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Radial Basis Function Pseudospectral Method for Solving Standard Fitzhugh-Nagumo Equation

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Abstract
In this article, a pseudospectral approach based on radial basis functions is considered for the solution of the standard Fitzhugh-Nagumo equation. The proposed radial basis function pseudospectral approach is truly mesh free. The standard Fitzhugh-Nagumo equation is approximated into ordinary differential equations with the help of radial kernels. An ODE solver is applied to solve the resultant ODEs. Shape parameter which decides the shape of the radial basis function plays a significant role in the solution. A cross-validation technique which is the extension of the statistical approach leave-one-out-cross-validation is used to find the shape parameter value. The presented method is demonstrated with the help of numerical results which shows a good understanding with the exact solution. The stability of the proposed method is demonstrated with the help of the eigenvalues method numerically.

Keywords- FN equation, Radial basis function, Meshless.

1. Introduction
Non-linear partial differential equations are extensively practice to simulate the physical phenomena in nature. The solution of non-linear partial differential equations provides useful information and better understanding of the physical phenomena that is modeled by these equations. One such partial differential equation is Fitzhugh-Nagumo equation which is widely used as a model of excitation.

The classical Fitzhugh-Nagumo equation (FN).

$$v_t = v_{xx} + v(v - \varsigma)(1 - v), 0 < \varsigma < 1 \tag{1}$$

where $v(x, t)$ is the unknown function, is a non-linear reaction-diffusion equation which model propagation of nerve impulse. The FN equation has been derived by Fitzhugh (1961) and separately by Nagumo et al. (1962). The FN equation has numerous application in the field of circuit theory, flame propagation, population genetics, neurophysiology and autocatalytic chemical reaction (Aronson and Weinberger, 1978; Argentina et al., 2000; Zorzano and Vazquez, 2003). The FN equation reduces to the real Newell-whitehead equation that describes the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures (Newell and Whitehead, 1969).

Various studies have established the numerical solution of FN equation. Nucci and Clarkson (1992) have found the new exact solution of FN equation using Jacobi-elliptic function. Shih et al. (2005) investigated the perturbed FN equation with the help of the method of approximate conditional symmetries. The author investigated the effect of variate electric potential across the cell membrane. Olmos and Shizgal (2009) proposed the Chebyshev multidomain algorithm with Chebyshev-Gauss-Lobatto nodes based on the pseudospectral approach to solve FN equation. The author compared the Chebyshev multidomain algorithm with two other methods and demonstrated the superiority of the algorithm with respect to accuracy and computational time. Van Gorder and Vajravelu (2010) established the variational technique to obtain the approximate analytic solution of Nagumo reaction-diffusion and Nagumo telegraph boundary value problems. Jiwari et al. (2014) investigated the solution of the equation using quadrature method.

In this paper, we have proposed a meshless technique based on radial basis function (RBF) for solving the FN equation. The mesh free methods have gained attention among researchers due to their mesh free and easy to implement nature even in higher dimension problems. These methods are comfortable even with irregular geometries. Mesh free methods eliminate the complexity of constructing the mesh as compared to other conventional methods. Kansa (1990) firstly used multiquadric (one of the RBF) to find the approximate solution of different types of partial differential equations. In recent years, these methods have been extensively used for the solution of different types of partial differential equations. (Fasshauer, 1996; Larson and Fornberg, 2003; Siraj-ul-Islam et al., 2008; Chen et al., 2014; Dehghan et al., 2015; Siraj-ul-Islam and Ahmad, 2017). In RBF methods, the multidimensional problem can be transformed into one dimensional by a univariate function with Euclidean norm.

Fasshauer (2005) established a collocation method based on RBF in combination with the pseudospectral approach (RBF-PS method). Pseudospectral methods are well known as highly accurate solvers for partial differential equations and using it with RBF, one can take advantage of the scattered multivariate nodes with complex geometries. Uddin and Ali (2012) solved the stiff non-linear partial differential equations using RBF-PS method. Uddin (2013) solved the equal width equation using the same technique. Recently, Rostamy et al. (2017) applied the RBF-PS approach to solve two dimensional hyperbolic telegraph equation.

In the current work, the RBF-PS method is proposed for the numerical simulation of Fitzhugh-Nagumo equation (1) with boundary and initial conditions as

$$v(a,t) = f_1(t),\ v(b,t) = f_2(t), \quad t \in [0,T] \tag{2}$$

$$v(x,0) = g(x),\ x \in [a,b] \tag{3}$$

The RBF-PS method is an efficient and accurate technique which needs two steps to approximate solution of the FN equation. In the first step, (1) transformed to ordinary differential equations (ODEs) by approximating the space derivatives with the help of RBFs and in the second step, the transformed equations are solved by MATLAB solvers. In RBF-PS, there are two different methods to deal with boundary conditions. Either basis function satisfies the boundary conditions or we need to impose it explicitly.

2. Implementation of RBFPS Method

In this section, proposed method is implemented for (1) with the given boundary conditions (2) and

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initial condition (3). Let the computational domain Ω be approximate into N ($x_k = 1,2,3, \dots N$) points. The RBF approximation $v_N(x,t)$ can be expressed as a linear combination of unknown interpolation coefficients ζ_k and RBFs ϕ_k as

$$v_N = \sum_{k=1}^{N} \zeta_k \, \phi(\|x_i - x_k\|) \tag{4}$$

where $\phi_k = \phi(r)$, r is the distance between the points x and x_k.

Equation (4) can written as

$$v_N = A\Upsilon \tag{5}$$

where $A_{ik} = \phi(\|x_i - x_k\|)$ are the radial basis functions at points and $\Upsilon = [\zeta_1, \zeta_2, \dots, \zeta_N]^T$ are the unknown interpolation coefficients.

Equation (4) after differentiation can be written as

$$\frac{d}{dx_i} v_N(x_i) = \sum_{k=1}^{N} \zeta_k \, \frac{d}{dx_i} \phi(\|x_i - x_k\|) \tag{6}$$

The matrix form of equation (6) evaluated on all the points x_i as

$$v_N' = A_x \Upsilon \tag{7}$$

where the entries of the derivative matrix A_x are $\frac{d}{dx}\phi(\|x_i - x_k\|)$. Because of the positive definite function, the interpolation matrix is invertible. so substituting the value of Υ from equation (5) to equation (7), we get

$$v_N' = A_x A^{-1} v_N = D_x v_N \tag{8}$$

where $D_x = A_x A^{-1}$ is the corresponding differential matrix. After differentiating again, we can write

$$v_N'' = A_{xx} A^{-1} v_N = D_{xx} v_N \tag{9}$$

where $D_{xx} = A_{xx} A^{-1}$ and $A_{xx} = \frac{d^2}{dx^2}\phi(\|x_i - x_k\|)$.

Using the matrix generated by RBF-PS procedure, equation (1) is transformed to

$$\frac{dv_N}{dt} = D_{xx} v_N + v_N(v_N - \varsigma)(1 - v_N) \tag{10}$$

As discussed in section 1, we are using second way of imposing the boundary conditions by replacing some rows of the right hand side of equation (10). Now equation (10) can be solved by any ODE solver. In the present case, we are using ODE45 for solving the obtained equation (10).

International Journal of Mathematical, Engineering and Management Sciences
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3. Selecting a Good Value of Shape Parameter and Stability Analysis

As discussed, some RBF contains shape parameter which controls the shape of the RBFs. It is already observed (Fasshauer,1996) that the result of the numerical solution effected by different aspects like the position of the data points, kind of RBF used, center points position and lastly the shape parameter. Researchers are still finding the best value of the shape parameter. Ferreira and Fasshauer (2007) extends the idea of Rippa (1999) (LOOCV technique) for finding the shape parameter value. LOOCV is out of sample testing technique used in statistics for a variety of parameter identification problems. Rippa determined the value of the shape parameter by minimizing the error function as

$$E = [E_1, E_2, E_3, \dots E_n]^t \tag{11}$$

where $E_k = \left|u(x_k) - v_N{}^k(x)\right|$ and $v_N{}^k(x)$ is the RBF interpolation to all points except x_k. Rippa used a simplified formula for finding the error to avoid the high cost.

$$E_k = \frac{\zeta_k}{A_{kk}^{-1}} \tag{12}$$

where ζ_k is the interpolation coefficient and A_{kk}^{-1} is the kth diagonal element of the inverse of the interpolation matrix. As the problem becomes an optimization problem, Rippa used Brent's method to find the optimal value of shape parameter for which the error function is minimum. As discussed in section 2, D_x is the differentiation matrix as

$$A{D_x}^t = (A_x)^t \tag{13}$$

Now minimising the error function $\|E\|$ for the matrix problem (13) as

$$E_k = \frac{(D_x)^t}{A_{kk}^{-1}} \tag{14}$$

They suggested the use of MATLAB function fminbnd to obtain the minimal of the error function. In this present study, we have used the Fasshauer criteria for choosing the shape parameter.

As discussed, the given equation (1) reduces to a system of ODEs (10) in time by discretization. The stability of the numerical approach for equation (1) depends on the stability of the reduced ordinary differential equations (10). If the system is not stable, then the numerical scheme may lead to non-converged solutions. The stability of the equation (10) depends upon the eigenvalues of the matrix D_{xx}. The condition to show the numerical scheme is stable is that $Re(\lambda_i) \leq 0$ for all i where $Re(\lambda_i)$ denotes the real part of the eigenvalues λ_i of the matrix D_{xx}.

4. Numerical Application

In this segment, we have solved two numerical problems with the help of the RBF-PS method. To verify the effectiveness of the method, various error norms are reported which can be defined as follows:

$$\text{RMS error} = \sqrt{\frac{\sum_{k=1}^{N}(v(x_k,t_n) - v_N(x_k,t_n))^2}{N}}.$$

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$$L_\infty \text{error} = \max_{1\le k\le N} |v(x_k, t_n) - v_N(x_k, t_n)|.$$

for $0 \le t_n \le T$, where v and v_N denotes the exact and the numerical solutions. In this paper, we used Cubic Matérn function as RBF given by $\varphi(r) = (15 + 15\epsilon r + 6(\epsilon r)^2 + (\epsilon r)^3)e^{-\epsilon r}$ as a basis function for approximation.

Example 1

Consider the non-linear FN equation (1) in the regular domain $(x, t) \in [A, B]\, X\, [0, T]$ with boundary condition

$$v(A, t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(A - \frac{2\varsigma - 1}{\sqrt{2}}t\right)\right).$$

$v(B, t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(B - \frac{2\varsigma-1}{\sqrt{2}}t\right)\right)$ and initial condition as $v(x, 0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right)$

The analytic solution of the given FN equation is

$$v(x, t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(x - \frac{2\varsigma - 1}{\sqrt{2}}t\right)\right).$$

The numerical solution has been obtained with $A = -10$, $B = 10$ and $\zeta = 0.75$ at different time levels. In Table 1, the maximum, relative and rms errors are calculated with $\zeta = 0.75$ at time 0.2, 0.5, 1, 1.5, 2, 3, 5 and the results are compared with Jiwari et al. (2014). 0.100059 is the shape parameter value calculated with proposed strategy. Table 2 shows that the approximated result attained by the given approach are in synchronization with the exact solution. Figure 1 represents the comparison of the numerical and exact solution at different times. Figure 2 represents the comparison of the surface plots of exact and numerical solution with N = 51 nodes which shows that the approximate results are in synchronization with the analytical solutions.

The matrix stability analysis is performed for different values of N. The eigenvalues for N=51 and N=71 have no imaginary part. Figure 3 represent the real and imaginary part of the eigenvalue for N=51 and N=71. The stability conditions satisfied as real part of all eigenvalues are non-positive.

Table 1. L_2, L_∞ and RMS error norms with ζ=0.75 and Δt=0.0001 at different t for example 1

t	L_∞	**RMS**	L_2	L_∞ **Jiwari et al. (2014)**	**RMS Jiwari et al. (2014)**	L_2 **Jiwari et al. (2014)**
0.2	1.7959E-06	1.3586E-07	9.1137E-07	4.7416E-05	1.5880E-05	2.3012E-06
0.5	7.9698E-06	5.2263E-07	3.5059E-06	1.2312E-04	3.8433E-05	5.5695E-06
1.0	2.6954E-05	1.4107E-06	9.4635E-06	2.6261E-04	8.1870E-05	1.1864E-05
1.5	5.7054E-05	2.4594E-06	1.6498E-05	4.2096E-04	1.3387E-04	1.9400E-05
2.0	9.8852E-05	3.6827E-06	2.4704E-05	5.9999E-04	1.9433E-04	2.8162E-05
3.0	2.2214E-04	7.0478E-06	4.7278E-05	1.0324E-03	3.4320E-04	4.9735E-05
5.0	6.5934E-04	2.0075E-05	1.3467E-04	2.3050E-03	7.8638E-04	1.1395E-04

Table 2. Absolute errors obtained for Example 1 with N=51

	***t*=1**	***t*= 3**	***t* =5**
x	**Abs. Error**	**Abs. Error**	**Abs. Error**
-10	4.48071E-05	0.000100337	8.83066E-05
-8	1.51021E-06	1.37322E-05	1.70647E-05
-6	1.81798E-07	8.18689E-07	2.20027E-06
-4	2.24975E-07	7.21170E-07	1.12395E-06
-2	1.88839E-07	6.54491E-07	7.42538E-07
0	1.20045E-07	5.71240E-07	6.71247E-07
2	6.34033E-08	4.83999E-07	8.57893E-07
4	1.99497E-08	2.59708E-07	1.92061E-06
6	2.59799E-09	2.40972E-06	1.67456E-05
8	6.87074E-07	2.41819E-05	0.000106304
10	2.38731E-05	0.000204314	0.000616341

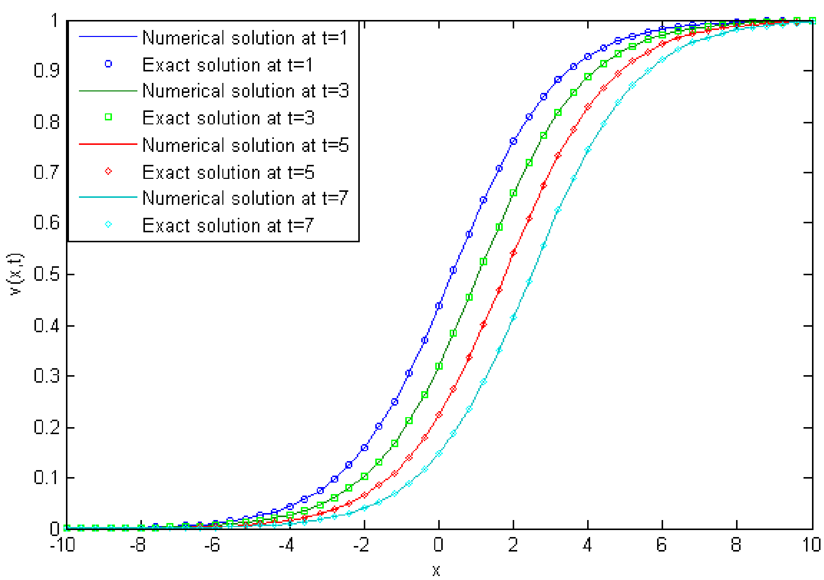

Figure 1. Comparison of results of Example 1 for different levels of time

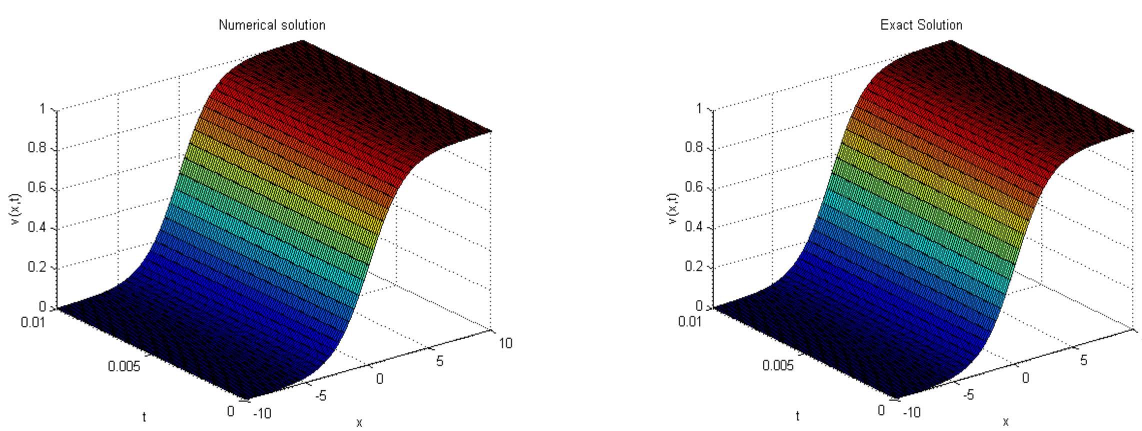

Figure 2. Analogize of solution for Example 1 for $t \leq 0.01$

International Journal of Mathematical, Engineering and Management Sciences
Vol. 5, No. 6, 1488-1497, 2020
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Example 2

Consider the non-linear FN equation (1) with $\zeta = -1$. It is a special type of FN equation known as real Newell-whitehead equation. The exact solution can be taken as

$$v(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-1}{2\sqrt{2}}\left(x - \frac{3}{\sqrt{2}}t\right)\right) \tag{15}$$

The equation is solved with initial and boundary conditions, extracted from equation (15). In this example, Table 3 presents the absolute error at different time levels. The error norms L_2, L_∞ and RMS errors are reported at distinct time measures with the help of Table 4. The comparison of the surface graph of numerical and analytic solution with N=21 is represented by Figure 4. As per the graph the numerical solution is in good accord with the analytic solution. The shape parameter value evaluated with the help of LOOCV strategy is 0.401232.

Table 3. Absolute errors obtained for Example 2 with N=21

	t =0.001	**t= 0.003**	**t =0.005**
x	**Absolute. Error**	**Absolute. Error**	**Absolute. Error**
0.1	1.03163E-10	6.85246E-09	3.14338E-08
0.2	2.22230E-11	1.34916E-10	1.86795E-09
0.3	1.45560E-11	7.38998E-12	5.84540E-11
0.4	9.02201E-12	4.85301E-12	4.81504E-12
0.5	4.66904E-12	3.17024E-13	5.10036E-13
0.6	5.31902E-12	1.55098E-12	2.19197E-12
0.7	2.97230E-11	1.38730E-11	7.51030E-11
0.8	3.07350E-11	1.67773E-10	2.30897E-09
0.9	1.32494E-10	8.45538E-09	3.89404E-08

Table 4. Various error norms for Example 2

Time (t)	**0.001**	**0.002**	**0.003**
L_∞	8.7088E-09	6.7952E-08	4.5460E-08
L_2	2.5076E-08	2.3994E-08	1.2648E-07
RMS	1.9004E-09	5.2359E-09	9.9202E-09

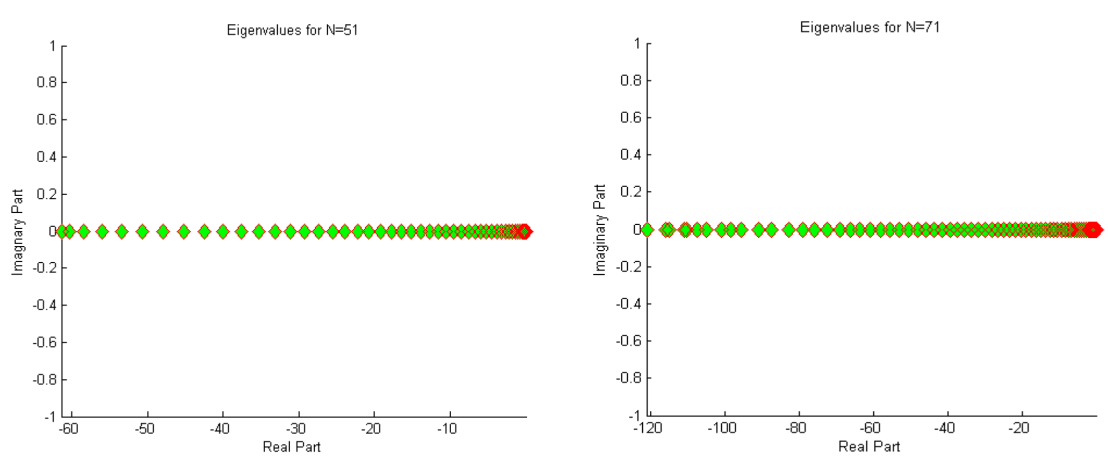

Figure 3. Eigen values of the corresponding matrix of Example 1 for N=51 and N=71

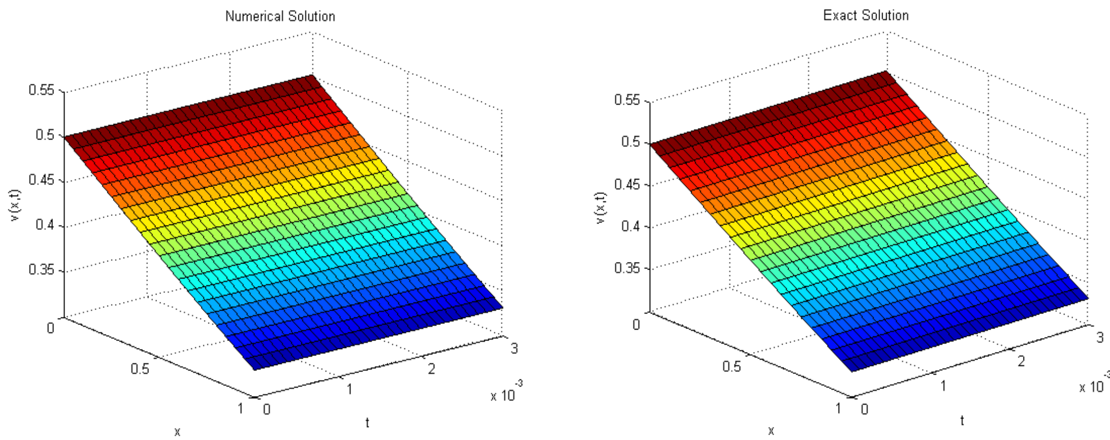

Figure 4. Comparison of results obtained in Example 2 with $\zeta = 1$ for $t \leq 0.003$

Example 3

In this example a two dimensional FN equation is considered. We consider (1) with $\zeta = 0.15$ over the square domain$[-100,100] \times [-100,100]$ as discussed by Moghaderi and Dehghan (2015) subject to the homogenous Neumann boundary conditions and initial condition as

$$v(x, y, 0) = \exp(-((x - 30)^2 + (y - 30)^2)/16).$$

The graph of the numerical solution using the proposed method on the square domain $[-100,100] \times [-100,100]$ with N=31 is shown in the Figure 5.

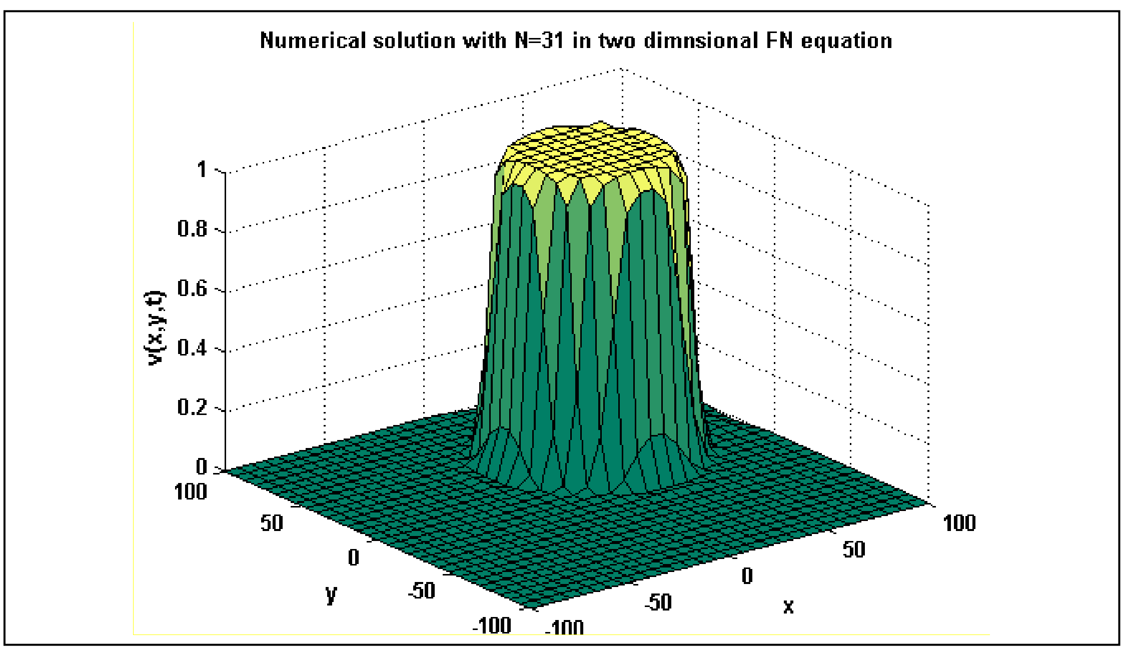

Figure 5. Numerical solution of Example 3 with N-31 and $\zeta = 0.15$ over the square domain

5. Conclusions

In the current study, radial basis function with pseudospectral approach is proposed to solve the standard Fitzhugh-Nagumo equation. The space derivatives are discretized by Radial basis function with pseudospectral approach that transform the given equation into a system of ordinary differential equations. The resultant ODEs are solved with the help of ODE solver. It is observed that in the implementation of the RBF, there is always a disagreement between the accuracy and numerical stability. Shape parameter value must be selected in such a way that there is a balance between the accuracy and the stability. Fasshauer criteria based on LOOCV strategy is applied for the choice of shape parameter. To validate the presented method, numerical problems are solved, compared with the exact solutions and error norms are presented. The method found to be accurate and fast.

Conflict of Interest

The authors declare no conflict of interest.

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