

Transient Solution of a Single Server Queuing Model with Correlated Reneging Using Runge-Kutta Method

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Abstract

In this paper, the concept of correlated reneging is introduced in queuing theory. The reneging considered so far is dependent on system size, but there are many real life situations where customers may renege due to exogenous factors other than the state of the system. Further, the reneging of customer may induce the other customers to renege at two successive time points. Such reneging is called correlated reneging. An M/M/1/K queuing model with correlated reneging is studied. Runge-Kutta method of fourth order is presented to obtain the transient solution of the model. Some performance measures like expected system size and expected waiting time in the system are studied.

Keywords- Queuing model, Correlated reneging, Transient solution, Runge-Kutta method, Transition marks.

1. Introduction

Queuing systems with impatient customers are quite useful in the performance analysis of systems like perishable inventory systems, hospitals having critical patients, telephone exchanges and computer communication systems with packet loss. The concept of impatient customers in queuing theory was introduced in the works of Haight (1957, 1959), Ancker Jr and Gafarian (1963a, 1963b), and Rao (1965; 1967). A number of queuing models with reneging and balking were studied since then. In recent past, Kumar and Sharma (2012) put forth a new concept of retention of reneging customers in queuing theory. It was envisaged that if the firms use some customer retention policies then there is a probability that a reneging customer can be retained. Kumar and Sharma (2018, 2019), Sharma et al. (2019) and Sudhesh and Azhagappan (2019) discussed some queuing systems with retention of reneging customers, and performed the transient analyses. Kumar and Soodan (2019) studied the transient behavior of a single server queuing model with correlated arrivals and reneging numerically.

The concept of correlation was first introduced by Mohan (1955) in gambler's ruin problem. Conolly (1968), Conolly and Hadidi (1969), Murari (1969), Mohan and Murrari (1972) studied some queuing models with correlated arrivals. Cidon et al. (1993) considered a queue in which service time is correlated to inter-arrival time with its application in communication systems. Vishnevskii and Dudin (2017) presented the applications of correlated arrival queuing models in telecommunication networks. Fiems and de Turck (2019) investigated a discrete-time queuing

system with session based arrivals. They evaluated the Taylor-series expansions of the joint probability generating function of the system state at slot boundaries. De Clercq and Walraevens (2020) performed the delay analysis of a priority queuing system having two kinds of arriving customers, viz. external arrivals and the correlated arrivals from other nodes.

In the renegeing considered so far, the renegeing of a customer was a function of system state. But, this may not fit well in many real life situations where the renegeing could be bursty based on exogenous factors, that is, renegeing depends on factors other than system state. For example, consider a central system of an online shopping company. The arrival of orders is similar to the arrival of customers, the dispatching of orders is analogous to the service of customers, and the orders cancelled before dispatching (i.e. being served) can be considered as renegeing customers. The cancellation of orders (i.e. renegeing of customers) could be bursty at times because of the reasons like delay in delivery, some other online shopping companies start offering discounts, getting bad reviews about the products etc. That is, if an order (a customer) is cancelled (renegeed) at any time instant, then there is an increased probability that an order may be cancelled at the next time instant. This kind of renegeing is referred to as correlated renegeing. In the above mentioned example, it is clearly seen that the renegeing of one customer induces the other customer to renege without any effect of system state. This motivates us to develop a queuing model with correlated renegeing.

In this paper, a finite capacity single server queuing model with correlated renegeing is studied. The transient analysis of the model is carried out. The remainder of the paper is arranged as follows: In section 2, the stochastic queuing model is described. In section 3, the mathematical model is presented. Section 4 deals with the transient solution of the model. In section 5, numerical illustration is provided. In section 6, performance measures are given. Finally, the conclusion and future work is provided in section 7.

2. Stochastic Queuing Model

We consider a finite capacity single server Markovian queuing model with correlated renegeing. The state-transition diagram of the queuing model is shown in Figure 1. The model under investigation is based on the following assumptions:

- (i) The customers arrive at a service facility one by one in accordance with Poisson process with parameter λ .
- (ii) There is a single queue and a single server. The service-times are independently, identically and exponentially distributed with parameter μ .
- (iii) The capacity of the system is finite (say, K), where $K=N+1$ and N is the queue capacity.
- (iv) After joining the queue and waiting for some time, a customer may get impatient and leave the queue (renege) without getting service. The renegeing of customers can take place only at the transition marks t_0, t_1, t_2, \dots where $\theta_r = t_r - t_{r-1}$, $r = 1, 2, 3, \dots$, are random variables with $P[\theta_r \leq x] = 1 - \exp(-\xi x)$; $\xi \geq 0, r = 1, 2, 3, \dots$ i.e. the distribution of inter-transition marks is negative exponential with parameter ξ .
- (v) The renegeing at two consecutive transition marks is governed by the following transition probability matrix:

to t_r

$$\begin{matrix} & & 0 & 1 \\ & & \parallel & \parallel \\ \text{from } t_{r-1} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \end{matrix}, \text{ where } p_{00} + p_{01} = 1 \text{ and } p_{10} + p_{11} = 1.$$

0 refers to no renegeing and 1 refers to the occurrence of renegeing. Thus, the renegeing at two consecutive transition marks is correlated.

3. Mathematical Model

Let us define the following probabilities as:

$Q_{0,r}(t)$ = Probability that at time t the queue length is zero, the server is idle, and r is an indicator whether a customer has renegeed or not in previous transition mark ($r=0$ refers to no renegeing and $r=1$ refers to the occurrence of renegeing at previous transition mark).

$P_{0,r}(t)$ = Probability that at time t the queue length is zero, the server is not idle, and r is an indicator whether a customer has renegeed or not in previous transition mark ($r=0$ refers to no renegeing and $r=1$ refers to the occurrence of renegeing at previous transition mark).

$P_{n,r}(t)$ = Probability that at time t the queue length is n ($1 \leq n \leq N$), the server is not idle, and r is an indicator whether a customer has renegeed or not in previous transition mark ($r=0$ refers to no renegeing and $r=1$ refers to the occurrence of renegeing at previous transition mark).

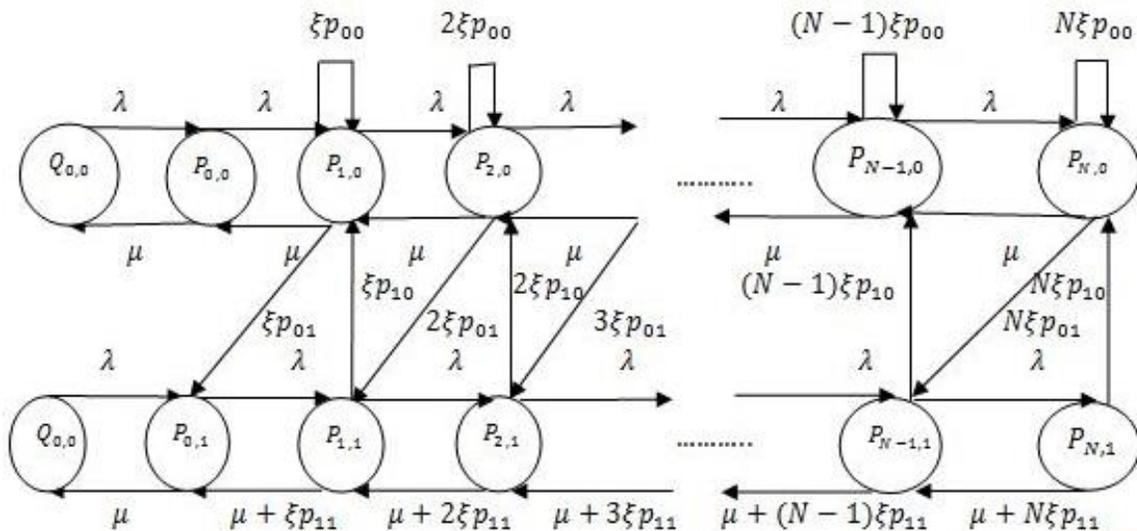


Figure 1: State transition diagram of the queuing model

The differential equations of the model are:

$$\frac{d}{dt} Q_{0,0}(t) = -\lambda Q_{0,0}(t) + \mu P_{0,0}(t) \quad (1)$$

$$\frac{d}{dt} Q_{0,1}(t) = -\lambda Q_{0,1}(t) + \mu P_{0,1}(t) \quad (2)$$

$$\frac{d}{dt} P_{0,0}(t) = -(\lambda + \mu)P_{0,0}(t) + \mu P_{1,0}(t) + \lambda Q_{0,0}(t) \quad (3)$$

$$\frac{d}{dt} P_{0,1}(t) = -(\lambda + \mu)P_{0,1}(t) + \mu P_{1,1}(t) + \lambda Q_{0,1}(t) + \xi[p_{11}P_{1,1}(t) + p_{01}P_{1,0}(t)] \quad (4)$$

$$\begin{aligned} \frac{d}{dt} P_{n,0}(t) = & -(\lambda + \mu + n\xi)P_{n,0}(t) + \mu P_{n+1,0}(t) + \lambda P_{n-1,0}(t) \\ & + n\xi[p_{00}P_{n,0}(t) + p_{10}P_{n,1}(t)] \quad , 1 \leq n < N \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{d}{dt} P_{n,1}(t) = & -(\lambda + \mu + n\xi)P_{n,1}(t) + \mu P_{n+1,1}(t) + \lambda P_{n-1,1}(t) \\ & + (n + 1)\xi[p_{01}P_{n+1,0}(t) + p_{11}P_{n+1,1}(t)] \quad , 1 \leq n < N \end{aligned} \quad (6)$$

$$\frac{d}{dt} P_{N,0}(t) = -(\mu + N\xi)P_{N,0}(t) + \lambda P_{N-1,0}(t) + N\xi[p_{00}P_{N,0}(t) + p_{10}P_{N,1}(t)] \quad (7)$$

$$\frac{d}{dt} P_{N,1}(t) = -(\mu + N\xi)P_{N,1}(t) + \lambda P_{N-1,1}(t) \quad (8)$$

The initial condition is $P_{0,0}(0) = 1$.

4. Transient Solution of the Model

Since the transient solution of the model is quite complex to obtain analytically, therefore, we apply the Runge-Kutta Method of fourth-order to obtain the transient solution. In Runge-Kutta method, for solving the differential equation $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$. The following formula $y_{i+1} = y_i + \left[\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \right] h$, where h is the step size and $k_1 = hf(t_i, y_i)$, $k_2 = hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right)$, $k_3 = hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right)$, $k_4 = hf(t_i + h, y_i + k_3)$ computes an approximate solution.

$$\text{Let } y = \begin{cases} Q_{0,k}(t) \text{ for } n = 0, k = 0,1 \\ P_{n-1,k}(t) \text{ for } n = 1,2,3, \dots, N + 1, k = 0,1 \end{cases}$$

$$\text{and } \begin{cases} \frac{dQ_{0,k}(t)}{dt} \text{ for } n = 0, k = 0,1 \\ \frac{dP_{n-1,k}(t)}{dt} \text{ for } n = 1,2,3, \dots, N + 1, k = 0,1 \end{cases} \text{ corresponds to } f(t, y). \text{ Therefore, the fourth-}$$

order Runge-Kutta method can be applied to solve the differential-difference equations of our model as given below:

$$\text{Consider a vector } \mathbf{PA} \text{ such that } \mathbf{PA}_{n,k}(t) = \begin{cases} Q_{0,k}(t) \text{ for } n = 0, k = 0,1 \\ P_{n-1,k}(t) \text{ for } n = 1,2,3, \dots, N + 1, k = 0,1 \end{cases}$$

Similarly the vectors $\mathbf{AA}_{n,k}$, $\mathbf{AB}_{n,k}$, $\mathbf{AC}_{n,k}$ and $\mathbf{AD}_{n,k}$ represent the multivariate version of the intermediate quantities. Vector \mathbf{AA} corresponds to quantity k_1 , therefore,

$$\begin{aligned} \mathbf{AA}_{n,k} &= (\Delta t) \frac{d\mathbf{PA}_{n,k}(t)}{dt} = [-\lambda \mathbf{PA}_{0,0}(t) + \mu \mathbf{PA}_{1,0}(t)] \Delta t; n = 0, k = 0 \\ \text{or} \\ &= [-\lambda \mathbf{PA}_{0,1}(t) + \mu \mathbf{PA}_{1,1}(t)] \Delta t; n = 0, k = 1 \\ \text{or} \\ &= [-(\lambda + \mu) \mathbf{PA}_{1,0}(t) + \mu \mathbf{PA}_{2,0}(t) + \lambda \mathbf{PA}_{0,0}(t)] \Delta t; n = 1, k = 0 \\ \text{or} \\ &= [-(\lambda + \mu) \mathbf{PA}_{1,1}(t) + \mu \mathbf{PA}_{2,1}(t) + \lambda \mathbf{PA}_{0,1}(t) + \xi [p_{11} \mathbf{PA}_{2,1}(t) + p_{01} \mathbf{PA}_{2,0}(t)]] \Delta t; \\ & \hspace{15em} n = 1, k = 1 \\ \text{or} \\ &= [-(\lambda + \mu + n\xi) \mathbf{PA}_{n+1,0}(t) + \mu \mathbf{PA}_{n+2,0}(t) + \lambda \mathbf{PA}_{n,0}(t) \\ & \quad + n\xi [p_{00} \mathbf{PA}_{n+1,0}(t) + p_{10} \mathbf{PA}_{n+1,1}(t)]] \Delta t; 1 < n \leq N, k = 0 \\ \text{or} \\ &= [-(\lambda + \mu + n\xi) \mathbf{PA}_{n+1,1}(t) + \mu \mathbf{PA}_{n+2,1}(t) + \lambda \mathbf{PA}_{n,1}(t) \\ & \quad + (n + 1)\xi [p_{01} \mathbf{PA}_{n+2,0}(t) + p_{11} \mathbf{PA}_{n+2,1}(t)]] \Delta t; 1 < n \leq N, k = 1 \\ \text{or} \\ &= [-(\mu + N\xi) \mathbf{PA}_{N+1,0}(t) + \lambda \mathbf{PA}_{N,0}(t) + N\xi [p_{00} \mathbf{PA}_{N+1,0}(t) + p_{10} \mathbf{PA}_{N+1,1}(t)]] \Delta t; \\ & \hspace{15em} n = N + 1, k = 0 \\ \text{or} \\ &= [-(\mu + N\xi) \mathbf{PA}_{N+1,1}(t) + \lambda \mathbf{PA}_{N,1}(t)] \Delta t; n = N + 1, k = 1. \end{aligned}$$

Vector \mathbf{PB} corresponds to quantity $(y + \frac{1}{2}k_1)$ such that $\mathbf{PB}_{n,k}(t + \frac{1}{2}\Delta t) = \mathbf{PA}_{n,k}(t) + \frac{1}{2}\mathbf{AA}_{n,k}$.

Vector \mathbf{AB} corresponds to quantity k_2 , therefore,

$$\begin{aligned} \mathbf{AB}_{n,k} &= (\Delta t) \frac{d\mathbf{PB}_{n,k}(t + \frac{1}{2}\Delta t)}{dt} = [-\lambda \mathbf{PB}_{0,0}(t + \frac{1}{2}\Delta t) + \mu \mathbf{PB}_{1,0}(t + \frac{1}{2}\Delta t)] \Delta t; \\ & \hspace{15em} n = 0, k = 0 \\ \text{or} \\ &= [-\lambda \mathbf{PB}_{0,1}(t + \frac{1}{2}\Delta t) + \mu \mathbf{PB}_{1,1}(t + \frac{1}{2}\Delta t)] \Delta t; n = 0, k = 1 \\ \text{or} \\ &= [-(\lambda + \mu) \mathbf{PB}_{1,0}(t + \frac{1}{2}\Delta t) + \mu \mathbf{PB}_{2,0}(t + \frac{1}{2}\Delta t) + \lambda \mathbf{PB}_{0,0}(t + \frac{1}{2}\Delta t)] \Delta t; \\ & \hspace{15em} n = 1, k = 0 \\ \text{or} \\ &= [-(\lambda + \mu) \mathbf{PB}_{1,1}(t + \frac{1}{2}\Delta t) + \mu \mathbf{PB}_{2,1}(t + \frac{1}{2}\Delta t) + \lambda \mathbf{PB}_{0,1}(t + \frac{1}{2}\Delta t) + \\ & \quad \xi [p_{11} \mathbf{PB}_{2,1}(t + \frac{1}{2}\Delta t) + p_{01} \mathbf{PB}_{2,0}(t + \frac{1}{2}\Delta t)]] \Delta t; n = 1, k = 1 \\ \text{or} \end{aligned}$$

$$= \left[-(\lambda + \mu + n\xi) \mathbf{PB}_{n+1,0} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PB}_{n+2,0} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PB}_{n,0} \left(t + \frac{1}{2} \Delta t \right) + n\xi \left[p_{00} \mathbf{PB}_{n+1,0} \left(t + \frac{1}{2} \Delta t \right) + p_{10} \mathbf{PB}_{n+1,1} \left(t + \frac{1}{2} \Delta t \right) \right] \right] \Delta t;$$

$1 < n \leq N, k = 0$

or

$$= \left[-(\lambda + \mu + n\xi) \mathbf{PB}_{n+1,1} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PB}_{n+2,1} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PB}_{n,1} \left(t + \frac{1}{2} \Delta t \right) + (n+1)\xi \left[p_{01} \mathbf{PB}_{n+2,0} \left(t + \frac{1}{2} \Delta t \right) + p_{11} \mathbf{PB}_{n+2,1} \left(t + \frac{1}{2} \Delta t \right) \right] \right] \Delta t;$$

$1 < n \leq N, k = 1$

or

$$= \left[-(\mu + N\xi) \mathbf{PB}_{N+1,0} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PB}_{N,0} \left(t + \frac{1}{2} \Delta t \right) + N\xi \left[p_{00} \mathbf{PB}_{N+1,0} \left(t + \frac{1}{2} \Delta t \right) + p_{10} \mathbf{PB}_{N+1,1} \left(t + \frac{1}{2} \Delta t \right) \right] \right] \Delta t; n = N + 1, k = 0$$

or

$$= \left[-(\mu + N\xi) \mathbf{PB}_{N+1,1} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PB}_{N,1} \left(t + \frac{1}{2} \Delta t \right) \right] \Delta t; n = N + 1, k = 1.$$

Vector **PC** corresponds to quantity $(y + \frac{1}{2}k_2)$ such that $\mathbf{PC}_{n,k} \left(t + \frac{1}{2} \Delta t \right) = \mathbf{PA}_{n,k}(t) + \frac{1}{2} \mathbf{AB}_{n,k}$.

Vector **AB** corresponds to quantity k_3 , therefore,

$$\mathbf{AC}_{n,k} = (\Delta t) \frac{d\mathbf{PC}_{n,k} \left(t + \frac{1}{2} \Delta t \right)}{dt}$$

$$= \left[-\lambda \mathbf{PC}_{0,0} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PC}_{1,0} \left(t + \frac{1}{2} \Delta t \right) \right] \Delta t; n = 0, k = 0$$

or

$$= \left[-\lambda \mathbf{PC}_{0,1} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PC}_{1,1} \left(t + \frac{1}{2} \Delta t \right) \right] \Delta t; n = 0, k = 1$$

or

$$= \left[-(\lambda + \mu) \mathbf{PC}_{1,0} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PC}_{2,0} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PC}_{0,0} \left(t + \frac{1}{2} \Delta t \right) \right] \Delta t;$$

$n = 1, k = 0$

or

$$= \left[-(\lambda + \mu) \mathbf{PC}_{1,1} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PC}_{2,1} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PC}_{0,1} \left(t + \frac{1}{2} \Delta t \right) + \xi \left[p_{11} \mathbf{PC}_{2,1} \left(t + \frac{1}{2} \Delta t \right) + p_{01} \mathbf{PC}_{2,0} \left(t + \frac{1}{2} \Delta t \right) \right] \right] \Delta t; n = 1, k = 1$$

or

$$= \left[-(\lambda + \mu + n\xi) \mathbf{PC}_{n+1,0} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PC}_{n+2,0} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PC}_{n,0} \left(t + \frac{1}{2} \Delta t \right) + n\xi \left[p_{00} \mathbf{PC}_{n+1,0} \left(t + \frac{1}{2} \Delta t \right) + p_{10} \mathbf{PC}_{n+1,1} \left(t + \frac{1}{2} \Delta t \right) \right] \right] \Delta t; 1 < n \leq N, k = 0$$

or

$$= \left[-(\lambda + \mu + n\xi) \mathbf{PC}_{n+1,1} \left(t + \frac{1}{2} \Delta t \right) + \mu \mathbf{PC}_{n+2,1} \left(t + \frac{1}{2} \Delta t \right) + \lambda \mathbf{PC}_{n,1} \left(t + \frac{1}{2} \Delta t \right) + (n+1)\xi \left[p_{01} \mathbf{PC}_{n+2,0} \left(t + \frac{1}{2} \Delta t \right) + p_{11} \mathbf{PC}_{n+2,1} \left(t + \frac{1}{2} \Delta t \right) \right] \right] \Delta t;$$

$1 < n \leq N, k = 1$

or

$$= \left[-(\mu + N\xi)PC_{N+1,0} \left(t + \frac{1}{2}\Delta t \right) + \lambda PC_{N,0} \left(t + \frac{1}{2}\Delta t \right) + N\xi \left[p_{00}PC_{N+1,0} \left(t + \frac{1}{2}\Delta t \right) + p_{10}PC_{N+1,1} \left(t + \frac{1}{2}\Delta t \right) \right] \right] \Delta t; n = N + 1, k = 0$$

or

$$= \left[-(\mu + N\xi)PC_{N+1,1} \left(t + \frac{1}{2}\Delta t \right) + \lambda PC_{N,1} \left(t + \frac{1}{2}\Delta t \right) \right] \Delta t; n = N + 1, k = 1.$$

Vector **PD** corresponds to quantity $(y + k_3)$ such that $PD_{n,k}(t) = PA_{n,k}(t) + \frac{1}{2}AC_{n,k}$.

Vector **AD** corresponds to quantity k_4 , therefore,

$$AD_{n,k} = (\Delta t) \frac{dPD_{n,k}(t)}{dt} = \left[-\lambda PD_{0,0}(t) + \mu PD_{1,0}(t) \right] \Delta t; n = 0, k = 0$$

or

$$= \left[-\lambda PD_{0,1}(t) + \mu PD_{1,1}(t) \right] \Delta t; n = 0, k = 1$$

or

$$= \left[-(\lambda + \mu)PD_{1,0}(t) + \mu PD_{2,0}(t) + \lambda PD_{0,0}(t) \right] \Delta t; n = 1, k = 0$$

or

$$= \left[-(\lambda + \mu)PD_{1,1}(t) + \mu PD_{2,1}(t) + \lambda PD_{0,1}(t) + \xi \left[p_{11}PD_{2,1}(t) + p_{01}PD_{2,0}(t) \right] \right] \Delta t; n = 1, k = 1$$

or

$$= \left[-(\lambda + \mu + n\xi)PD_{n+1,0}(t) + \mu PD_{n+2,0}(t) + \lambda PD_{n,0}(t) + n\xi \left[p_{00}PD_{n+1,0}(t) + p_{10}PD_{n+1,1}(t) \right] \right] \Delta t; 1 < n \leq N, k = 0$$

or

$$= \left[-(\lambda + \mu + n\xi)PD_{n+1,1}(t) + \mu PD_{n+2,1}(t) + \lambda PD_{n,1}(t) + (n + 1)\xi \left[p_{01}PD_{n+2,0}(t) + p_{11}PD_{n+2,1}(t) \right] \right] \Delta t; 1 < n \leq N, k = 1$$

or

$$= \left[-(\mu + N\xi)PD_{N+1,0}(t) + \lambda PD_{N,0}(t) + N\xi \left[p_{00}PD_{N+1,0}(t) + p_{10}PD_{N+1,1}(t) \right] \right] \Delta t; n = N + 1, k = 0$$

or

$$= \left[-(\mu + N\xi)PD_{N+1,1}(t) + \lambda PD_{N,1}(t) \right] \Delta t; n = N + 1, k = 1.$$

Using the vectors $AA_{n,k}$, $AB_{n,k}$, $AC_{n,k}$ and $AD_{n,k}$ the probabilities can be computed recursively using the equation.

$$P_{n,k}(t + \Delta t) = P_{n,k}(t) + \frac{1}{6} (AA_{n,k} + 2AB_{n,k} + 2AC_{n,k} + AD_{n,k}); n = 0, 1, 2, \dots, N + 1, k = 0, 1.$$

5. Numerical Illustration

Now we obtain the time-dependent probabilities of the model by using the Runge-Kutta Method as developed above. We take $\lambda = 1.8, \mu = 2.5, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, \xi = 0.1$, and $N = 2$. The system starts at time $t = 0$ with initial condition $P_{0,0}(0) = 1$. Here, we obtain the state probabilities for time $t = 0.1$ as under:

$$\begin{aligned}
 Q_{0,0}(0.1) &= Q_{0,0}(0) + \frac{1}{6}(AA_{0,0} + 2AB_{0,0} + 2AC_{0,0} + AD_{0,0}) = 0 + \frac{1}{6}[0.25 + \\
 &\quad 2(0.17375) + 2(0.19779375) + 0.1302379425] = 0.1872209070, \\
 Q_{0,1}(0.1) &= Q_{0,0}(0) + \frac{1}{6}(AA_{0,1} + 2AB_{0,1} + 2AC_{0,1} + AD_{0,1}) = 0 + \frac{1}{6}[0 + 2 * 0 + 2 * \\
 &\quad 0.0000225 + 0.00001593] = 0.00001155, \\
 P_{0,0}(0.1) &= P_{0,0}(0) + \frac{1}{6}(AA_{1,0} + 2AB_{1,0} + 2AC_{1,0} + AD_{1,0}) = 1 + \frac{1}{6}(-0.43 + \\
 &\quad 2(-0.29255) + 2(-0.33663675) + 0.2153323725) = 0.682715681, \\
 P_{0,1}(0.1) &= P_{0,1}(0) + \frac{1}{6}(AA_{1,1} + 2AB_{1,1} + 2AC_{1,1} + AD_{1,1}) = 0 + \frac{1}{6}(0 + 2 * \\
 &\quad 0.00018 + 2 * 0.00007992 + 0.0002482706) = 0.0001280117, \\
 P_{1,0}(0.1) &= P_{1,0}(0) + \frac{1}{6}(AA_{2,0} + 2AB_{2,0} + 2AC_{2,0} + AD_{2,0}) = 0 + \frac{1}{6}(0.18 + 2 * \\
 &\quad 0.1182 + 2 * 0.1387438 + 0.0848200608) = 0.1299181368, \\
 P_{1,1}(0.1) &= P_{1,1}(0) + \frac{1}{6}(AA_{2,1} + 2AB_{2,1} + 2AC_{2,1} + AD_{2,1}) = 0 + \frac{1}{6}(0 + 2 * 0 + 2 * \\
 &\quad 0.0000162 + 0.0000101736) = 0.0000070956.
 \end{aligned}$$

Sum of probabilities: $Q_{0,0}(0.1) + Q_{0,1}(0.1) + P_{0,0}(0.1) + P_{0,1}(0.1) + P_{1,0}(0.1) + P_{1,1}(0.1) = 1.000001382$. Thus, we have obtained all the time-dependent probabilities of the model by applying Runge-Kutta Method.

To validate our model, we use the ‘ode45’ function of MATLAB software to obtain the transient numerical results for different values of parameters.

6. Performance Measures

We study the following performance measures:

- (i) Expected system size $L_s(t) = \sum_{n=0}^N (n + 1)[P_{n,0}(t) + P_{n,1}(t)]$.
- (ii) Expected waiting time in the system $W_s(t) = \frac{L_s(t)}{\mu[1 - Q_{0,0}(t) - Q_{0,1}(t)]}$.

The variations in these performance measures with respect to time are shown in figures 2 and 3. Figure 2 shows the variation in expected system size ($L_s(t)$) with time. It can be seen that the expected system size increases with time to some extent and then asymptotically reaches the steady-state. Similarly, in figure 3, the variation in expected waiting time of a customer in the system ($W_s(t)$) is shown. The waiting time also increases with time, and then asymptotically reaches the steady-state. The values of the parameters are taken as: $\lambda = 5.2$, $\mu = 6$, $\xi = 0.3$, $N = 20$, $p_{00} = 0.8$, $p_{01} = 0.2$, $p_{10} = 0.7$ and $p_{11} = 0.3$.

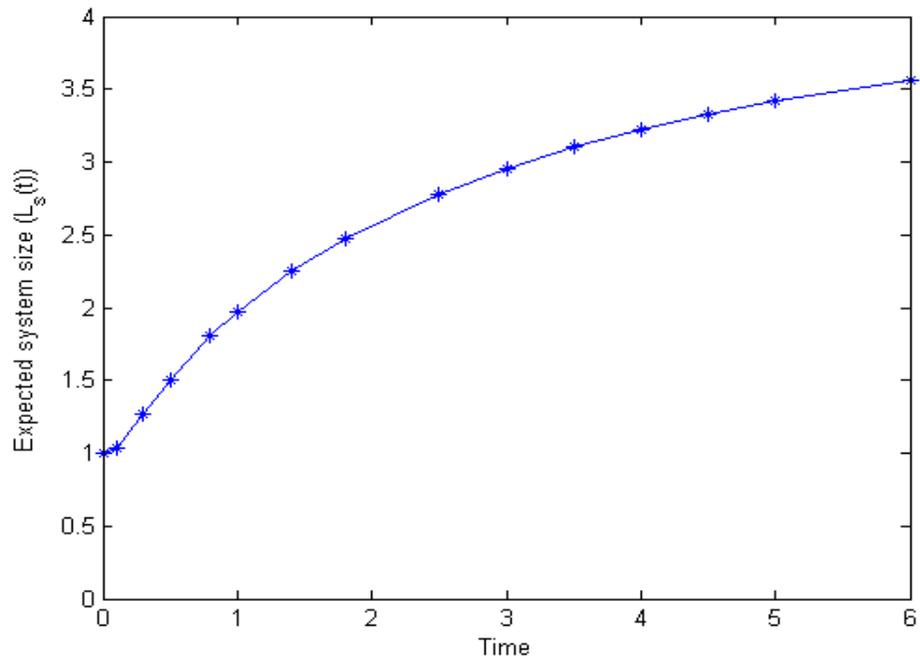


Figure 2. Expected system size vs. time

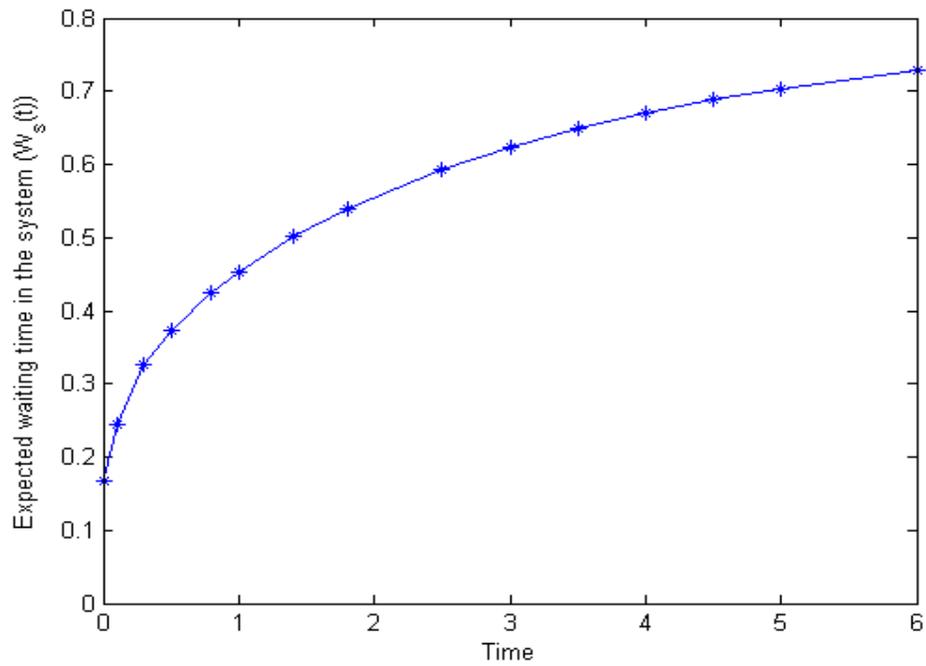


Figure 3. Expected waiting time in the system vs. Time

7. Conclusions and Future Work

In this paper, the concept of correlated reneging has been introduced in queuing theory. A single server, finite capacity queuing system with correlated reneging is studied. The transient solution of the model is obtained using Runge-Kutta Method. A numerical example is provided to illustrate the functioning of the model.

In future, the model can be extended to include the effect of balking. The same work can be extended to include generally distributed inter-arrival and service times. The multi-server analogue of the model can also be obtained. Same model can be extended where arrivals are also correlated in nature.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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