdot \tilde{B}_H) = R(\tilde{A}_H) \cdot R(\tilde{B}_H)$.

- **Division:** $R(\tilde{A}_H \div \tilde{B}_H) = \frac{R(\tilde{A}_H)}{R(\tilde{B}_H)}$ if $R(\tilde{B}_H) \neq 0$.

Note: The above operations are based on the graded mean integration (GMI) method, which yields crisp values for fuzzy quantities. If computations are to be carried out using α -cut or parametric representations instead, the corresponding interval arithmetic rules should be applied to preserve the fuzziness throughout the calculation.

3. Inventory Management Model

The notation used in the proposed inventory model is presented in **Table 1**.

Table 1. Notation used in the proposed inventory model.

Notation	Description
$I(t), S(t), \bar{D}(t), \theta(t)$	Fuzzy inventory level, shortage level, demand rate and deterioration rate at time t .
$I(t), S(t), D(t), \theta(t)$	Corresponding representative crisp inventory level, shortage level, demand rate and deterioration rate at time t .
$\tilde{a}, \tilde{b}, \tilde{c}$	Fuzzy coefficients of the quadratic demand function.
a, b, c	Corresponding representative crisp coefficients of the quadratic demand function.
t_1	Time at which the inventory level becomes zero.
t_2	Total replenishment-cycle length.
$I_{\max}, S_{\max}$	Maximum inventory and maximum shortage levels.
Q	Replenishment quantity per cycle.
$\tilde{\alpha}, \alpha$	Fuzzy deterioration-rate coefficient and its representative crisp value.
$\tilde{O}, \tilde{h}, \tilde{S}_c, \tilde{d}$	Fuzzy ordering, holding, shortage and deterioration cost parameters.
O, h, S_c, d	Corresponding representative crisp cost parameters.
$\bar{TAC}(t_1, t_2), TAC(t_1, t_2)$	Fuzzy total average cost per unit time and its representative crisp value.

This section presents the mathematical formulation of the inventory model with fuzzy quadratic demand, time-dependent deterioration, hexagonal fuzzy cost parameters and a total average cost minimization. The inventory pattern is shown in **Figure 1**.

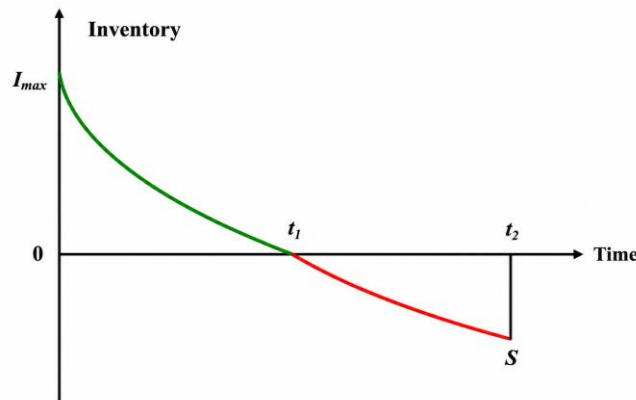


Figure 1. Inventory pattern during the depletion and shortage periods.

The inventory system is initially formulated using fuzzy parameters. These parameters are then converted into their representative crisp values using the ranking function $R(\cdot)$, leading to a deterministic optimization problem.

In practical inventory systems involving healthcare and perishable products, demand may vary nonlinearly due to seasonal changes, patient requirements and market conditions. Therefore, a quadratic demand function is considered to represent such time-varying demand behaviour. Similarly, the deterioration of medicines, food products and other temperature-sensitive items may increase with storage duration. Hence, a time-dependent deterioration rate is adopted to describe the gradual loss of quality over time.

Based on these practical considerations, the following assumptions are made:

- The lead time is assumed to be negligible.
- The demand rate is represented by a fuzzy quadratic function of time. After applying the ranking function, the corresponding representative crisp form is expressed as

$$R(\tilde{D}(t)) = R(\tilde{a} + \tilde{b}t + \tilde{c}t^2) = a + bt + ct^2 \quad (1)$$

where, $R(\tilde{a}) = a$, $R(\tilde{b}) = b$, $R(\tilde{c}) = c$.

- The time-dependent deterioration rate is given by

$$R(\tilde{\theta}(t)) = \theta(t) = \alpha t, \quad R(\tilde{\alpha}) = \alpha \quad (2)$$

- All shortages occurring during the shortage period are assumed to be completely backlogged.
- The time-dependent holding-cost rate is defined as

$$H(t) = R(\tilde{h})t = ht, \quad R(\tilde{h}) = h \quad (3)$$

4. Inventory Formulation

This section develops the mathematical formulation of the proposed inventory model. The replenishment cycle consists of two stages: a positive-inventory period and a shortage period. The inventory levels corresponding to these stages are derived first. Subsequently, the ordering, holding, shortage and deterioration costs are formulated. Based on these cost components, the total average cost function and the related optimization problem are established.

4.1 Inventory Dynamics

Immediately after replenishment at $t = 0$, the inventory level reaches its maximum value, denoted by $I_{\max}$. During the interval $0 \leq t \leq t_1$, the on-hand inventory gradually decreases because of customer demand and product deterioration.

At $t = t_1$, the inventory level becomes zero, marking the end of the positive-inventory period and the beginning of the shortage period. During $t_1 \leq t \leq t_2$, shortages accumulate because of continued customer demand. At the end of the replenishment cycle, when $t = t_2$, the shortage level reaches its maximum value, denoted by $S_{\max}$.

The inventory pattern over the complete replenishment cycle is illustrated in **Figure 1**. The on-hand inventory level during $0 \leq t \leq t_1$ is denoted by $I(t)$, whereas the accumulated shortage level during $t_1 \leq t \leq t_2$ is denoted by $S(t)$.

4.2 Inventory Phase ($0 \leq t \leq t_1$)

During the positive-inventory phase, the inventory level decreases because of demand and deterioration. Hence, the governing differential equation is

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -D(t), \quad 0 \leq t \leq t_1 \quad (4)$$

The corresponding boundary conditions are $I(0) = I_{\max}$, $I(t_1) = 0$.

Substituting the demand function in Equation (1) and the deterioration rate in Equation (2) into Equation (4) and applying the boundary condition $I(t_1) = 0$, the first-order approximate inventory level is obtained as

$$I(t) = a(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) + \frac{c}{3}(t_1^3 - t^3) + \frac{\alpha}{2} \left[\frac{a}{3}(t_1^3 - t^3) + \frac{b}{4}(t_1^4 - t^4) + \frac{c}{5}(t_1^5 - t^5) \right]. \quad (5)$$

Equation (5) is obtained by applying a first-order Taylor approximation to the exponential terms arising from the integrating-factor solution. The neglected terms are of order $O((\alpha t^2)^2)$. Therefore, the approximation is appropriate when $\alpha t_1^2 < 1$. Under this condition, the higher-order terms have only a limited influence on the inventory level and the resulting total average cost. However, the approximation error may increase as αt_1^2 becomes larger. The approximation simplifies the subsequent cost formulation while retaining the leading influence of the time-dependent deterioration rate.

4.3 Shortage Phase ($t_1 \leq t \leq t_2$)

During the shortage phase, shortages accumulate at the same rate as customer demand. Therefore,

$$\frac{dS(t)}{dt} = D(t), \quad t_1 \leq t \leq t_2 \quad (6)$$

The corresponding boundary condition is $S(t_1) = 0$.

Solving Equation (6) subject to the boundary condition $S(t_1) = 0$, the shortage level is obtained as

$$S(t) = a(t - t_1) + \frac{b}{2}(t^2 - t_1^2) + \frac{c}{3}(t^3 - t_1^3) \quad (7)$$

Substituting $t = 0$ into Equation (5), the maximum inventory level is obtained as

$$I_{\max} = at_1 + \frac{b}{2}t_1^2 + \frac{c}{3}t_1^3 + \frac{\alpha}{2} \left(\frac{a}{3}t_1^3 + \frac{b}{4}t_1^4 + \frac{c}{5}t_1^5 \right) \quad (8)$$

Substituting $t = t_2$ into Equation (7), the maximum shortage level is obtained as

$$S_{\max} = a(t_2 - t_1) + \frac{b}{2}(t_2^2 - t_1^2) + \frac{c}{3}(t_2^3 - t_1^3) \quad (9)$$

Therefore, the replenishment quantity per cycle is

$$Q = I_{\max} + S_{\max}.$$

4.4 Ordering Cost

The ordering cost incurred during each replenishment cycle is

$$OC = R(\tilde{O}) = O \quad (10)$$

4.5 Holding Cost

Since the holding-cost rate varies linearly with time, the holding cost incurred during the positive-inventory period is expressed as

$$HC = h \int_0^{t_1} t I(t) dt = h \left[\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \frac{c}{10}t_1^5 + \alpha \left(\frac{a}{20}t_1^5 + \frac{b}{24}t_1^6 + \frac{c}{28}t_1^7 \right) \right] \quad (11)$$

here, h denotes the representative crisp holding-cost coefficient.

4.6 Shortage Cost

The shortage cost incurred during the shortage period is expressed as

$$SC = S_c \int_{t_1}^{t_2} S(t) dt = S_c \left[\frac{a}{2} (t_2 - t_1)^2 + \frac{b}{6} (2t_2^3 - 3t_1 t_2^2 + t_1^3) + \frac{c}{12} (3t_2^4 - 4t_1 t_2^3 + t_1^4) \right] \quad (12)$$

here, S_c denotes the representative crisp shortage-cost coefficient.

4.7 Deterioration Cost

The deterioration cost represents the value of inventory lost due to deterioration during the positive-inventory period. It is expressed as

$$DC = d \int_0^{t_1} \alpha I(t) dt = d\alpha \left(\frac{a}{6} t_1^3 + \frac{b}{8} t_1^4 + \frac{c}{10} t_1^5 \right) \quad (13)$$

Here, d denotes the representative crisp deterioration cost per unit.

4.8 Total Average Cost

The fuzzy total average cost is converted into its representative crisp value using the ranking function. The total average cost per unit time is defined as

$$R(\overline{TAC}(t_1, t_2)) = TAC(t_1, t_2) = \frac{1}{t_2} [OC + HC + SC + DC] \quad (14)$$

Dividing the total cost per cycle by the cycle length t_2 gives the total average cost per unit time. Substituting Equations (10)-(13) into Equation (14), we obtain

$$TAC(t_1, t_2) = \frac{1}{t_2} \left[\begin{aligned} &O + h \cdot \left(\frac{a}{6} t_1^3 + \frac{b}{8} t_1^4 + \frac{c}{10} t_1^5 + \alpha \left[\frac{a}{20} t_1^5 + \frac{b}{24} t_1^6 + \frac{c}{28} t_1^7 \right] \right) \\ &+ S_c \cdot \left(\frac{a}{2} [(t_2 - t_1)^2] + \frac{b}{6} [2t_2^3 - 3t_1 t_2^2 + t_1^3] + \frac{c}{12} [3t_2^4 - 4t_1 t_2^3 + t_1^4] \right) \\ &+ d \cdot \alpha \left(\frac{a}{6} t_1^3 + \frac{b}{8} t_1^4 + \frac{c}{10} t_1^5 \right) \end{aligned} \right] \quad (15)$$

4.9 Mathematical Formulation

The inventory optimization problem is formally expressed as

$$\min_{t_2} TAC(t_1, t_2) \quad (16)$$

subject to

$$t_1 > 0, \quad t_2 > t_1 \quad (17)$$

The depletion time t_1 is determined separately from the specified maximum inventory level $I_{\max}$ and is subsequently treated as fixed during the CGASA iterations. Therefore, t_2 is the principal decision variable. The condition $t_2 > t_1$ ensures that the total replenishment cycle extends beyond the positive-inventory period and includes a nonzero shortage period.

5. Proposed CGASA Solution Procedure and Convergence Analysis

This section presents the proposed Convergent Gradient Adaptive Search Algorithm (CGASA) for minimizing the total average cost with respect to the replenishment-cycle length. The solution procedure, pseudocode, methodological comparison, flowchart and convergence analysis are presented in the following subsections.

5.1 Solution Procedure

In the proposed procedure, t_2 is optimized while t_1 is treated as fixed. Starting from a feasible initial value, CGASA applies a gradient-based update and verifies whether the objective function decreases. The learning rate is adjusted when necessary and a β -based correction is used for local refinement. The iterations continue until the prescribed stopping criterion is satisfied.

Step 1: Initialization

- The value of t_1 is obtained numerically using Mathematica by computing the unique positive root. This value represents the time at which the available inventory becomes zero.
- The initialization $t_2^{(0)} = t_1 + \delta$ (with $\delta = \rho \cdot t_1$ and $\rho = 0.1$) guarantees feasibility ($t_2 > t_1$) from the first iteration.
- The initial learning rate η is set in a data-based way as:

$$\eta^{(0)} = \frac{1}{(\|\nabla TAC\| + \varepsilon)}.$$

where, $\varepsilon > 0$ is a small constant.

- Finally, the main control parameters of CGASA are the learning rate η , the increase and decrease factors γ_{increase} and γ_{decrease} , the fine-tuning parameter β and convergence tolerance ε .

Step 2: Dynamic Gradient Descent

Compute the partial derivatives:

$$\frac{\partial TAC}{\partial t_1}, \frac{\partial TAC}{\partial t_2}.$$

Update t_2 :

$$t_2^{(k+1)} = t_2^k - \eta \left(\frac{\partial TAC}{\partial t_2} \right)_{t_2=t_2^{(k)}}.$$

Descent-check and η reduction (to avoid violation): If $TAC(t_2^{k+1}) \geq TAC(t_2^k)$, then, reduce the learning rate and recompute: $\eta^k \leftarrow \gamma_{\text{decrease}} \eta^k$, $0 < \gamma_{\text{decrease}} < 1$,

$$t_2^{(k+1)} = t_2^k - \eta \left(\frac{\partial TAC}{\partial t_2} \right).$$

Repeat this reduction until $TAC(t_2^{(k+1)}) < TAC(t_2^{(k)})$.

The adaptive learning rate mechanism adjusts the learning rate dynamically based on the gradient norm:

$$\eta^{(k+1)} = \eta^k \times \text{AdaptFactor},$$

where the adaptation factor is defined as

$$\text{AdaptFactor} = \begin{cases} \gamma_{\text{increase}}, & \text{if } \|\nabla TAC(t^{(k)})\| > \|\nabla TAC(t^{(k-1)})\|, \\ \gamma_{\text{decrease}}, & \text{otherwise.} \end{cases}$$

After each adaptive update, the learning rate is restricted to the admissible interval $[\eta_{\min}, \eta_{\max}]$, where $0 < \eta_{\min} \leq \eta^{(k+1)} \leq \eta_{\max} < 2/L$.

Typical values are $\gamma_{\text{increase}} = 1.1$, $\gamma_{\text{decrease}} = 0.9$ and $\|\nabla TAC(k)\| = \sqrt{\left(\frac{\partial TAC}{\partial t_1}\right)^2 + \left(\frac{\partial TAC}{\partial t_2}\right)^2}$.

The values $\gamma_{increase} = 1.1$, $\gamma_{decrease} = 0.9$ are used to make moderate adjustments of 10% to the learning rate. Such controlled changes help to avoid sudden variations in the step size and promote a stable descent process. The β -based correction is updated according to the behaviour of the gradient norm and therefore serves as a deterministic local-refinement mechanism instead of a fixed tuning parameter. The bound $\eta_{max} < 2/L$ is consistent with the standard stability condition for gradient-based methods applied to smooth objective functions.

Step 3: Fine-Tuning Decision Variable

Refine the updated value of t_2 using the β -based correction rule:

$$t_2^{(k+1)} = t_2^{(k+1)} - \left[\frac{\beta^{(k+1)}}{2} \times (t_2^{(k+1)} - t_2^k) \right].$$

At the first iteration, the refinement parameter $\beta^{(0)}$ is initialized deterministically as

$$\beta^{(0)} = \frac{\|\nabla TAC(t^{(0)})\|}{1 + \|\nabla TAC(t^{(0)})\|}, 0 < \beta^{(0)} < 1.$$

Update rule for $\beta^{(k+1)}$

$$\beta^{(k+1)} = \begin{cases} 1.05 \beta^{(k)}, & \text{if } \|\nabla TAC(t^{(k)})\| > \|\nabla TAC(t^{(k-1)})\|, \\ 0.95 \beta^{(k)}, & \text{otherwise.} \end{cases}$$

Step 4: Exploration–Exploitation Coefficient

Compute the exploration–exploitation coefficient θ_k as:

$$\theta_k = \frac{\left(\left\| \frac{\partial TAC}{\partial t_1} \right\|^2 + \left\| \frac{\partial TAC}{\partial t_2} \right\|^2 \right)}{(\|\Delta t_1\| + \|\Delta t_2\|)}.$$

The coefficient θ_k is used only as a diagnostic measure to interpret the search behaviour of CGASA. A higher θ_k indicates exploration, whereas a lower θ_k indicates stabilization near the optimum.

Step 5: Convergence and Final Output

The iterations are stopped when

$$|TAC(k+1) - TAC(k)| < \varepsilon.$$

Output:

- Computed t_1 .
- Optimal value t_2^* .
- Final TAC values.

5.2 Pseudocode of the Proposed CGASA Procedure

The compact pseudocode of the proposed CGASA procedure is presented in **Table 2**.

Table 2. Pseudocode of the proposed CGASA procedure.

Step	CGASA operation
Input	Model parameters and fuzzy cost parameters, ε , η_{min} , η_{max} , $\gamma_{increase}$ and $\gamma_{decrease}$.
1	Convert fuzzy cost parameters into crisp values using the graded mean integration method.
2	Compute t_1 as the unique positive root and initialize $t_2^{(0)} = t_1 + \delta$ (with $\delta = \rho \cdot t_1$ and $\rho = 0.1$).
3	Set $\eta^{(0)} = \frac{1}{(\ \nabla TAC\ + \varepsilon)}$, and $\beta^{(0)} = \frac{\ \nabla TAC(t^{(0)})\ }{1 + \ \nabla TAC(t^{(0)})\ }$, and initialize $k = 0$.
4	Compute $TAC(t_2^{(k)})$ and $\partial TAC / \partial t_2$.
5	Update t_2 using the gradient direction and apply the descent-check correction.

Table 2 continued...

6	Update η within $[\eta_{min}, \eta_{max}]$ using $\gamma_{increase}$ and $\gamma_{decrease}$.
7	Apply the β -based refinement and compute θ_k as a diagnostic measure.
8	If $ TAC(k+1) - TAC(k) < \epsilon$, stop; otherwise, set $k \leftarrow k+1$ and repeat Steps 4–7.
Output	Computed t_1 , optimal t_2^* and final TAC value.

5.3 Comparison of Existing Methods and the Proposed CGASA

A comparison of the existing methods and the proposed CGASA is presented in **Table 3**.

Table 3. Comparison of NHFP, PHFPA, the classical gradient method and CGASA.

Feature	NHFP	PHFPA	Classical gradient method	Proposed CGASA
Method type	Neutrosophic hesitant fuzzy programming.	Pythagorean hesitant fuzzy programming.	Gradient-based method.	Adaptive gradient-based method.
Uncertainty handling	Neutrosophic hesitant information.	Pythagorean hesitant information.	Not included directly.	Hexagonal fuzzy parameters converted using GMI.
Initial value	Based on the adopted fuzzy programming procedure.	Based on the adopted fuzzy programming procedure.	User selected.	Feasible initialization using $t_2^{(0)} = t_1 + \delta$ (with $\delta = \rho \cdot t_1$ and $\rho = 0.1$).
Learning rate	Not applicable.	Not applicable.	Fixed.	Data-based and adaptively updated.
Descent correction	Not applicable.	Not applicable.	Not included.	Included.
Step-size bounds	Not applicable.	Not applicable.	Not included.	Restricted to an admissible interval.
Local refinement	Based on the respective programming framework.	Based on the respective programming framework.	Not included.	β -based refinement.
Role in this study	Numerical comparison.	Numerical comparison.	Methodological comparison.	Proposed TAC minimization procedure.

Note: NHFP and PHFPA are fuzzy programming approaches and are not directly comparable with gradient-based algorithms. They are included for numerical comparison, whereas the classical gradient method is used to highlight the algorithmic features of CGASA.

5.4 Flowchart of the Proposed CGASA Procedure

The computational steps of the proposed CGASA procedure are illustrated in **Figure 2**.

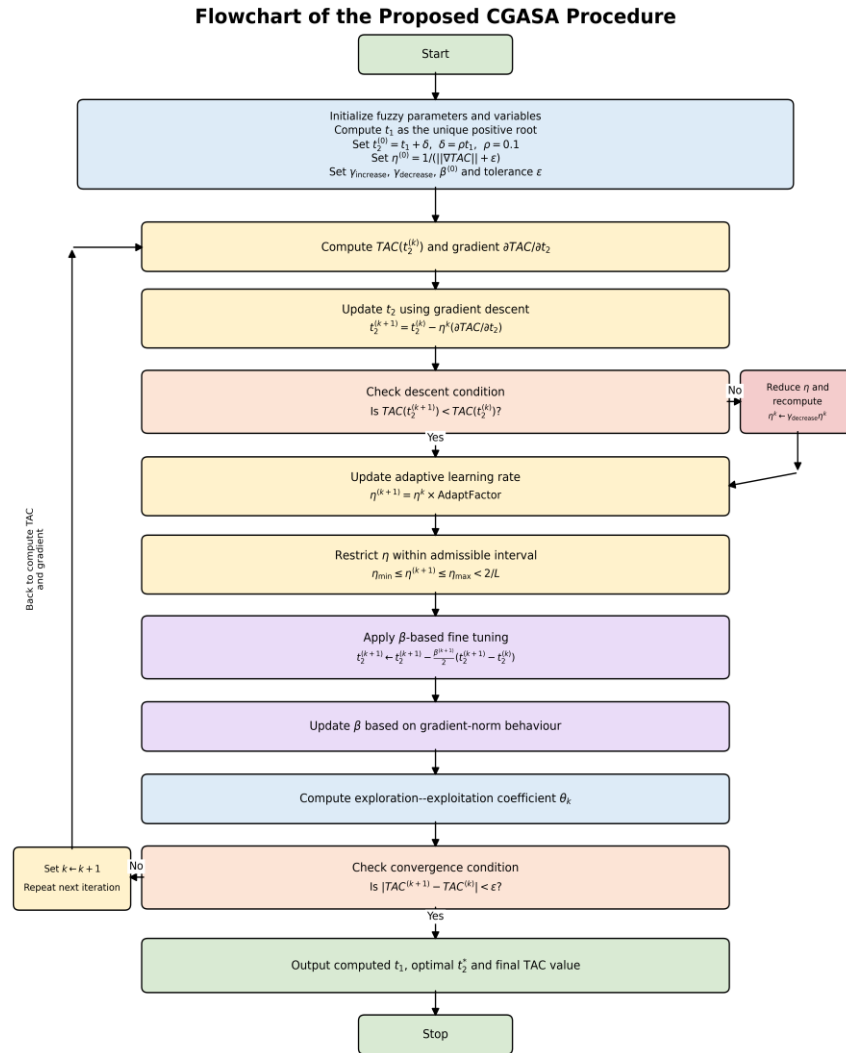


Figure 2. Flowchart of the proposed CGASA procedure.

5.5 Analytic Properties of TAC

Let $R(*)$ denote the defuzzification operator applied to the fuzzy parameters. By linearity and positive homogeneity, the representative crisp parameters are O, h, S_c, d, a, b, c and α . For a fixed depletion time $t_1 > 0$, the cycle length $t_2 > t_1$ is treated as the decision variable.

Define the shortage integral as

$$\Phi(t_1, t_2) = \frac{a}{2}(t_2 - t_1)^2 + \frac{b}{6}(2t_2^3 - 3t_1t_2^2 + t_1^3) + \frac{c}{12}(3t_2^4 - 4t_1t_2^3 + t_1^4).$$

Since t_1 is fixed, the ordering, holding and deterioration terms are independent of t_2 . Collecting these terms gives

$$A = O + h \left[\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \frac{c}{10}t_1^5 + \alpha \left(\frac{a}{20}t_1^5 + \frac{b}{24}t_1^6 + \frac{c}{28}t_1^7 \right) \right] + d\alpha \left(\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \frac{c}{10}t_1^5 \right).$$

Hence, the total average cost can be written in the compact form

$$TAC(t_2) = \frac{A+S_c \Phi(t_1, t_2)}{t_2}, \quad t_2 > t_1 > 0 \quad (18)$$

Continuous Differentiability on (t_1, ∞)

The function $\Phi(t_1, t_2)$ is a polynomial in t_2 , while $\frac{1}{t_2}$ is infinitely differentiable for $t_2 > 0$. Therefore, $TAC(t_2)$ is infinitely differentiable on (t_1, ∞) . The first two derivatives of $\Phi(t_2)$ are

$$\Phi'(t_2) = (t_2 - t_1)(a + bt_2 + ct_2^2),$$

$$\Phi''(t_2) = a + b(2t_2 - t_1) + c(3t_2^2 - 2t_1t_2).$$

Let $G(t_2) = A + S_c \Phi(t_1, t_2)$. Using the quotient rule, we obtain

$$TAC'(t_2) = \frac{G'(t_2)t_2 - G(t_2)}{t_2^2} = \frac{S_c(t_2 \Phi'(t_2) - \Phi(t_2)) - A}{t_2^2} \quad (19)$$

$$t_2 \Phi'(t_2) - \Phi(t_2) = \frac{a}{2}(t_2^2 - t_1^2) + b\left(\frac{2}{3}t_2^3 - \frac{1}{2}t_1t_2^2 - \frac{1}{6}t_1^3\right) + c\left(\frac{3}{4}t_2^4 - \frac{2}{3}t_1t_2^3 - \frac{1}{12}t_1^4\right) \quad (20)$$

Strong Convexity on $[t_1, T_{\max}]$

Differentiating Equation (19) and simplifying over a common denominator t_2^3 yields

$$TAC''(t_2) = \frac{2A+S_c\left[at_1^2+b\left(\frac{2}{3}t_2^3+\frac{1}{3}t_1^3\right)+c\left(\frac{3}{2}t_2^4-\frac{2}{3}t_1t_2^3+\frac{1}{6}t_1^4\right)\right]}{t_2^3}, \quad t_2 > t_1 \quad (21)$$

For the inventory model, $A, S_c, a, b, c > 0$ and $t_2 \geq t_1 > 0$. Every term on the right-hand side of Equation (21) is positive. Hence, $TAC(t_2)$ is strictly convex. On the compact interval $[t_1, T_{\max}]$, a positive lower bound for $TAC''(t_2)$ is,

$$m = \frac{2A+S_c\left(at_1^2+\frac{b}{3}t_1^3+\frac{c}{6}t_1^4\right)}{T_{\max}^3} + \frac{2S_cb}{3} + \frac{5S_cc t_1}{6} > 0 \quad (22)$$

Therefore, $TAC''(t_2) \geq m > 0$ on $t_2 \in [t_1, T_{\max}]$. Thus, the TAC function is m -strongly convex on $[t_1, T_{\max}]$ and has a unique minimizer on this interval.

Lipschitz Continuity of the Gradient on $[t_1, T_{\max}]$

Since $TAC''(t_2)$ is continuous on the compact interval $[t_1, T_{\max}]$, it is bounded on this interval. Hence, $TAC'(t_2)$ is Lipschitz continuous. From the equivalent form of Equation (21), a valid upper bound for the Lipschitz constant is

$$L = \frac{2A+S_c\left(at_1^2+\frac{b}{3}t_1^3+\frac{c}{6}t_1^4\right)}{t_1^3} + \frac{2S_cb}{3} + S_cc\left(\frac{3}{2}T_{\max} - \frac{2}{3}t_1\right) \quad (23)$$

Accordingly,

$$|TAC'(y) - TAC'(x)| \leq L|y - x|, \quad x, y \in [t_1, T_{\max}] \quad (24)$$

These properties provide the analytical basis for the convergence conditions used in CGASA. Strict convexity ensures a unique optimal cycle length, while the Lipschitz bound supports the admissible learning-rate condition $0 < \eta_{\max} < 2/L$.

Theorem. Convergence of CGASA Iterates with an Adaptive Learning Rate

Let $TAC: [t_1, T_{\max}] \rightarrow \mathbb{R}$ be continuously differentiable and m -strongly convex, with $m > 0$ and suppose that its derivative is L -Lipschitz continuous on $[t_1, T_{\max}]$. Assume that all iterates generated by CGASA remain in $[t_1, T_{\max}]$ and that the unique minimizer t^* lies in the interior of this interval $(t_1, T_{\max})$. Thus, for all $x, y \in [t_1, T_{\max}]$, $TAC(y) \geq TAC(x) + TAC'(x)(y - x) + \frac{m}{2}|y - x|^2$. The specific adaptive update rule does not affect the proof, provided that each updated learning rate is restricted to the admissible interval $0 < \eta_{\min} \leq \eta^{(k)} \leq \eta_{\max} < \frac{2}{L}$. The iterates are generated by

$$t^{(k+1)} = t^{(k)} - \eta^{(k)} TAC'(t^{(k)}).$$

Then the sequence $\{t^{(k)}\}$ converges to the unique minimizer t^* of TAC . Moreover,
 $|TAC'(t^{(k)})| \rightarrow 0, \quad t^{(k)} \rightarrow t^*, \quad TAC(t^{(k)}) \rightarrow TAC(t^*) \quad \text{as } k \rightarrow \infty.$

Proof. Since TAC' is L -Lipschitz continuous, the descent lemma gives

$$TAC(t^{(k+1)}) \leq TAC(t^{(k)}) + TAC'(t^{(k)})(t^{(k+1)} - t^{(k)}) + \frac{L}{2}|t^{(k+1)} - t^{(k)}|^2.$$

Using the update rule,

$$t^{(k+1)} = t^{(k)} - \eta^{(k)} TAC'(t^{(k)}),$$

we obtain $TAC(t^{(k+1)}) \leq TAC(t^{(k)}) - \left(\eta^{(k)} - \frac{L}{2}(\eta^{(k)})^2\right) |TAC'(t^{(k)})|^2$.

Define $\delta^{(k)} = \eta^{(k)} - \frac{L}{2}(\eta^{(k)})^2$.

Since $0 < \eta_{\min} \leq \eta^{(k)} \leq \eta_{\max} < 2/L$, we have $\delta^{(k)} > 0$.

Moreover, because the function $q(\eta) = \eta - (L/2)\eta^2$ is continuous and positive on the compact interval $[\eta_{\min}, \eta_{\max}]$,

$$\delta_{\min} = \min_{\eta \in [\eta_{\min}, \eta_{\max}]} \left(\eta - \frac{L}{2}\eta^2 \right) > 0,$$

and hence $\delta^{(k)} \geq \delta_{\min}$ for every k . Therefore,

$$TAC(t^{(k)}) - TAC(t^{(k+1)}) \geq \delta^{(k)} |TAC'(t^{(k)})|^2 \geq \delta_{\min} |TAC'(t^{(k)})|^2 \geq 0.$$

Thus, $\{TAC(t^{(k)})\}$ is non-increasing and bounded below by $TAC(t^*)$; consequently, it converges. Summing the preceding inequality from $k = 0$ to infinity gives

$$\delta_{\min} \sum_{k=0}^{\infty} |TAC'(t^{(k)})|^2 \leq TAC(t^{(0)}) - \lim_{k \rightarrow \infty} TAC(t^{(k)}) < \infty.$$

Hence, $\sum_{k=0}^{\infty} |TAC'(t^{(k)})|^2 < \infty$ and therefore $|TAC'(t^{(k)})| \rightarrow 0$.

To establish convergence of the iterates rigorously, strong convexity implies the strong monotonicity of the derivative:

$$(TAC'(t^{(k)}) - TAC'(t^*)) (t^{(k)} - t^*) \geq m |t^{(k)} - t^*|^2.$$

Since t^* is the unique minimizer and $TAC'(t^*) = 0$, the Cauchy-Schwarz inequality yields $m |t^{(k)} - t^*| \leq |TAC'(t^{(k)})|$.

Because $|TAC'(t^{(k)})| \rightarrow 0$, it follows that $t^{(k)} \rightarrow t^*$. Finally, continuity of TAC gives $TAC(t^{(k)}) \rightarrow TAC(t^*)$. Therefore,

$$\lim_{k \rightarrow \infty} t^{(k)} = t^*, \quad |TAC'(t^{(k)})| \rightarrow 0, \quad \text{and} \quad TAC(t^{(k)}) \rightarrow TAC(t^*).$$

Practical implication: Strong convexity guarantees that the total average cost has a unique and stable minimum. In practical terms, the optimal replenishment time t_2^* yields the lowest TAC, while any deviation from t_2^* increases the TAC. Therefore, the obtained solution supports a stable and reliable replenishment decision.

6. Computational Example

The numerical example is examined in the context of a hospital pharmacy inventory system. The two items represent non-emergency temperature-sensitive medical products, such as routine vaccines, insulin supplies and diagnostic kits. Demand may vary according to patient flow and scheduled hospital requirements, while these products may deteriorate or expire during storage. Shortages are assumed to be completely backlogged because pending non-emergency demand can be fulfilled during the next replenishment cycle, subject to a service-delay penalty.

The basic parameter values, with minor modifications for the present computational analysis, are adapted from the earlier study (Das and Islam, 2024). The maximum inventory level I_{max} is assumed. The hospital pharmacy setting is illustrative and does not represent data collected from a specific hospital. Since the present model employs modified parameter settings, a different cost formulation and the proposed CGASA solution procedure, the reported TAC values are computed independently. The input parameter values considered for items 1 and 2 are presented in **Table 4**.

Table 4. Input parameter values for items 1 and 2.

Parameter	Item 1	Item 2
Fuzzy ordering cost $\tilde{O}$	(980,990,1000,1010,1020,1030;0.9)	(1070,1080,1090,1100,1110,1120;0.9)
Fuzzy holding-cost coefficient $\tilde{h}$	(16,18,20,22,24,26;0.8)	(17,19,21,23,25,27;0.8)
Fuzzy shortage-cost coefficient $\tilde{S}_c$	(30,35,40,45,50,55;0.86)	(32,37,42,47,52,57;0.8)
Fuzzy deterioration cost $\tilde{d}$	(14,16,18,20,22,24;0.9)	(15,17,19,21,23,25;0.9)
Constant demand coefficient (a)	70	75
Linear demand coefficient (b)	50	55
Quadratic demand coefficient (c)	12	15
Deterioration-rate coefficient (α)	0.5	0.55
Maximum inventory level I_{max}	17.35	17.60

The fuzzy cost parameters are converted into their representative crisp values using the GMI method. For each item, the depletion time t_1 is obtained from the specified I_{max} and CGASA is applied to determine the optimal replenishment-cycle length t_2 and the corresponding minimum TAC. **Tables 5** and **6** present the iteration results for items 1 and 2, respectively.

Table 5. CGASA inventory optimization results for item 1.

Iteration	t_1	t_2	Gradient	η	β	TAC	θ_k	ΔTAC
1.	0.227584	0.250000	-14259.272	0.000077	0.000004	2117.2999	203326840.26	1660.9517
2.	0.227584	1.249998	1297.1785	0.000085	0.000004	1998.0778	16815292.23	119.2221
3.	0.227584	1.149930	1100.8767	0.000093	0.000004	1905.9483	12973305.96	92.1295
4.	0.227584	1.056513	888.7980	0.000103	0.000004	1842.2541	9521872.72	63.6942
5.	0.227584	0.973551	667.3078	0.000113	0.000004	1804.8039	6499094.28	37.4502
6.	0.227584	0.905034	451.6657	0.000124	0.000005	1787.3419	3998996.65	17.4620
7.	0.227584	0.854021	265.4201	0.000137	0.000005	1781.4603	2136362.94	5.8816
8.	0.227584	0.803233	130.3333	0.000150	0.000005	1780.2276	953683.13	1.2327
9.	0.227584	0.795457	51.7295	0.000165	0.000005	1780.1428	344107.34	0.0848

Table 6. CGASA inventory optimization results for item 2.

Iteration	t_1	t_2	Gradient	η	β	TAC	θ_k	ΔTAC
1.	0.216043	0.250000	-15387.539	0.000071	0.000003	2288.3252	236776384.36	1817.9540
2.	0.216043	1.249998	1372.0436	0.000079	0.000004	2164.7780	19193068.88	123.5472
3.	0.216043	1.151916	1163.9303	0.000086	0.000004	2069.3477	14801672.17	95.4303
4.	0.216043	1.060391	940.2692	0.000095	0.000004	2003.2376	10870345.54	66.1101
5.	0.216043	0.979059	707.8419	0.000105	0.000004	1964.0979	7439345.02	39.1397
6.	0.216043	0.911709	482.2131	0.000115	0.000004	1945.5603	4607281.73	18.5376
7.	0.216043	0.861239	286.9659	0.000127	0.000005	1939.1243	2492547.26	6.4360
8.	0.216043	0.828201	143.9209	0.000139	0.000005	1937.6954	1136433.91	1.4289
9.	0.216043	0.801762	58.9536	0.000153	0.000005	1937.5743	423192.42	0.1211

Table 7 presents the optimal results for items 1 and 2.

Table 7. Comparison of optimal results obtained using different methods for items 1 and 2.

Method	$t_1^{(1)}$	$t_2^{(1)}$	TAC ₁	Reduction for Item 1 (%)	$t_1^{(2)}$	$t_2^{(2)}$	TAC ₂	Reduction for Item 2 (%)
CGASA (proposed)	0.227584	0.795457	1780.1428	–	0.216043	0.801762	1937.5743	–
NHFP	0.227584	0.769849	1788.8342	0.4859	0.216043	0.761307	1937.5743	0.0000
PHFPA	0.227584	0.770024	1781.5693	0.0801	0.216043	0.765361	1938.1793	0.0312

Table 7 compares the optimal results obtained using CGASA, NHFP and PHFPA for the two numerical items. For Item 1, the proposed method yields a TAC approximately 0.4859% and 0.0801% lower than those obtained using NHFP and PHFPA, respectively. For Item 2, CGASA and NHFP yield the same TAC up to four decimal places, although their optimal replenishment-cycle lengths differ because the TAC function is nearly flat over $0.76 \leq t_2 \leq 0.80$. The proposed method also gives a TAC approximately 0.0312% lower than that obtained using PHFPA. Since NHFP and PHFPA use different fuzzy-programming frameworks, the comparison demonstrates numerical consistency and competitive performance rather than direct algorithmic superiority. The iteration results indicate that the algorithm progressively reduces the TAC while maintaining the feasibility condition $t_2 > t_1$.

Figure 3 presents a graphical comparison of the TAC values obtained using CGASA, NHFP and PHFPA for items 1 and 2.

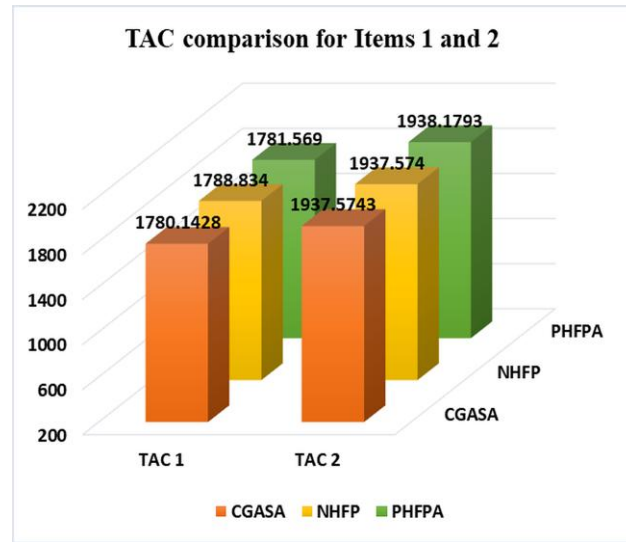


Figure 3. TAC comparison among CGASA, NHFP and PHFPA.

7. Sensitivity Analysis

This section examines the effects of changes in the ordering cost, holding-cost coefficient, deterioration-rate coefficient and shortage-cost coefficient on TAC. Each parameter is varied separately, while the remaining parameters are kept at their baseline values. The corresponding results obtained using CGASA, NHFP and PHFPA are presented in Figures 4-11.

7.1 Effect of Ordering Cost on TAC

Figures 4 and 5 present the effect of ordering cost on TAC for items 1 and 2, respectively. For all three methods, the total average cost increases as the ordering cost rises. This behaviour is expected because a higher fixed replenishment expense directly increases the cost incurred in each inventory cycle.

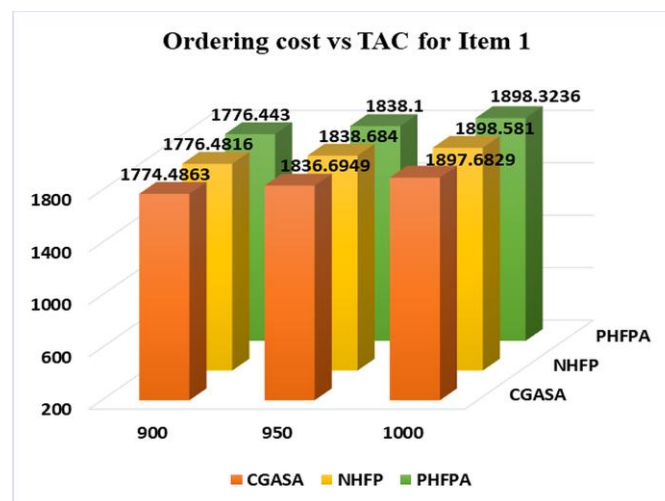


Figure 4. Effect of ordering cost on TAC for item 1.

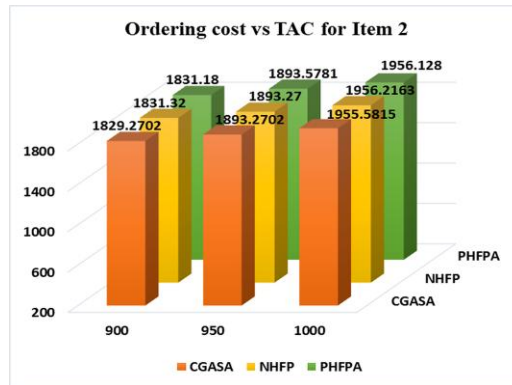


Figure 5. Effect of ordering cost on TAC for item 2.

7.2 Effect of Holding Cost on TAC

Figures 6 and 7 show the effect of the holding-cost coefficient on the total average cost. An increase in the holding-cost coefficient results in a gradual rise in the total average cost for both items because the cost of maintaining positive inventory becomes higher. The variation in TAC over the selected holding-cost range is moderate compared with that observed for ordering cost.

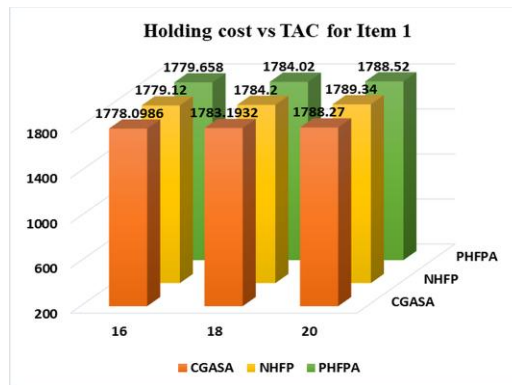


Figure 6. Effect of holding cost on TAC for item 1.

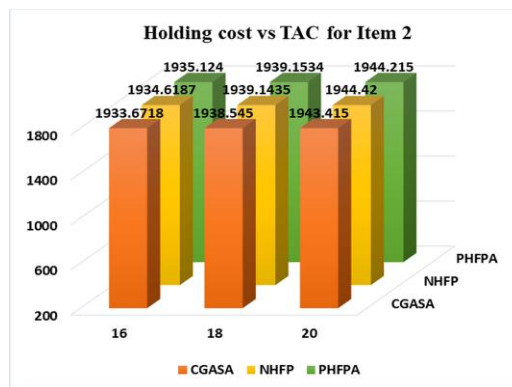


Figure 7. Effect of holding cost on TAC for item 2.

7.3 Effect of Deterioration Rate and Shortage Cost on TAC

Figures 8 and 9 show that TAC increases gradually with the deterioration-rate coefficient. However, the change remains small over the selected range, indicating that moderate variations in the deterioration-rate coefficient have a limited effect under the parameter settings considered.

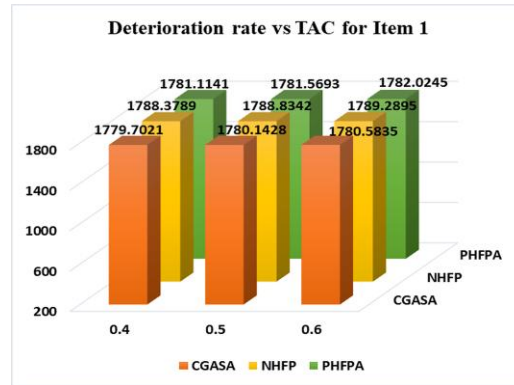


Figure 8. Effect of deterioration rate on TAC for item 1.

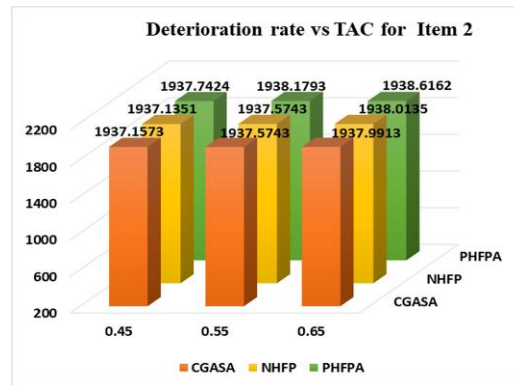


Figure 9. Effect of deterioration rate on TAC for item 2.

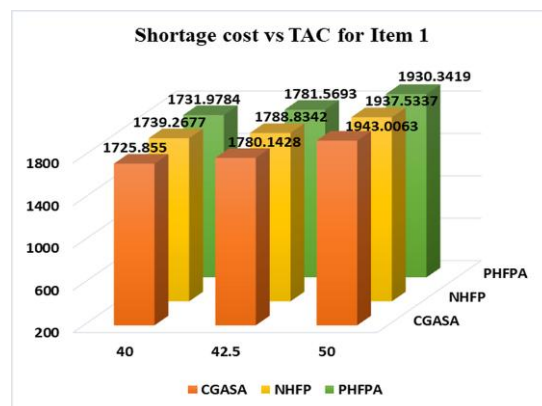


Figure 10. Effect of shortage cost on TAC for item 1.

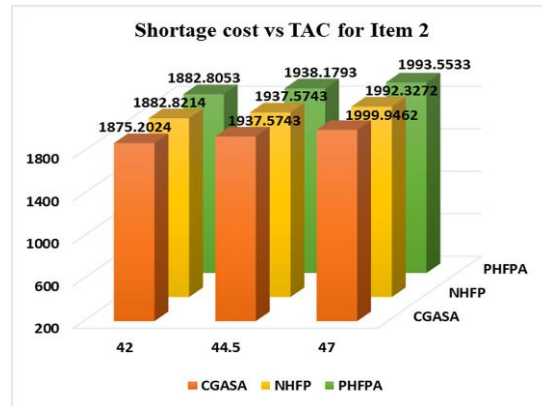


Figure 11. Effect of shortage cost on TAC for item 2.

Figures 10 and **11** show that TAC increases with the shortage-cost coefficient. The increase is more pronounced than that observed for the deterioration-rate coefficient because a higher shortage cost raises the penalty associated with backlogged demand. Thus, shortage cost has a noticeable influence on replenishment decisions for both items.

Overall, ordering cost and shortage cost have a more pronounced influence on TAC, whereas the effects of holding cost and deterioration rate are comparatively moderate over the selected ranges. These observations are specific to the numerical parameter settings considered in this study and should not be interpreted as universal relationships for all inventory systems.

7.4 Computational Complexity and Iteration-Based Performance

Each CGASA iteration requires the evaluation of TAC and its gradient with respect to the replenishment-cycle length. Since these expressions contain low-degree polynomial and rational terms, the computational effort per iteration remains limited for a fixed parameter set, while the overall effort increases with the number of iterations.

The proposed procedure was implemented in MATLAB and follows a deterministic update scheme. It does not require random initialization, stochastic sampling or repeated trial-based parameter selection. For the numerical examples considered, the algorithm reached stable solutions within 8–10 iterations. The same computational structure can be applied to other parameter settings by replacing the relevant cost parameters and updating the corresponding TAC expression and its derivative.

Table 8 presents a comparison of the proposed CGASA and the fixed-learning-rate gradient descent method based on the final total average cost and the number of iterations. Since both methods follow a gradient-based solution structure, the comparison provides a direct indication of their computational performance for the problem considered.

Table 8. Indicative iteration-based performance comparison.

Method	TAC	Iterations to converge
Proposed CGASA	1780.14	8 – 10
Gradient Descent (fixed η)	1782.66	20+

The comparison shows that CGASA attains a lower TAC in fewer reported iterations than the fixed-learning-rate gradient descent method.

8. Conclusion

In this study, a fuzzy inventory optimization model is developed for deteriorating items with quadratic demand, time-dependent holding cost and fully backlogged shortages. Uncertainty in the ordering, holding, shortage and deterioration costs is represented using hexagonal fuzzy numbers. The graded mean integration method is applied to obtain the corresponding representative crisp values and the resulting TAC function is optimized using the proposed Convergent Gradient Adaptive Search Algorithm.

The proposed procedure uses a feasible initial solution together with adaptive learning-rate adjustment, descent verification, bounded updates and β -based local refinement. The convergence analysis establishes the conditions under which the sequence of replenishment-cycle estimates approaches the unique minimum. In the numerical study, the minimum TAC values obtained for Items 1 and 2 are 1780.1428 and 1937.5743, respectively. The results obtained using NHFP and PHFPA are included for numerical comparison and indicate that the CGASA solutions are consistent with the corresponding results.

The sensitivity study shows that all four examined parameters influence TAC, although the extent of their effects differs over the selected ranges. Ordering cost and shortage cost produce comparatively larger changes, whereas the effects of the holding-cost and deterioration-rate coefficients remain moderate. In particular, the response to shortage cost indicates the importance of accounting for the cost of backlogged demand when determining the replenishment policy.

8.1 Managerial Implications

The results of the proposed model have practical relevance for inventory decisions involving healthcare products and other deteriorating items. In hospital pharmacy systems, medicines, vaccines and temperature-sensitive products may lose quality over time, while shortages may affect patient service. The proposed model can help managers determine suitable replenishment decisions by jointly considering demand variation, deterioration, holding cost and shortage cost. For perishable supply chains involving food, vegetables, dairy products and other deteriorating items, the model supports cost-effective inventory planning by balancing ordering, holding, deterioration and shortage-related costs. Thus, the CGASA-based framework can assist managers in reducing the total inventory cost while maintaining service reliability under uncertain cost conditions.

8.2 Limitations and Future Scope

The present model assumes quadratic demand, time-dependent deterioration, negligible lead time and fully backlogged shortages. Future studies may extend the proposed framework by considering stochastic demand, partial backlogging, variable lead time, carbon-emission policies, real-time data and multi-level inventory networks.

Conflicts of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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AI Disclosure

Generative AI tools were used only for language refinement and grammatical correction. All scientific content, mathematical analysis and results were developed and verified by the authors.

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