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The Reliability of a System Involving Change Points

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Abstract
The reliability of a system having some change points is presented. The technique of calculation is based on a previously developed TFCF procedure for evaluating the reliability for i.i.d. component. It involves the use of some auxiliary functions to set up a set of recursive relations. The resultant equations are solved numerically. An extension to the more general TSCSTFCF procedure and its application to start-up demonstration tests is given. Also, in case of testing, the possibility of carrying out simultaneous tests on a set of units is considered.

Keywords- Reliability, Change points, Start-up demonstration tests, Point mass distribution.

Notation

TSCSTFCF	total successes consecutive successes total failures consecutive failures
k_s	total number of successes
k_f	total number of failures
k_{cs}	length of run of successes
k_{cf}	length of run of failures
$T_{n,s}$	the total number of successes up to the n'th test
$T_{n,f}$	the total number of failures up to the n'th test
$L_{n,s}$	the length of the largest run of successes up to n
X_n	the output at the n-th test: =1 success, =0 failure
N	the random number of tests till stopping
u[.]	the unit step function
$\delta[.]$	the delta function.

1. Introduction

The way of evaluating the reliability of systems consisting of n identical components is commonly known. An example of a reference book on the basic theory of consecutive k-out-of-n systems is of Kuo and Zou (2003). A comparison of various procedures for performing tests that are based on this theory has been presented by Gera (2018). Matters become more cumbersome when we must deal with systems that have various parts with different reliabilities. Some work has been carried out for systems with non-identical elements (see for instance Smith and Griffith, 2011). The application of the theory to start-up demonstration tests has been presented by Balakrishnan et al. (2014) and others.

We divide the system into several parts (at so-called "change points") so that the components within each part are identical from the point of view of their reliabilities. Eryilmaz (2016) handled the case of a single change point. An extension to the case of two change points has been presented by Peng and Xiao (2018). They developed non-recursive formulas for calculating the survival function and for the reliability of such systems. An example of a pipeline which transports oil from one location to the other has been shown. Due to different environmental

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conditions along the various parts of the pipeline, there are assumed different reliabilities for each of its parts.

A general method for analyzing the survival function and the point mass distribution function for systems having components (or tests) with identical and independent reliabilities (probabilities of success) has been introduced by Gera (2010, 2011, 2019) – the TSCSTFCF procedure. It has been used for start-up demonstration tests. The procedure is based on some auxiliary functions which lead to a set of recursive relations that are normally solved numerically. This method is extended here to include change points.

2. A Single Change Point

A change point is given at $n=n_1$ within the distribution of the probabilities of success throughout the different tests:

$$p(n') = \begin{Bmatrix} p_1 & 1 \le n' \le n_1 \\ p_2 & n_1 < n' \end{Bmatrix} \tag{1}$$

A consecutive k_{cs}-out-of-n G:system is considered.

Let $L_{n,s}$ be the length of the largest run of successes up to n and let $T_{n,s}$ be the total number of successes up to the n. X_n is the output at the n'th test:

$$X_n = \begin{Bmatrix} 1 & success\ at\ n \\ 0 & failue\ at\ n \end{Bmatrix} \tag{2}$$

Define the auxiliary functions for r=0,1:

$$f_r(i,n) = P\{T_{ns} = i, L_{ns} < k_{cs}, X_n = r\} \tag{3}$$

Gera (2011, 2019) presented interconnecting relations for a more general case (TSCSTFCF). Meanwhile we confine the presentation to the CS (consecutive successes) procedure.

For every i,n

$$f_0(i,n) = \begin{Bmatrix} q_1 \cdot [f_0(i,n-1) + f_1(i,,n-1)] & n \le n_1 \\ q_2 \cdot [f_0(i,n-1) + f_1(i,n-1)] & n > n_1 \end{Bmatrix} \tag{4}$$

Also,

$n \le n_1$:

$$f_1(i,n) = \sum_{a=1}^{\min(k_{cs}-1,i,n_1)} p_1^{\,a} \cdot f_0(i-a,n-a) \tag{5}$$

where,

$f_1(0,0)=1$

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and

$n > n_1$:

$$f_1(i,n) = \sum_{a=1}^{\min(n-n_1,k_{cs}-1,i)} p_2^{\,a} \cdot f_0(i-a,n-a) + \sum_{b=1}^{\min(k_{cs}-1-(n-n_1),i,n-n_1)} p_2^{\,n-n_1} \cdot p_1^{\,b} \cdot f_0\big(i-b-(n-n_1), n-b-(n-n_1)\big) \tag{6}$$
$$+ p_2^{\,n-n_1} \cdot p_1^{\,n_1} \cdot \delta[i-n] \cdot u\big[k_{cs}-1-i\big]$$

where, δ[.] is the delta function and u[.] is the unit step function.

The boundary conditions for the system of equations (4)-(6) are:

$$f_0(0,n) = \begin{Bmatrix} q_1^{\,n} & n \le n_1 \\ q_1^{\,n_1} q_2^{\,n-n_1} & n > n_1 \end{Bmatrix} \tag{7}$$
$$f_1(0,n) = 0$$

$$f_0(1,1) = 0$$
$$f_1(1,n) = \begin{Bmatrix} p_1 q_1^{\,n-1} & n \le n_1 \\ p_2 q_1^{\,n_1} q_2^{\,n-n_1-1} & n > n_1 \end{Bmatrix} \tag{8}$$

Thereafter,

$$P\{N > n\} = \sum_{i=0}^{n} \{f_0(i,n) + f_1(i,n)\} \tag{9}$$

and the reliability of a G: system up to n is given by

$$R^{(G)}(n) = 1 - P\{N > n\} \tag{10}$$

Example A: Calculation of $R^{(G)}(n)$ for p_1=0.75, p_2=0.5; change point at n_1=3, k_{cs}=3. 's' stands for success and 'f' stands for failure, '1' may be any of them (Table 1).

Table 1. $R^{(G)}(n)$ for n_1=3, k_{cs}=3, p_1=0.75, p_2=0.5.

N	1	2	3	4	5	6	7
P{N>n}	1	1	0.5781	0.5078	0.4609	0.4297	0.3936
P{N=n}	0	0	0.4219 'sss'	0.0703 'fsss'	0.0469 '1fsss'	0.0313 '11fsss'	0.0361 '$\overline{sss}\,fsss$'
$R^{(G)}(n)$	0	0	0.4219	0.4922	0.5391	0.5703	0.6064

The above equality (9) may be modified to include also the possibility of having a total number of successes (k_s) till n. This is the total successes consecutive successes procedure (TSCS).

Example B: Like example A with the addition of k_s=4 (Table 2).

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Table 2. $R^{(G)}(n)$ for n_1=3, k_{cs}=3, p_1=0.75, p_2=0.5, k_s=4.

n	1	2	3	4	5	6	7
P{N>n}	1	1	0.5781	0.5078	0.4258	0.3242	0.2295
P{N=n}	0	0	0.4219 'sss'	0.0703 'fsss'	0.0820 '1fsss', 'ssfss'	0.1016 'sffsss', 'fsfsss' 'fffsss','ssffss 'sfsfss','fssfss' 'ssfsfs','sfssfs'	0.0947
$R^{(G)}(n)$	0	0	0.4219	0.4922	0.5747	0.6758	0.7705

Obviously, the TSCS procedure exhibits increased values of the reliability compared to those of the CS system.

3. Two Change Points

A generalization to two change points at n_1, n_2 is presented. Accordingly,

$$p(n)=\begin{Bmatrix} p_1 & 1\le n\le n_1 \\ p_2 & n_1<n\le n_1+n_2 \\ p_3 & n_1+n_2<n \end{Bmatrix} \tag{11}$$

The auxiliary functions (3) are as before for the CS procedure. The modified interconnecting relations are:

$$\begin{aligned} &For\ \ n\le n_1: \\ &f_1(i,n)=\sum_{a=1}^{\min(k_{cs}-1,i,n_1)} p_1^{\,a}\cdot f_0(i-a,n-a) \end{aligned} \tag{12}$$

where, $f_0(0,0)=1$.

$$\begin{aligned} &For\ \ n_1<n\le n_1+n_2: \\ &f_1(i,n)=\sum_{a=1}^{\min(n-n_1,k_{cs}-1,i)} p_2^{\,a}\cdot f_0(i-a,n-a) \\ &\quad+\sum_{b=n-n_1+1}^{\min(k_{cs}-1-(n-n_1),i,n-1)} p_2^{\,n-n_1}\cdot p_1^{\,b}\cdot f_0\big(i-b-(n-n_1),n-b-(n-n_1)\big) \\ &\quad+p_2^{\,n-n_1}p_1^{\,n_1}\cdot\delta[i-n]\cdot u\big[k_{cs}-1-i\big] \end{aligned} \tag{13}$$

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$$
\begin{aligned}
&For\ \ n > (n_1 + n_2): \\
&f_1(i,n) = \sum_{a=1}^{\min[n-(n_1+n_2),i,k_{cs}-1]} p_3^{\,a} \cdot f_0(i-a, n-a) \\
&\quad + \sum_{b=1}^{\min(n-n_1,k_{cs}-1-[n-(n_1+n_2,i)])} p_3^{\,n-(n_1+n_2)} p_2^{\,b} \cdot f_0\big(i-b-[n-(n_1+n_2), n-b-[n-(n_1+n_2)]]\big) \\
&\quad + \sum_{c=1}^{\min(n-1,k_{cs}-1-(n-n_{11}))} p_3^{\,n-(n_1+n_2)} p_2^{\,n_2} p_1^{\,c} \cdot f_0\big(i-c-(n-n_1), n-c-(n-n_1)\big) \\
&\quad + p_3^{\,n-(n_1+n_2)} p_2^{\,n_2} p_1^{\,n_1} \cdot \delta[i-n] \cdot u[k_{cs}-1-i]
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&For\ \ n > 1: \\
&f_0(i,n) = \begin{Bmatrix} q_1[f_0(i,n-1) + f_1(i,n-1)] & 1 \le n \le n_1 \\ q_2[f_0(i,n-1) + f_1(i,n-1)] & n_1 < n \le n_1 + n_2 \\ q_3[f_0(i,n-1) + f_1(i,n-1)] & n_1 + n_2 < n \end{Bmatrix}
\end{aligned}
\tag{15}
$$

The boundary conditions for the system of equations (12)-(15) are:

$$f_0(1,1) = 0$$

$$
f_0(0,n) = \begin{Bmatrix} q_1^{\,n} & n \le n_1 \\ q_1^{\,n_1} q_2^{\,n-n_1} & n_1 + n_2 \ge n > n_1 \\ q_1^{\,n_1} q_2^{\,n_2} q_3^{\,n-(n_1+n_2)} & n > n_1 + n_2 \end{Bmatrix}
\tag{16}
$$

$$
f_1(1,n) = \begin{Bmatrix} p_1 q_1^{\,n-1} & n \le n_1 \\ p_2 q_1^{\,n_1} q_2^{\,n-n_1-1_1} & n_1 + n_2 \ge n > n_1 \\ p_3 q_1^{\,n_1} q_2^{\,n_2} q_3^{\,n-(n_1+n_2)-1} & n > n_1 + n_2 \end{Bmatrix}
\tag{17}
$$

It may be noted that for a CF system (F: system), the reliability is given by:

$$
R^{(F)}(n) = 1 - \sum_{n'=1}^{n} P\{N = n'\}
\tag{18}
$$

The auxiliary functions are evaluated as before but with the proper interchanging of the roles of the p_r and q_r variables.

Example C: The reliability of an oil pipeline system. It is considered as an F: system composed of three parts. Considering various combinations of the parameters of the system, we get identical results to those of Peng and Xiao (2018) as presented in their Table 1.

Example D: CF system, k_{cf}=3, n_1=5, n_2=9, n_3=6. The reliability is evaluated at n=20 (Figure 1).

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R1 is the reliability due to $p_1(x) = 0.88-x$, $p_2=0.92$, $p_3=0.95$
R2 is the reliability due to $p_1=0.88$, $p_2(x) = 0.92-x$, $p_3=0.95$
R3 is the reliability due to $p_1=0.88$, $p_2=0.92$. $p_3(x) = 0.95-x$.

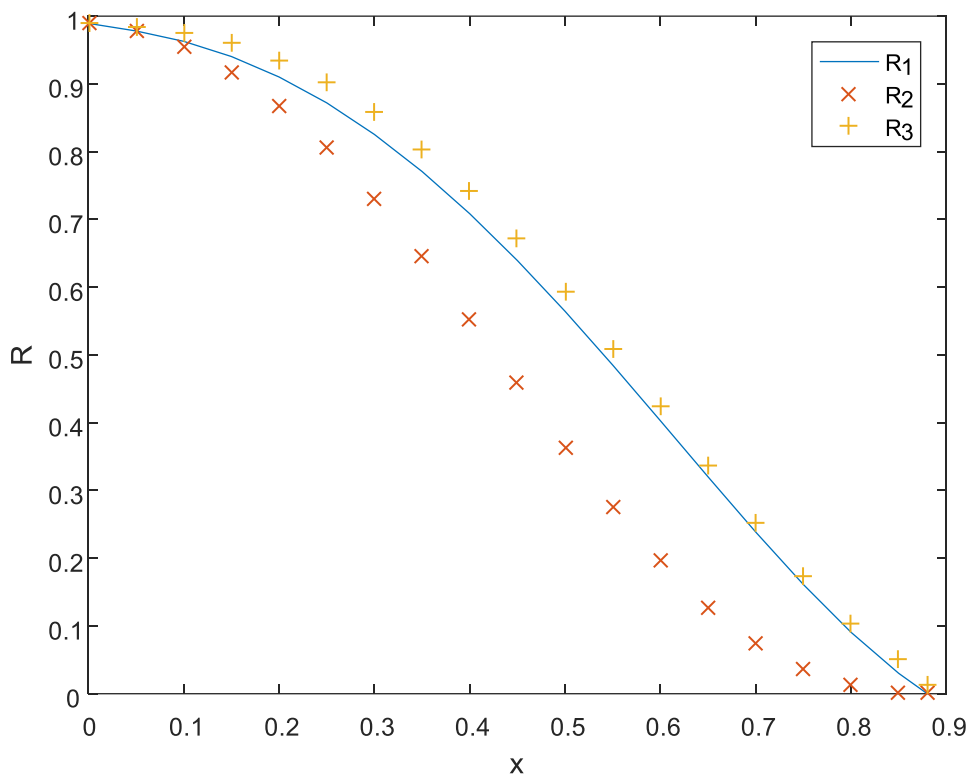

Figure 1. The change in the value of R(20) due to changes in p_1, p_2, p_3, $k_{cf}=3$, $n_1=5$, $n_2=9$, $n_3=6$.

4. Extension of the Procedure

Like for the case of a single change point, we may generalize the above procedure to include also the total number of failures (the TFCF procedure).

Let k_f denote the total number of failures. The tail distribution of N is given in this case by:

$$P\{N > n\} = \sum_{i=\max(0,n-k_f+1)}^{\min(k_s-1,n)} \{f_0(i,n) + f_1(i,n)\} \tag{19}$$

Example D: The reliability of a TFCF system, n=20, various values of k_f, k_{cf} (Table 3).

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Table 3. $R^{(F)}(20)$, various values of n_1, n_2, k_f,k_{cf}.

k_f	k_{cf}	n_1	n_2	n_3	p_1	p_2	p_3	R(20)
-	3	5	9	6	0.88	0.92	0.95	0.9893
5	-	"	"	"	0.88	0.92	0.95	0.9813
5	3	"	"	"	0.88	0.92	0.95	0.9744
-	3	"	"	"	0.92	0.92	0.92	0.9915
5	-	"	"	"	0.92	0.92	0.92	0.9817
5	3	"	"	"	0.92	0.92	0.92	0.9743
-	3	6	8	6	0.88	0.92	0.95	0.9883
5	-	"	"	"	0.88	0.92	0.95	0.9794
5	3	"	"	"	0.88	0.92	0.95	0.9719
-	4	5	9	6	0.88	0.92	0.95	0.9991
6	-	"	"	"	0.88	0.92	0.95	0.9962
6	4	"	"	"	0.88	0.92	0.95	0.9955

5. Application to Start-Up Demonstration Tests

Start-up demonstration sets of tests are often assumed to be i.i.d.(identical independent). In contrast, the probabilities of success of each test may change due to different environmental conditions. Thus, we may apply the above theory also to such tests. Here we may apply a generalized version of the former CSCF, TSCS and TFCF procedures – the TSCSTFCF procedure, Gera (2011, 2018). The set of tests is stopped if either there is a total number of successes or a run of consecutive successes or if we observe a certain total number of failures or a run of failures. Accordingly, a decision regarding the acceptance or rejection of the tested unit is then taken. The aim here is to shorten as much as possible the number of tests subject to some constraints.

Referring to (19), the point distribution of N will be given by (for n>1):

$$P\{N=n\} = P\{N>n-1\} - P\{N>n\} \tag{20}$$

The generalization of the auxiliary functions and the interconnecting relations between them has already been given before for the i.i.d. case.

$n > i \geq 1$:

$$f_0(i,n) = \begin{cases} \sum_{a_1=1}^{\min(k_{cf}-1,n)} q_1^{a_1} \cdot f_1(i,n-a_1) & n \leq n_1 \\ \sum_{a_2=1}^{\min(k_{cf}-1,n-n_1)} q_2^{a_2} \cdot f_1(i,n-a_2) + \sum_{a_1=1}^{\min(n_1,k_{cf}-1-(n-n_1))} q_2^{n-n_1} q_1^{a_1} \cdot f_1(i,n_1-a_1) & n > n_1 \end{cases} \tag{21}$$

where $f_1(0,0) = 1$

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$n > i \geq 1$:

$$f_1(i,n) = \begin{cases} \sum_{b_1=1}^{\min(k_{cs}-1,n)} p_1^{b_1} \cdot f_0(i-b_1, n-b_1) & n \leq n_1 \\ \sum_{b_2=1}^{\min(k_{cs}-1,n-n_1)} p_2^{b_2} \cdot f_0(i-b_2, n-b_2) + \sum_{b_1=1}^{\min(n_1,k_{cs}-1-(n-n_1))} p_2^{n-n_1} p_1^{b_1} \cdot f_0(i-b_1-(n-n_1), n_1-b_1) & n > n_1 \end{cases}$$

$$where \quad f_0(0,0) = 1 \tag{22}$$

Like before, appropriate boundary conditions must be taken into account. Then, use is made of (19) and (20).

Example E: CS procedure. The point mass distribution for some values of n. p_1=0.75, p_2=0.5, n_1=2, k_{cs}=2 (Table 4).

Table 4. P{N=n} for .p_1=0.75, p_2=0.5, n_1=2, k_{cs}=2.

n	1	2	3	4	5
P{N=n}	0	0.5625 'ss'	0.0938 'fss'	0.0625 'ffss'	0.0273 '~(ss)fss'

Example F: TSCSTFCF procedure. p_1=0.75, p_2=0.5, n_1=3, k_{cs}=k_{cf}=2, without (1) and with the addition of the constraint k_s=k_f=3 (2) (Table 5).

Table 5. P{N=n} for p_1=0.75, p_2=0.5: CSCF (k_{cs}=k_{cf}=2) (1), TSCSTFCF (k_{cs}=k_{cf}=2, k_s=k_f=3) (2).

n	1	2	3	4	5	6	7
P{N=n} (1)	0	0.625 'ss','ff'	0.1875 'sff','fss'	0.0938 'fsff','sfss'	0.0469	0.0234	0.0117
P{N=n} (2)	0	0.625	0.1875	0.0938	0.0938	0	0

Considering two change points, we derive the following generalization of the previous interconnecting relations:

For n > i ≥ 1:

$$f_0(i,n) = \begin{cases} \sum_{a_1=1}^{\min(k_{cf}-1,n)} q_1^{a_1} \cdot f_1(i, n-a_1) & n \leq n_1 \\ \sum_{a_2=1}^{\min(k_{cf}-1,n-n_1)} q_2^{a_2} \cdot f_1(i, n-a_2) + \sum_{a_1=1}^{\min(n_1,k_{cf}-1-(n-n_1 1))} q_2^{n-n_1} q_1^{a_1} \cdot f_1(i, n_1-a_1) & n_1 < n \leq (n_1+n_2) \\ \sum_{a_3=1}^{\min(k_{cf}-1,n-(n_1+n_2))} q_3^{a_3} \cdot f_1(i, n-a_3) + \sum_{a_2=1}^{\min(k_{cf}-1-[n-(n_1+n_2)],n_2)} q_3^{n-(n_1+n_2)} q_2^{a_2} \cdot f_1(i, n_1+n_2-a_2) + S_3 & (n_1+n_2) < n \end{cases} \tag{23}$$

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where,

$$S_3 = \sum_{a_1=1}^{\min(k_{cf}-1-(n-n_1),n_1-1)} q_3^{\,n-(n_1+n_2)} q_2^{\,n_2} q_1^{\,a_1} \cdot f_1(i, n_1 - a_1)$$

and

$$f_1(i,n) = \left\{ \begin{array}{ll} \sum_{b_1=1}^{\min(k_{cs}-1,n)} p_1^{\,b_1} \cdot f_0(i-b_1, n-b_1) & n \le n_1 \\ \sum_{b_2=1}^{\min(k_{cs}-1,n-n_1)} p_2^{\,b_2} \cdot f_0(i-b_2, n-b_2) + \sum_{b_1=1}^{\min(n_1,k_{cs}-1-(n-n_1 1))} p_2^{\,n-n_1} p_1^{\,b_1} \cdot f_0(i-b_1-(n-n_1), n_1-b_1) & n_1 < n \le (n_1+n_2) \\ \sum_{b_3=1}^{\min(k_{cs}-1,n-(n_1+n_2))} p_3^{\,b_3} \cdot f_0(i-b_3, n-b_3) + \sum_{b_2=1}^{\min(k_{cs}-1-[n-(n_1+n_2)],n_2)} p_3^{\,n-(n_1+n_2)} p_2^{\,b_2} \cdot f_0(i-b_2-n+(n_1+n_2), n_1+n_2-b_2) + T_3 & (n_1+n_2) < n \end{array} \right\} \tag{24}$$

$$T_3 = \sum_{b_1=1}^{\min(k_{cs}-1-(n-n_1),n_1-1)} p_3^{\,n-(n_1+n_2)} p_2^{\,n_2} p_1^{\,b_1} \cdot f_0(i-b_1-n+n_1, n_1-b_1).$$

Everywhere, $f_0(0,0)=1$.

The boundary conditions are given by:

$$f_0(0,n) = \left\{ \begin{array}{cc} q_1^{\,n} & n \le n_1 \\ q_1^{\,n_1} q_2^{\,n-n_1} & n_1 < n \le n_1+n_2 \\ q_1^{\,n_1} q_2^{\,n_2} q_3^{\,n-(n_1+n_2)} & n_1+n_2 < n \end{array} \right\} \tag{25}$$

$$f_1(n,n) = \left\{ \begin{array}{cc} p_1^{\,n} & n \le n_1 \\ p_1^{\,n_1} p_2^{\,n-n_1} & n_1 < n \le n_1+n_2 \\ p_1^{\,n_1} p_2^{\,n_2} p_3^{\,n-(n_1+n_2)} & n_1+n_2 < n \end{array} \right\}.$$

$$f_0(i,1) = q_1 \cdot \delta[i],$$
$$f_1(i,1) = p_1 \cdot \delta[i-1].$$

Example G: Two change points. TFCF. Reliability R(20) for various values of n_r and of p_r (Table 6).

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Table 6. The reliability of a two change points system TFCF.

k_f	k_{cf}	n_1	n_2	n_3	p_1	p_2	p_3	R
-	3	5	9	6	0.88	0.92	0.95	0.9893
5	-	5	9	6	0.88	0.92	0.95	0.9813
5	3	5	9	6	0.88	0.92	0.95	0.9744
-	3	5	9	6	0.92	0.92	0.92	0.9915
5	-	5	9	6	0.92	0.92	0.92	0.9817
5	3	5	9	6	0.92	0.92	0.92	0.9763
-	3	6	8	6	0.88	0.92	0.95	0.9883
5	-	6	8	6	0.88	0.92	0.95	0.9794
5	3	6	8	6	0.88	0.92	0.95	0.9719
-	4	5	9	6	0.88	0.92	0.95	0.9991
6	-	5	9	6	0.88	0.92	0.95	0.9962
6	4	5	9	6	0.88	0.92	0.95	0.9955

Example H: Two change points, TSCSTFCF. P{N>20} for various values of n_r and of p_r. n_1=5, n_2=9, n_3=6 (Table 7).

Table 7. P{N>20} for a two change points system, TSCSTFCF n_1=5, n_2=9, n_3=6.

k_s	k_{cs}	k_f	k_{cf}	p_1	p_2	p_3	P{N>20}
18	-	-	-	0.92	0.92	0.92	0.2121
18	-	-	-	0.88	0.92	0.95	0.2168
18	15	5	3	0.88	0.92	0.95	0.1851

6. Several Parallel Tests

Instead of testing a single unit, it is often preferable to test more units in parallel if we have such spare units at our disposal Gera (2015, 2018). The number of units is denoted by *M*. Instead of the previous runs of successes or failures, here we deal with planar squares and rectangles. We thus look for the occurrence of a rectangle Mxk_{cs} of successes and/or a rectangle Mxk_{cf} of failures. Let $R_{M,n,s}$ be the maximal area of rectangles formed by squares till performing the *n*'th stage of parallel tests and let $R_{M,n,f}$ be the appropriate area for failures. In analogy to before, $T_{n,s}$ presents the total number of successes till the *n*'th stage for all *M* units and likewise for T_{nf}.

Like before, consider a single change point (1).

For any $M \geq m \geq 1$, $2^M \geq j \geq 1$, and any $2^M \geq r \geq 0$,

$$V(m, j) = \begin{Bmatrix} 0 & (2r+1)u \geq j > 2ru \\ 1 & (2r+2)u \geq j > (2r+1)u \end{Bmatrix} \tag{26}$$

For instance, for *M*=2 units,

$$V = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

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For any column j, we denote the relevant column vector by $\underline{V}_j$ and by $\|\cdot\|$ its L_1 norm.

We need to add a variable 'j' to the definition of the previous presented auxiliary functions as follows:

$$f(i,j,n) = P\left\{T_{n,s} = i, R_{M,n,s} < Mk_{cs}, R_{M,n,f} < Mk_{cf}, \underline{X}_n = \underline{V}_j\right\} \tag{27}$$

The following inter-relations are derived:

$i \geq 1, n > 1$:

$$f(i,1,n) = \begin{Bmatrix} q_1{}^2 \sum_{j'=1}^{4} f(i, j''n-1) & n \leq n_1 \\ q_2{}^2 \sum_{j'=1}^{4} f(i, j', n-1 & n > n_1 \end{Bmatrix} \tag{28}$$

$i > 1, n > 1$:

$$f(i,2,n) = f(i,3,n) = \begin{Bmatrix} q_1 p_1 \cdot \sum_{j'=1}^{4} f(i-1, j', n-1) & n \leq n_1 \\ q_2 p_2 \cdot \sum_{j'=1}^{4} f(i-1, j', n-1) & n > n_1 \end{Bmatrix} \tag{29}$$

$For\ \ n \leq n_1$:

$$f(i,4,n) = \sum_{a=1}^{\min(k_{cs}-1, n_1)} p_1{}^{2a} \cdot \sum_{j'=1}^{3} f(i-2a, j', n-a) \tag{30}$$

$For\ \ n > n_1$:

$$\begin{aligned} f(i,4,n) = & \sum_{a=1}^{\min(n-n_1, k_{cs}-1)} p_2{}^{Ma} \cdot \sum_{j'=1}^{3} f(i-2a, j', n-a) \\ & + \sum_{b=1}^{\min(k_{cs}-1-(\ ))} p_2{}^{2(n-n_1)} \cdot p_1{}^{2b} \cdot \sum_{j'=1}^{3} f\left(i-2b-2(n-n_1), n-b-(n-n_1)\right) \\ & + p_2{}^{2(n-n_1)} \cdot p_1{}^{2n_1} \cdot \delta[i-2n] \cdot u\left[(k_{cs}-1)-n\right] \end{aligned} \tag{31}$$

Boundary conditions:

$i = 0, j > 0, n \geq 1$:

$$f(0,j,n) = \begin{Bmatrix} q_1{}^{Mn} \cdot \delta[j-1] & n \leq n_1 \\ q_1{}^{Mn_1} \cdot q_2{}^{M(n-n_1)} \cdot \delta[j-1] & n > n_1 \end{Bmatrix} \tag{32}$$

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$i \geq 0,\ j > 0, n = 1:$

$$f\left(i = \left\|\underline{V}_j\right\|, j, 1\right) = p_1^{\left\|\underline{V}_j\right\|} q_1^{M - \left\|\underline{V}_j\right\|} \tag{33}$$

Considering a G: system,

$$P\{N > n\} = \sum_{i=0}^{n} \sum_{j=1}^{2\wedge M} f(i, j, n) \tag{34}$$

and use (10): $R^{(G)}(n) = 1 - P\{N > n\}$.

Example: M=2, n_1=4, p_1=0.5, p_2=0.25, k_{cs}=2 (Table 8).

Table 8. Reliability for planar CS system with a change point M=2, , n_1=4, p_1=0.5, p_2=0.25, k_{cs}=2.

$$S_2 = \begin{pmatrix} S \\ S \end{pmatrix},\ \overline{S}_2 = \begin{pmatrix} \overline{S} \\ S \end{pmatrix}.$$

n	1	2	3	4	5	6	7
P{N>n}	1	0.9375	0.8906	0.8438	0.8328	0.8302	0.8271
P{N=n}	0	0.0625 $S_2 S_2$	0.0469 $\overline{S_2} S_2 S_2$	0.011 $1\overline{S_2} S_2 S_2$	0.0026 $\overline{S}_2 S_2 \overline{S}_2 S_2 S_2$ $1\overline{S}_2 \overline{S}_2 S_2 S_2$	0.0031	0.003
$R^{(G)}(n)$	0	0.0625	0.1094	0.1562	0.1672	0.1698	0.1729

7. Conclusion

The TSCSTFCF procedure for evaluating the reliability of consecutive k-out-of-n systems of components (or tests) has been generalized to include change points within the model. Whereas initially it has been assumed that all components (or tests) have identical reliabilities (or probabilities of success), here we considered systems which are comprised of sets of components (tests) each of which have different reliabilities. The change points indicate the distinction between the parts.

A previously developed method of using some auxiliary functions to evaluate the reliability of a CF system with change points has been presented. A set of recursive equations is created which are solved numerically. A generalization to TFCF systems has been given. A further extension to start-up demonstration tests involving the TSCSTFCF procedures has been added.

A practical application of the technique may be for instance for a long distance oil pipeline system. Such a line may be divided into several sections according to different environmental conditions. Each part exhibits a different value for its reliability. Likewise, sets of tests may be set up according to the difficulty of passing each test.

It should be added that the components (tests) were assumed to be independent. However, normally we meet systems that involve some dependence between the components. Also, we assumed that the components either work or fail. Actually, we often observe degradation in the

operation of some components (or partial success in a set of tests). Future work should include both dependent and multi-state components (tests).

The possibility of running several tests in parallel has also been examined. Obviously, this may yield some shortening of the testing procedure. An example has been provided for the case of two parallel tests. Again, the case of dependency and of multi-state outputs may be incorporated into the model within future work.

Conflict of Interest
The author confirms that there is no conflict of interest to declare for this publication.

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