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Reliability Study of <*n, f*, 2> Systems: A Generating Function Approach

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Abstract
In this paper we carry out a reliability study of the <*n, f,* 2> systems with independent and identically distributed components ordered in a line. More precisely, we obtain the generating function of structure's reliability, while recurrence relations for determining its signature vector and reliability function are also provided. For illustration purposes, several numerical results are presented and some figures are constructed and appropriately commented.

Keywords- Samaniego's signature, Generating function, <*n, f, k*> system.

1. Introduction

In the field of Reliability Engineering, a considerable research activity has been steered towards the investigation of operating features of consecutive-type structures, due to their applications in optimization of telecommunication networks, complex infrared detecting systems, oil pipeline systems, vacuum systems in accelerators or spacecraft relay stations.

A consecutive–*k*–out–of–*n*: *F* system consisting of *n* linearly ordered units (or components), fails if and only if at least *k* consecutive ones fail. Several recent research studies related to those structures have been appeared in the literature and a variety of generalizations have been also proposed for delivering more moldable features. As a paradigm, we mention the *r*–within–consecutive *k*–out–of–*n*: *F* system ($1 < r \le k \le n$), which was established by Tong (1985). The particular structure fails if and only if there exist *k* consecutive components which include among them, at least *r* failed components (see also Griffith, 1986; Triantafyllou and Koutras, 2011). Further interesting extensions of the traditional consecutive–*k*–out–of–*n*: *F* system embrace the *m*–consecutive–*k*–out–of–*n*: *F* system ($n \ge mk$) (see, e.g. Makri and Philippou, 1996; Eryilmaz et al., 2011), the sparsely connected consecutive-*k* systems (see, e.g. Zhao et al., 2007; Shen and Cui, 2015), the *m*-consecutive-*k*-out-of-*n: F-r-S- interrupted* systems (see Dafnis et al., 2019) or the *m*-consecutive-*k*, *l*-out-of-*n* systems (see Cui et al., 2015). The generalized consecutive–*k*–out–of–*n* systems with more than two possible working states, which are known as multi-state structures have also practical usefulness (see, e.g. Koutras, 1997; Zhao and Cui, 2010; Eryilmaz and Tuncel, 2016; Eryilmaz et al., 2016). Furthermore, the reliability structures, which involve two common failure criteria, have attracted a lot of research interest in the literature. For instance, the (n, f, k) system established by Tung (1982), has been studied by Chang et al. (1999) via a Markov chain approach, while Zuo et al. (2000) or Triantafyllou and Koutras (2014) investigated some interesting reliability attributes and applications of it. Additional extensions of (n, f, k) structures have been proposed; Gera (2004) provided a system which looks after two different failure criteria. More precisely, the abovementioned structure combines the well-known *f*–out–of–*n*: *G* and the classical consecutive *k*–out–of–*n*: *G* structure. Moreover, Mohan et al. (2009)

proposed an innovative scheme based on a *m*-consecutive *k*–out–of–*n*: *F* and a consecutive k_c –out–of–*n*: *F* system, while Cui and Xie (2005) introduced the so-called $((n_1, n_2, ..., n_N), f, k)$ structure involving *N* modules with the *i-th* module consisted of n_i units in parallel (see, also Eryilmaz and Tuncel, 2015). In addition, Dafnis et al. (2010) proposed the so-called *m*-consecutive-*k*, *r*-out-of-*n*: *FS* system, which fails of there are at least *m* runs of failed components of lengths k_i such that the *i-th* failure run is followed by less than k_i+*r* working components. For a detailed and up-to-date survey on the consecutive–*k*–out–of–*n*: *F* systems and their generalizations, we suggest the detailed reviews of Chao et al. (1995), Eryilmaz (2010) and Triantafyllou (2015) and the well-written monographs by Kuo and Zuo (2003) and Chang et al. (2000). A survey of reliability approaches in various fields of engineering and physical sciences is provided by Ram (2013).

In this manuscript, we bring off a reliability study of the $< n, f, k >$ structure introduced by Cui et al. (2006) which operates under two common failure criteria. The $< n, f, k >$ structure contains *n* components and fails if, and only if, there exist at least *f* failed components and at least *k* consecutive failed components. It is often to trade with structures, which fail if two or even more successive units have stopped working. In other words, the breakdown of adjacent units seems to burden the system more than it can afford. Based on this remark, we next steer our study towards the reliability characteristics of such structures. In fact, if we assume that $k = 2$, then the resulting $< n, f, k >$ structure belongs to the abovementioned category. In Section 2, we derive the generating function of the $< n, f, 2 >$ system with independent and identically distributed components ordered in a line, while recurrence relations for determining its reliability function and its signature vector are introduced and proved in Section 3.

2. The Generating Function of the < *n*, *f*, 2 >Systems

In this section, we employ a generating function approach in order to investigate some reliability characteristics of the linear $< n, f, 2 >$ structure. To achieve that, the Markov chain methodology established by Koutras (1996) is implemented (see, also Fu and Lou, 2003; Zhao and Cui, 2009) and an explicit expression for determining the generating function of the reliability function of the linear $< n, f, 2 >$ structure with independent and identically distributed (*i.i.d.* hereafter) components.

Let us first point out the necessary methodological background that will be proved useful for the study of the $< n, f, k >$ structure. We assume that R_n, $n = 0,1,...$ corresponds to the reliability function of an imbeddable in a Markov chain structure consisting of *n* components. The general form for evaluating the reliability function R_n of the abovementioned system is given as

$$R_n = \boldsymbol{\pi}_0' \Lambda^n \mathbf{u} = 1 - \boldsymbol{\pi}_0' \Lambda^n \mathbf{e}_N \tag{1}$$

where, Λ is the transition probability matrix related to the system and $\boldsymbol{\pi}_0 = (1, 0, 0, ..., 0)'$, $\boldsymbol{u} = (1,1,...,1,0)'$, $\mathbf{e}_N = (0,0,...,0,1)'$.

In addition, if $R(z;p)=\sum_{n=0}^{\infty}R_n z^n$ corresponds to the generating function of the sequence $\{R_n, n=0,1,\ldots\}$, then $R(z;p)$ be expressed as

$$R(z;p)=\boldsymbol{\pi}_0'\left(I-z\Lambda\right)^{-1}\mathbf{u}=\frac{1}{1-z}-\frac{1}{\det(I-z\Lambda)}e_{1,N} \tag{2}$$

where, $e_{i,j}$ denotes the (i,j) element of the adjoint of matrix $I-z\Lambda$. Consequently, should one be able to evaluate the last entry of the first row of the matrix $\left(I-z\Lambda\right)^{-1}$, the generating function of the reliability function of the corresponding system can be deduced after some algebraic maneuvering.

Let us next denote by T the lifetime of a reliability system with n components and $T_1,T_2,\ldots,T_n$ its components' lifetimes. If we assume that $T_1,T_2,\ldots,T_n$ are independent and identically distributed, the signature of the system is defined as the probability vector $(s_1(n),s_2(n),\ldots,s_n(n))$ with

$$s_i(n)=P(T=T_{i:n}),\, i=1,2,\ldots,n,$$

where, $T_{1:n}\le T_{2:n}\le\ldots\le T_{n:n}$ denote the respective order statistics of the random sample $T_1,T_2,\ldots,T_n$. It is evident that the signature vector of a reliability structure has turned to be a crucial performance metric for appraising the behavior of coherent structures. Moreover, the signature vector seems to be able to bolster up the evaluation of reliability polynomial or the stochastic comparisons between different structures' lifetimes (see, e.g. Samaniego, 1985; Kochar et al., 1999). For some recent advances on the evaluation of the signature vector of a reliability structure, one may refer to Kumar and Ram (2018, 2019) and Kumar et al. (2019).

We next study the linear $<n,f,k>$ structure, where $f>k$. Please mention that for the special case $f\le k$, the linear $<n,f,k>$ structure reduces to the traditional consecutive *k*-out-of-*n*: *F* system. Applying the equation (1) and taking into account the Markov chain representation of the linear $<n,f,k>$ system proposed by Cui et al. (2006), we may determine the reliability function R_n, $n=k,k+1,\ldots$ of the $<n,f,k>$ structure. Proposition 1 provides an explicit expression for determining the generating function $R(z;p)=\sum_{n=k}^{\infty}R_n z^n$ of the sequence $\{R_n, n=k,k+1,\ldots\}$. It is worth mentioning that the below mentioned result, which calls for several complex algebraic operations, seems to be the main theoretical outcome of the present manuscript, since it supports the proof of the remaining propositions and consequently all numerical or figurative illustrations displayed later on.

Proposition 1. The generating function of the reliability function of the linear $<n,f,2>$ $(f>2)$ structure with *i.i.d.* components can be determined as

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$$R(z;p)=\frac{(1-pz)^{f-1}(1-pz-pqz^2)+(qz)^f\left\{qp^{f-1}z^f-(1-pz)\left(1-\sum_{i=1}^{f-2}(pz)^i\right)\right\}}{(1-z)(1-pz)^{f-1}(1-pz-pqz^2)},$$
(3)

where, $p, q=1-p$ correspond to the common reliability and unreliability of the components.

Proof. Taking advantage of the blocked form of the transition probability matrix of the linear $<n, f, k>$ structure proposed by Cui et al. (2006), the corresponding transition probability matrix of the linear $<n, f, 2>$ structure can be viewed as a special case of the general from and written as

$$\mathbf{\Lambda}=\begin{bmatrix} A^{(1)}_{(f+1)\times(f+1)} & B^{(1)}_{(f+1)\times f} & \mathbf{0}_{(f+1)\times(f-2)} & 0 \\ A^{(2)}_{f\times(f+1)} & \mathbf{0}_{f\times f} & B^{(2)}_{f\times(f-2)} & C^{(1)}_{f\times 1} \\ \mathbf{0}_{(f-2)\times(f+1)} & \mathbf{0}_{(f-2)\times f} & B^{(3)}_{(f-2)\times(f-2)} & C^{(2)}_{(f-2)\times 1} \\ \mathbf{0}_{1\times(f+1)} & \mathbf{0}_{1\times f} & \mathbf{0}_{1\times(f-2)} & 1 \end{bmatrix}=\left(\begin{array}{c|c|c|c} \overbrace{p\ 0\ 0 \dots 0\ 0}^{f+1} & \overbrace{q\ 0\ 0 \dots 0\ 0}^{f} & \overbrace{0\ 0\ 0 \dots 0\ 0}^{f-2} & 0 \\ 0\ p\ 0 \dots 0\ 0 & 0\ q\ 0 \dots 0\ 0 & 0\ 0\ 0 \dots 0\ 0 & 0 \\ \vdots\quad \ddots\quad \vdots & \ddots & \vdots\quad \vdots & \vdots \\ 0\ 0\ 0 \dots p\ 0 & 0\ 0\ 0 \dots 0\ q & 0\ 0\ 0 \dots 0\ 0 & 0 \\ 0\ 0\ 0 \dots 0\ p & 0\ 0\ 0 \dots 0\ q & 0\ 0\ 0 \dots 0\ 0 & 0 \\ \hline 0\ p\ 0 \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & q\ 0\ 0 \dots 0\ 0 & 0 \\ 0\ 0\ p \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & 0\ q\ 0 \dots 0\ 0 & 0 \\ \vdots\quad \ddots\quad \vdots & \vdots\quad \vdots & \vdots\quad \ddots\quad \vdots & \vdots \\ 0\ 0\ 0 \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & 0\ 0\ 0 \dots 0\ q & 0 \\ 0\ 0\ 0 \dots p\ 0 & 0\ 0 \dots\ 0\ 0 & 0\ 0\ 0 \dots 0\ 0 & q \\ 0\ 0\ 0 \dots 0\ p & 0\ 0 \dots\ 0\ 0 & 0\ 0\ 0 \dots 0\ 0 & q \\ \hline 0\ 0\ 0 \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & p\ q\ 0 \dots 0\ 0 & 0 \\ 0\ 0\ 0 \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & 0\ p\ q \dots 0\ 0 & 0 \\ \vdots & \vdots & \ddots\quad \vdots & \vdots \\ 0\ 0\ 0 \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & 0\ 0\ 0 \dots p\ q & 0 \\ 0\ 0\ 0 \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & 0\ 0\ 0 \dots 0\ p & q \\ \hline 0\ 0\ 0 \dots 0\ 0 & 0\ 0 \dots\ 0\ 0 & 0\ 0\ 0 \dots 0\ 0 & 1 \end{array}\right)_{N\times N}$$
(4)

where, $N=3f$ denotes the order of matrix Λ. The $(1, N)$ element of the adjoint of the matrix $I-z\Lambda$ is given by

$$e_{1,N}=(-1)^{N+1}M_{N,1},\tag{5}$$

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where, $M_{N,1}$ corresponds to the minor determinant which is deduced by deleting the last row and the first column of the square matrix $I - z\Lambda$. More precisely, the determinant $M_{N,1}$ takes on the following form

$$
M_{N,1} = \left| \begin{array}{ccccc|cccccc|cccccc|c}
\multicolumn{5}{c}{\overbrace{\hspace{4cm}}^{f}} & \multicolumn{6}{c}{\overbrace{\hspace{4cm}}^{f}} & \multicolumn{6}{c}{\overbrace{\hspace{4cm}}^{f-2}} & \\
0 & 0 & \dots & 0 & 0 & -qz & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
1-pz & 0 & \dots & 0 & 0 & 0 & -qz & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 1-pz & \dots & 0 & 0 & 0 & 0 & -qz & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & & \vdots & & & \ddots & & & \vdots & & \vdots & & \vdots & & \\
0 & 0 & \dots & 1-pz & 0 & 0 & 0 & 0 & \dots & 0 & -qz & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1-pz & 0 & 0 & 0 & \dots & 0 & -qz & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\hline
-pz & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & -qz & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & -pz & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & -qz & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & -qz & \dots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \ddots & & & \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -qz & 0 \\
0 & 0 & \dots & -pz & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -qz \\
0 & 0 & \dots & 0 & -pz & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & -qz \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1-pz & -qz & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1-pz & -qz & \dots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & & & \vdots & & & \ddots & \vdots & & \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1-pz & -qz & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1-pz & -qz
\end{array} \right|_{(N-1)\times(N-1)}
$$

We next expand the determinant according to the elements of its first row and by casting a careful glance at the resulted expression, we conclude that $M_{N,1}$ may we written as

$$
M_{N,1} = (-1)^{f+1}\beta \left| \begin{array}{cc} aI_f & D_{f\times(2f-2)} \\ \hline F_{(2f-2)\times f} & G_{(2f-2)\times(2f-2)} \end{array} \right|,
$$

with $a = 1 - pz,\ \beta = -qz$ and

$$
D_{f\times(2f-2)} = \begin{pmatrix}
\overbrace{\beta\ 0\ 0 \dots 0\ 0}^{f-1} & \overbrace{0\ 0\ 0 \dots 0\ 0}^{f-2} & 0 \\
0\ \beta\ 0 \dots 0\ 0 & 0\ 0\ 0 \dots 0\ 0 & 0 \\
\vdots \quad \ddots \quad \vdots & \vdots \quad\quad \vdots & \vdots \\
0\ 0\ 0 \dots 0\ \beta & 0\ 0\ 0 \dots 0\ 0 & 0 \\
0\ 0\ 0 \dots 0\ \beta & 0\ 0\ 0 \dots 0\ 0 & 0
\end{pmatrix}, \quad F_{(2f-2)\times f} = \gamma \begin{pmatrix} I_f \\ \mathbf{0}_{(f-2)\times f} \end{pmatrix}, \gamma = -pz,
$$

$$
G_{(2f-2)\times(2f-2)} = \left(\begin{array}{ccccccccccccc}
\multicolumn{6}{c}{\overbrace{\qquad\qquad\qquad}^{f-1}} & & \multicolumn{6}{c}{\overbrace{\qquad\qquad\qquad}^{f-2}} \\
0 & 0 & 0 \ldots & 0 & 0 & 0 & \beta & 0 & 0 \ldots & 0 & 0 & & 0 \\
1 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & \beta & 0 \ldots & 0 & 0 & & 0 \\
0 & 1 & 0 \ldots & 0 & 0 & 0 & 0 & 0 & \beta \ldots & 0 & 0 & & 0 \\
\vdots & & \ddots & & & \vdots & \vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 \ldots & 1 & 0 & 0 & 0 & 0 & 0 \ldots & 0 & \beta & & 0 \\
0 & 0 & 0 \ldots & 0 & 1 & 0 & 0 & 0 & 0 \ldots & 0 & 0 & & \beta \\
0 & 0 & 0 \ldots & 0 & 0 & 1 & 0 & 0 & 0 \ldots & 0 & 0 & & \beta \\
\hline
0 & 0 & 0 \ldots & 0 & 0 & 0 & a & \beta & 0 \ldots & 0 & 0 & & 0 \\
0 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & a & \beta \ldots & 0 & 0 & & 0 \\
\vdots & & & & & \vdots & \vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & 0 & 0 \ldots & a & \beta & & 0 \\
0 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & 0 & 0 \ldots & 0 & a & & \beta
\end{array} \right) .
$$

We next recall the so-called Schur complement (Cottle, 1974) and the quantity $M_{N,1}$ is rewritten as

$$
\begin{aligned}
M_{N,1} &= (-1)^{f+1}\beta \left| G_{(2f-2)\times(2f-2)} - (1/a) F_{(2f-2)\times f} D_{f\times(2f-2)} \right| \cdot \left| aI_f \right| \\
&= (-1)^{f+1}\beta a^f \left(-\delta \left| H_{(2f-3)\times(2f-3)} \right| + (-1)^f \beta \left| L_{(2f-3)\times(2f-3)} \right| \right),
\end{aligned} \tag{6}
$$

where, $\delta = \beta\gamma / a$ and

$$
H_{(2f-3)\times(2f-3)} = \left(\begin{array}{cccccccccccc}
\multicolumn{6}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{f-2}} & \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{f-2}} & \\
-\delta & 0 & 0 \ldots 0 & 0 & 0 & 0 & \beta & 0 \ldots & & 0 & 0 & 0 \\
1-\delta & & 0 \ldots 0 & 0 & 0 & 0 & 0 & \beta \ldots & & 0 & 0 & 0 \\
0 & 1-\delta \ldots & 0 & 0 & 0 & 0 & 0 & 0 \ldots & & 0 & 0 & 0 \\
\vdots & \ddots & & & \vdots & \vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 \ldots 1-\delta & & 0 & 0 & 0 & 0 \ldots & & 0 & \beta & 0 \\
0 & 0 & 0 \ldots 0 & 1-\delta & & 0 & 0 & 0 \ldots & & 0 & 0 & \beta \\
0 & 0 & 0 \ldots 0 & 0 & 1 & 0 & 0 & 0 \ldots & & 0 & 0 & \beta \\
\hline
0 & 0 & 0 \ldots 0 & 0 & 0 & a & \beta & 0 \ldots & & 0 & 0 & 0 \\
0 & 0 & 0 \ldots 0 & 0 & 0 & 0 & a & \beta \ldots & & 0 & 0 & 0 \\
\vdots & & & & \vdots & \vdots & & \ddots & & & \vdots & \\
0 & 0 & 0 \ldots 0 & 0 & 0 & 0 & 0 & 0 \ldots & & a & \beta & 0 \\
0 & 0 & 0 \ldots 0 & 0 & 0 & 0 & 0 & 0 \ldots & & 0 & a & \beta
\end{array} \right) ,
$$

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$$
L_{(2f-3)\times(2f-3)} = \left(\begin{array}{cccccc|cccccc}
\multicolumn{5}{c}{\overbrace{\hspace{6em}}^{f-1}} & & \multicolumn{5}{c}{\overbrace{\hspace{6em}}^{f-3}} & \\
1-\delta & 0 \dots 0 & & & 0 & \beta & 0 \dots & & 0 & 0 & 0 \\
0 & 1-\delta \dots 0 & & & 0 & 0 & \beta \dots & & 0 & 0 & 0 \\
\vdots & \ddots & & & \vdots & \vdots & \ddots & & \vdots & \vdots \\
0 & 0 & 0 \dots 0 & & 0 & 0 & 0 & \dots 0 & \beta & 0 \\
0 & 0 & 0 \dots 1-\delta & & 0 & 0 & & \dots 0 & 0 & \beta \\
0 & 0 & 0 \dots 0 & & 1 & 0 & 0 & \dots 0 & 0 & \beta \\
\hline
0 & 0 & 0 \dots 0 & & 0 & \beta & 0 & \dots 0 & 0 & 0 \\
0 & 0 & 0 \dots 0 & & 0 & a & \beta & \dots 0 & 0 & 0 \\
\vdots & & & & \vdots & \vdots & \ddots & & \vdots & \vdots \\
0 & 0 & 0 \dots 0 & & 1 & 0 & 0 & \dots a & \beta & \beta \\
0 & 0 & 0 \dots 0 & & 1 & 0 & 0 & \dots 0 & a & \beta
\end{array}\right).
$$

Taking advantage of the block form of the matrix $L_{(2f-3)\times(2f-3)}$, we readily deduce the following

$$
\left|L_{(2f-3)\times(2f-3)}\right| = \left|R_{(f-1)\times(f-1)}\right| \cdot \left|S_{(f-2)\times(f-2)}\right| = \beta^{f-2}, \tag{7}
$$

where,

$$
R_{(f-1)\times(f-1)} = \begin{pmatrix}
1-\delta & 0 \dots 0 & 0 \\
0 & 1-\delta \dots 0 & 0 \\
\vdots & \ddots & \vdots \\
0 \quad 0 & 0 \dots 0 & 0 \\
0 \quad 0 & 0 \dots 1-\delta \\
0 \quad 0 & 0 \dots 0 & 1
\end{pmatrix}, \quad
S_{(f-2)\times(f-2)} = \begin{pmatrix}
\beta & 0 & \dots & 0 & 0 & 0 \\
a & \beta & \dots & 0 & 0 & 0 \\
\vdots & & \ddots & & \vdots & \\
0 & 0 & \dots & a & \beta & \beta \\
0 & 0 & \dots & 0 & a & \beta
\end{pmatrix}.
$$

The above expression for the determinant of matrix $L_{(2f-3)\times(2f-3)}$ is readily deduced since it can be viewed as

$$
L_{(2f-3)\times(2f-3)} = \begin{bmatrix} R_{(f-1)\times(f-1)} & LM_{(f-1)\times(f-2)} \\ \mathbf{0}_{(f-1)\times(f-1)} & S_{(f-2)\times(f-2)} \end{bmatrix},
$$

where, $LM_{(f-1)\times(f-2)}$ corresponds to the upper-right block of matrix $L_{(2f-3)\times(2f-3)}$.

We next apply the Schur complement once again in order to obtain a closed expression for the determinant of $H_{(2f-3)\times(2f-3)}$. Note that the aforementioned block matrix can be viewed as

$$H_{(2f-3)\times(2f-3)} = \begin{pmatrix} U_{(f-2)\times(f-2)} & V_{(f-2)\times(f-1)} \\ W_{(f-1)\times(f-2)} & Y_{(f-1)\times(f-1)} \end{pmatrix},$$

where,

$$U_{(f-2)\times(f-2)} = \begin{pmatrix} \overbrace{-\delta \quad 0 \quad 0 \ldots 0 \quad 0 \quad 0}^{f-2} \\ 1-\delta \quad 0 \ldots 0 \quad 0 \quad 0 \\ 0 \quad 1-\delta \ldots 0 \quad 0 \quad 0 \\ \vdots \qquad \ddots \qquad \vdots \\ 0 \quad 0 \quad 0 \ldots 1-\delta \quad 0 \\ 0 \quad 0 \quad 0 \ldots 0 \quad 1-\delta \end{pmatrix}, \quad V_{(f-2)\times(f-1)} = \begin{pmatrix} \overbrace{0 \quad \beta \quad 0 \ldots \quad 0 \quad 0}^{f-2} \quad 0 \\ 0 \quad 0 \quad \beta \ldots \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \ldots \quad 0 \quad 0 \quad 0 \\ \vdots \qquad\qquad \vdots \quad \vdots \\ 0 \quad 0 \quad 0 \ldots \quad 0 \quad \beta \quad 0 \\ 0 \quad 0 \quad 0 \ldots \quad 0 \quad 0 \quad \beta \end{pmatrix}$$

$$W_{(f-1)\times(f-2)} = \begin{pmatrix} 0 & 0 & 0 \ldots 0 & 0 & 1 \\ 0 & 0 & 0 \ldots 0 & 0 & 0 \\ 0 & 0 & 0 \ldots 0 & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 \ldots 0 & 0 & 0 \\ 0 & 0 & 0 \ldots 0 & 0 & 0 \end{pmatrix}, \quad Y_{(f-1)\times(f-1)} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & \beta \\ a & \beta & 0 & \ldots & 0 & 0 & 0 \\ 0 & a & \beta & \ldots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \ldots & a & \beta & 0 \\ 0 & 0 & 0 & \ldots & 0 & a & \beta \end{pmatrix}.$$

Therefore, the determinant of matrix $H_{(2f-3)\times(2f-3)}$ can be expressed as

$$\left|H_{(2f-3)\times(2f-3)}\right| = \left|U_{(f-2)\times(f-2)}\right| \cdot \left|Y_{(f-1)\times(f-1)} - W_{(f-1)\times(f-1)} U^{-1}_{(f-2)\times(f-2)} V_{(f-2)\times(f-1)}\right|. \tag{8}$$

Observing that

$$U_{(f-2)\times(f-2)} = -\delta \cdot Z_{(f-2)\times(f-2)},$$

where,

$$Z_{(f-2)\times(f-2)} = \begin{pmatrix} \overbrace{1 \quad 0 \quad 0 \ldots \quad 0 \quad 0}^{f-2} \\ -1/\delta \quad 1 \quad 0 \ldots \quad 0 \quad 0 \\ 0 \quad -1/\delta \quad 1 \ldots \quad 0 \quad 0 \\ \vdots \qquad \ddots \qquad \vdots \\ 0 \quad 0 \quad 0 \ldots \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \ldots -1/\delta \quad 1 \end{pmatrix}$$

we readily conclude the following

$$\left|U_{(f-2)\times(f-2)}\right| = (-\delta)^{f-2}$$

(9)

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It goes without saying that should one determine the inverse of matrix $Z_{(f-2)\times(f-2)}$, the expression $(Y_{(f-1)\times(f-1)} - W_{(f-1)\times(f-1)}U^{-1}_{(f-2)\times(f-2)}V_{(f-2)\times(f-1)})$ shall be written in a more convenient form. However, $Z^{-1}_{(f-2)\times(f-2)}$ can be expressed as the product of Gaussian elementary transformation matrices (see, e.g. Golub and Van Loan, 1996; Vanderbril et al., 2008) as follows

$$Z^{-1}_{(f-2)\times(f-2)} = \prod_{i=1}^{f-3} M_i, \quad \text{where} \quad M_i = I_{f-2} + (1/\delta)\cdot \mathbf{e}_{i+1}\cdot \mathbf{e}'_i .$$

After some straightforward algebraic manipulations, we get

$$Z^{-1}_{(f-2)\times(f-2)} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1/\delta & 1 & 0 & \dots & 0 & 0 \\ 1/\delta^2 & 1/\delta & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 1/\delta^{f-3} & 1/\delta^{f-4} & 1/\delta^{f-5} & \dots & 1/\delta & 1 \end{pmatrix}$$

and consequently, the following ensues

$$Y_{(f-1)\times(f-1)} - W_{(f-1)\times(f-1)}U^{-1}_{(f-2)\times(f-2)}V_{(f-2)\times(f-1)}$$
$$= \begin{bmatrix} 0 & \beta/\delta^{f-2} & \beta/\delta^{f-3} & \dots & \beta/\delta^3 & \beta/\delta^2 & \beta(1/+1/\delta) \\ a & \beta & 0 & \dots & 0 & 0 & 0 \\ 0 & a & \beta & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & \alpha & \beta & 0 \\ 0 & 0 & 0 & & 0 & a & \beta \end{bmatrix}$$

By expanding he determinant of the matrix appeared in the last formula through the items of its first row, we conclude that

$$\left| Y_{(f-1)\times(f-1)} - W_{(f-1)\times(f-1)}U^{-1}_{(f-2)\times(f-2)}V_{(f-2)\times(f-1)} \right| = \beta(-1)^f \left(\sum_{i=1}^{f-2} \frac{1}{\delta^{f-i-1}} a^i (-\beta)^{f-i-2} + a^{f-2} \right). \tag{10}$$

Combining formulae (6)-(10) we reach the following

$$M_{N,1} = (-1)^{f+1}\beta a^f \left((-\delta)^{f-1}\beta(-1)^f \left(\sum_{i=1}^{f-2} \frac{1}{\delta^{f-i-1}} a^i (-\beta)^{f-i-2} + a^{f-2} \right) + (-1)^f \beta^{f-1} \right)$$

and, on substituting the last expression in (5), the desired expression we are looking for determining the generating function of the reliability function of the linear $< n, f, 2 >$ system is readily deduced by the aid of (2).

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3. Recurrence Relations for the Reliability Function and the Signature Vector of the $< n, f, 2 >$ Systems

We next focus on two important performance characteristics of the linear $< n, f, 2 >$ structure consisting of *i.i.d*. components. More precisely, we exploit the general result proved in Section 2 in order to establish recurrence relations not only for evaluating the reliability function of the underlying system but also for determining the coordinates of the respective signature vector.

The following proposition provides a recursive scheme in terms of the design parameter *n* for determining the reliability function of the linear $< n, f, 2 >$ structure.

Proposition 2. (i) The reliability function R_n of the linear $< n, f, 2 >$ structure ($f > 2$) consisting of *i.i.d*. components serves the next reccurence

$$R_n = \sum_{x=1}^{f} \left\{ \left(-(-1)^x \binom{f-1}{x} + (-1)^{x-1} \binom{f-1}{x-1} - (-1)^{x-2} \binom{f-1}{x-2} \right) p^x \right.$$
$$\left. + \left((-1)^{x-1} \binom{f-1}{x-1} + (-1)^{x-3} \binom{f-1}{x-3} \right) p^{x-1} + (-1)^{x-3} \binom{f-1}{x-3} p^{x-2} \right\} R_{n-x} \qquad (11)$$
$$- (-1)^{f-2} (f-1)(p^{f-1} - p^f) R_{n-f-1} - (-1)^{f-1} (p^f - p^{f+1}) R_{n-f-2}, \quad n > f+1$$

where, *p* corresponds to the common reliability of its components and $\binom{a}{b}$ is simply the amount of different ways for picking up *b* objects out of *a* distinct elements ($a \geq b \geq 0$).

(ii) The sufficient group for setting forth the recursive relation (11) is given as

$$R_n = \begin{cases} 1, & \text{if } 0 < n < \max(f, 2), \\ 0, & \text{if } f = 0 \text{ or } n = 0, \\ \sum_{j=0}^{[(n+1)/2]} \binom{n-j+1}{j} p^{n-j} (1-p)^j, & \text{if } f = 1 \text{ or } f = 2, \end{cases} \qquad (12)$$

where, the quantity [*x*] gives the integer part of *x*.

Proof. (i) Applying (3) and the well-known identity

$$(1 - pz)^{f-1} = \sum_{i=0}^{f-1} \binom{f-1}{i} (-pz)^i ,$$

we may write

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$$(1-z)(1-pz-pqz^2)\sum_{i=0}^{f-1}\binom{f-1}{i}(-pz)^i\sum_{n=0}^{\infty}R_n z^n$$
$$=(1-pz-pqz^2)\sum_{i=0}^{f-1}\binom{f-1}{i}(-pz)^i$$
$$+(qz)^f\left[qz(pz)^{f-1}-(1-pz)\left(1-\sum_{i=1}^{f-2}(pz)^i\right)\right].$$

We next observe that the maximum power exponent on the right-hand side of the above relation equals to 2*f*. Consequently, taking into account that the coefficients of z^n on the left-hand side should not appear for $n \geq 2f+1$, the above equality can be rewritten as

$$(1-z)(1-pz-pqz^2)\sum_{n=0}^{\infty}R_n z^n\left(1+(f-1)(-pz)+\binom{f-1}{2}(-pz)^2+\binom{f-1}{3}(-pz)^3+\ldots\right.$$
$$\left.+\binom{f-1}{f-3}(-pz)^{f-3}+\binom{f-1}{f-2}(-pz)^{f-2}+\binom{f-1}{f-1}(-pz)^{f-1}\right)=0.$$

We next collect together the reliability terms having the same low index and the following recurrence relation for evaluating the reliability R_n arises

$$R_n-(1+pf)R_{n-1}-\left(-p(f-1)-p^2-p^2(f-1)-p^2\binom{f-1}{2}\right)R_{n-2}$$
$$-\left(-p+p^2+p^2\binom{f-1}{2}+p^3(f-1)+p^3\binom{f-1}{2}+p^3\binom{f-1}{3}\right)R_{n-3}-\ldots$$
$$-\left(-(-1)^{f-3}\binom{f-1}{f-3}p^{f-2}+(-1)^{f-1}p^{f-1}+(-1)^{f-3}\binom{f-1}{f-3}p^{f-1}+(-1)^{f-1}p^f-(-1)^{f-2}\binom{f-1}{f-2}p^f\right)$$
$$-\left(-(-1)^{f-2}\binom{f-1}{f-2}p^{f-1}+(-1)^{f-2}\binom{f-1}{f-2}p^f\right)R_{n-f-1}-(-1)^f(p^f-p^{f+1})R_{n-f-2}=0.$$

The outcome we are looking for is readily deduced by applying some simple algebraic operations

(ii) In order to apply the recurrence for determining the reliability function of the linear $<n, f, 2>$ structure ($f > 2$) provided in (11), a suitable set of initial conditions is needed. It is straightforward that in case of $f=0$ or $n=0$, the structure does not operate, namely its reliability function equals to 0. Moreover, if the total number of components is no greater than the parameter *f*, then we readily deduce that the resulting structure never fails. In other words, it is evident that if $n<f$, then one of the failure criteria of the linear $<n, f, 2>$ structure cannot be fulfilled. This means that, given that $n<f$, the reliability function of the linear $<n, f, 2>$ system is equal to 1. Finally, if the parameter *f* does not exceed value 2, then the underlying structure behaves actually as the traditional consecutive-2-out-of-*n*: *F* system. Consequently, for such cases the reliability function of the linear

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$< n, f, 2 >$ structure could be computed via well-known results appeared already in the literature (see, e.g. Chiang and Niu, 1981; Derman et al., 1982; Triantafyllou and Koutras, 2008b).

Please note that a different recursive scheme for evaluating the reliability of the linear and circular $< n, f, k >$ structures has been proposed by Cui et al. (2006). However, the recurrence proved in the present manuscript seems to be more favorable in the sense that it operates exclusively with respect to the total number of components, in contrast to the scheme proposed by Cui et al. (2006) which takes into consideration both design parameters *n* and *f*. Consequently, when dealing with the special case $k = 2$ of the general structure $< n, f, k >$, it seems convenient to calculate the reliability of the linear $< n, f, 2 >$ system by employing the result provided previously. Table 1 displays the reliability polynomial of linear $< n, f, 2 >$ structures for different values of its design parameters.

Figures 1-4 display the reliability polynomial of the linear $< n, f, 2 >$ systems consisting of *i.i.d.* components under a specified value of the design parameter *f* versus the common reliability of its components *p*. Indeed, the following figures reveal the impact of the design parameter *n* on the reliability of the resulting linear $< n, f, 2 >$ structure.

Table 1. Reliability polynomial of <*n*, *f*, 2> systems for different values of *n*, *f*.

n	f	**Reliability polynomial**	n	f	**Reliability polynomial**
3	3	$p^3 - 3p^2 + 3p$	6	5	$5p^6 - 24p^5 + 45p^4 - 40p^3 + 15p^2$
4	3	$3p^4 - 8p^3 + 6p^2$	7	5	$15p^7 - 70p^6 + 126p^5 - 105p^4 + 35p^3$
5	3	$5p^5 - 12p^4 + 7p^3 + p^2$	8	5	$35p^8 - 160p^7 + 280p^6 - 224p^5 + 70p^4$
6	3	$6p^6 - 12p^5 + 3p^4 + 4p^3$	9	5	$69p^9 - 310p^8 + 530p^7 - 410p^6 + 121p^5 + p^4$
7	3	$6p^7 - 9p^6 - 3p^5 + 6p^4 + p^3$	10	5	$120p^{10} - 530p^9 + 885p^8 - 660p^7 + 180p^6 + 6p^5$
8	3	$6p^8 - 8p^7 - 2p^6 + 5p^4$	11	5	$190p^{11} - 825p^{10} + 1345p^9 - 965p^8 + 240p^7 + 15p^6 + p^5$
9	3	$7p^9 - 13p^8 + 11p^7 - 15p^6 + 10p^5 + p^4$	12	5	$281p^{12} - 1202p^{11} + 1921p^{10} - 1340p^9 + 320p^8 + 14p^7 + 7p^6$
10	3	$9p^{10} - 22p^9 + 27p^8 - 24p^7 + 5p^6 + 6p^5$	6	6	$-p^6 + 6p^5 - 15p^4 + 20p^3 - 15p^2 + 6p$
11	3	$11p^{11} - 28p^{10} + 28p^9 - 6p^8 - 20p^7 + 15p^6 + p^5$	7	6	$-6p^7 + 35p^6 - 84p^5 + 105p^4 - 70p^3 + 21p^2$
12	3	$12p^{12} - 26p^{11} + 7p^{10} + 36p^9 - 49p^8 + 14p^7 + 7p^6$	8	6	$-21p^8 + 120p^7 - 280p^6 + 336p^5 - 210p^4 + 56p^3$
4	4	$-p^4 + 4p^3 - 6p^2 + 4p$	9	6	$-56p^9 + 315p^8 - 720p^7 + 840p^6 - 504p^5 + 126p^4$
5	4	$-4p^5 + 15p^4 - 20p^3 + 10p^2$	10	6	$-126p^{10} + 700p^9 - 1575p^8 + 1800p^7 - 1050p^6 + 252p^5$
6	4	$-10p^6 + 36p^5 - 45p^4 + 20p^3$	11	6	$-251p^{11} + 1380p^{10} - 3065p^9 + 3445p^8 - 1965p^7 + 456p^6 + p^5$
7	4	$-19p^7 + 66p^6 - 78p^5 + 31p^4 + p^3$	12	6	$-455p^{12} + 2478p^{11} - 5439p^{10} + 6020p^9 - 3360p^8 + 750p^7 + 7p^6$
8	4	$-30p^8 + 100p^7 - 110p^6 + 36p^5 + 5p^4$	7	7	$p^7 - 7p^6 + 21p^5 - 35p^4 + 35p^3 - 21p^2 + 7p$
9	4	$-42p^9 + 134p^8 - 136p^7 + 34p^6 + 10p^5 + p^4$	8	7	$7p^8 - 48p^7 + 140p^6 - 224p^5 + 210p^4 - 112p^3 + 28p^2$
10	4	$-55p^{10} + 170p^9 - 165p^8 + 40p^7 + 5p^6 + 6p^5$	9	7	$28p^9 - 189p^8 + 540p^7 - 840p^6 + 756p^5 - 378p^4 + 84p^3$
11	4	$-70p^{11} + 215p^{10} - 215p^9 + 75p^8 - 20p^7 + 15p^6 + p^5$	10	7	$84p^{10} - 560p^9 + 1575p^8 - 2400p^7 + 2100p^6 - 1008p^5 + 210p^4$
12	4	$-88p^{12} + 274p^{11} - 293p^{10} + 136p^9 - 49p^8 + 14p^7 + 7p^6$	11	7	$210p^{11} - 1386p^{10} + 3850p^9 - 5775p^8 + 4950p^7 - 2310p^6 + 462p^5$
5	5	$p^5 - 5p^4 + 10p^3 - 10p^2 + 5p$	12	7	$462p^{12} - 3024p^9 + 8316p^{10} - 12320p^9 + 10395p^8 - 4752p^7 + 924p^6$

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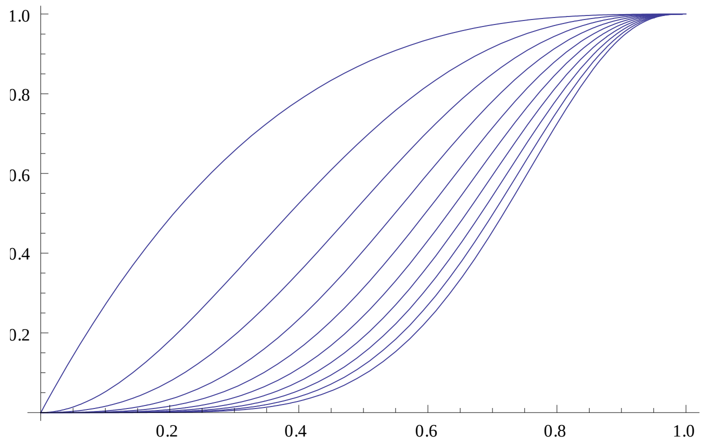

Figure 1. The reliability polynomial of the linear <n,3,2> for n = 3,4,...,12.

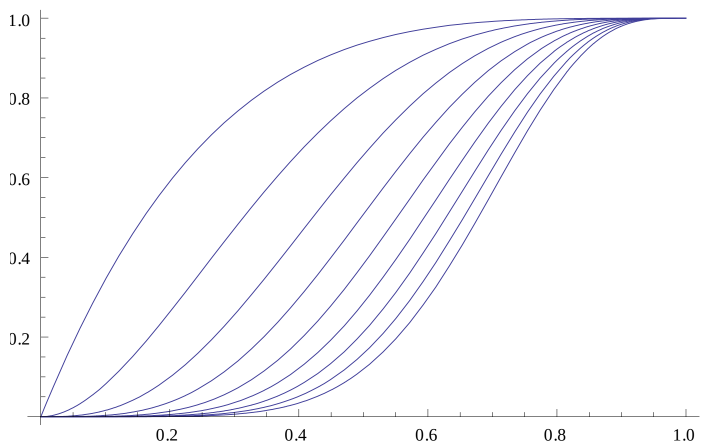

Figure 2. The reliability polynomial of the linear <n,4,2> for n = 4,5,...,12.

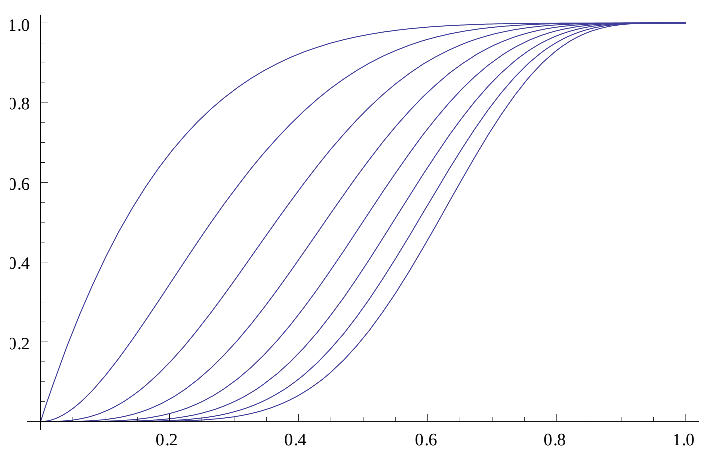

Figure 3. The reliability polynomial of the linear <n,5,2> for n = 5,6,...,12.

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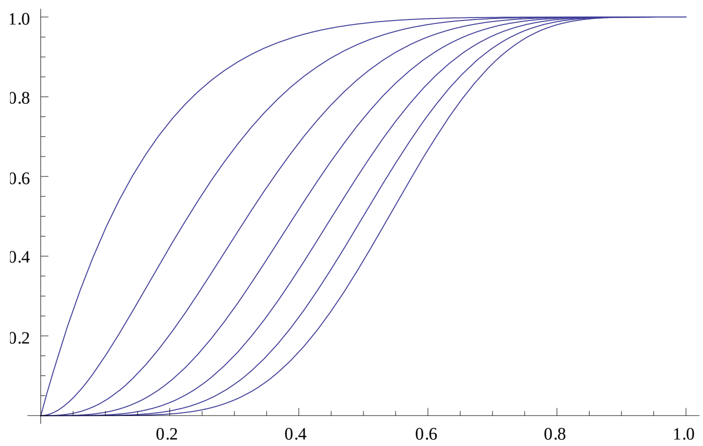

Figure 4. The reliability polynomial of the linear <*n*,6,2> for *n* = 6,7,...,12.

Based on the above figures, it is readily observed that the reliability of the linear $< n, f, 2 >$ systems decreases as its total number of components increases.

Figures 5 and 6 display the reliability polynomial of the linear $< n, f, 2 >$ systems consisting of *i.i.d.* components under a specified value of the design parameter *n* versus the common reliability of its components *p*. Indeed, the following figures reveal the impact of the design parameter *f* on the reliability of the resulting linear $< n, f, 2 >$ structure**.**

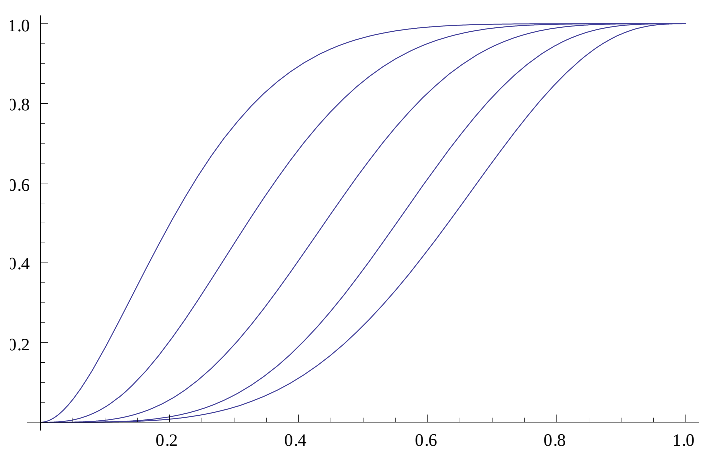

Figure 5. The reliability polynomial of the linear <8,*f*,2> for *f* = 3,4,5,6,7.

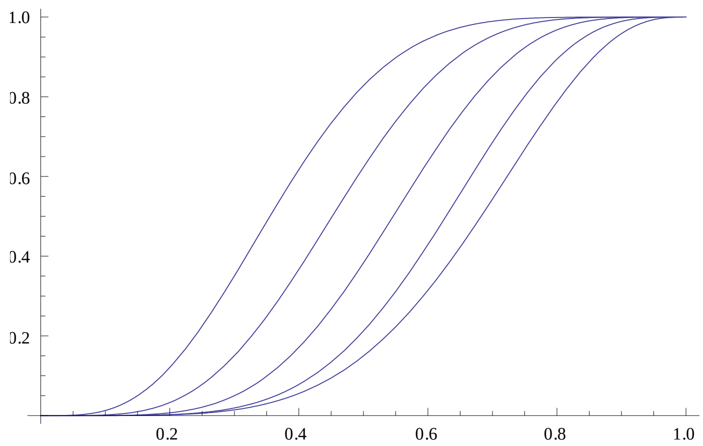

Figure 6. The reliability polynomial of the linear <10,*f*,2> for *f* = 3,4,5,6,7.

Based on the above figures, it is easily deduced that the reliability of the linear $<n, f, 2>$ systems increases as the design parameter *f* increases.

In addition, Figure 7 depicts the reliability polynomial of several well-known consecutive-type structures. In order to accomplish some figure-based comparisons, we consider different reliability systems of order *n* = 6 consisting of *i.i.d.* components. More precisely, the reliability structures appeared in Figure 7 are given below

- the linear $<6,3,2>$ system (*Green color line*)
- the linear $(6,3,2)$ system (*Red color line*)
- the 3-out-of-6*: F* system (*Blue color line*)
- the consecutive 2-out-of-6: *F* system (*Orange color line*)
- the 2-within-consecutive-3-out-of-6: *F* system (*Black color line*)

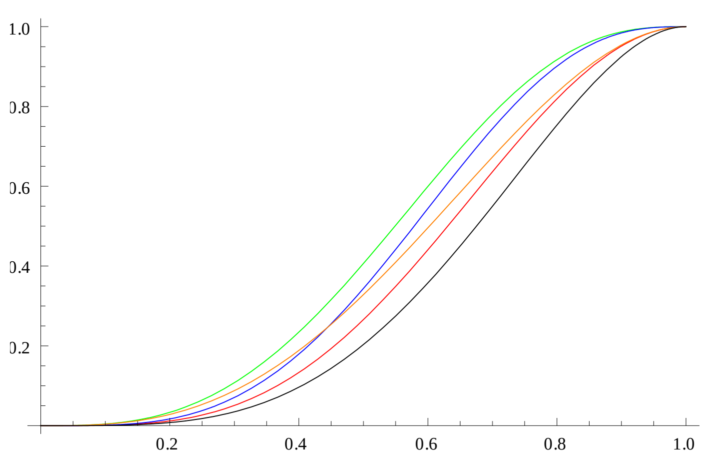

Figure 7. The reliability polynomial of several consecutive-type systems of order *n* = 6.

Based on the above figure, some useful remarks are stated as follows

- the linear $< 6,3,2 >$ system outperforms all competing structures of order 6 in the case examined
- the 2-within-consecutive-3-out-of-6: *F* system seems to exhibits the weakest performance in comparison with the other structures of order 6 appeared in the same figure
- the linear $(6,3,2)$ system outperforms only the 2-within-consecutive-3-out-of-6: *F* system, while it seems to be worse compared to the remaining structures of order 6 appeared in Figure 7.

The following proposition provides a recurrence relation in terms of the design parameter *n* for determining the coordinates of the signature vector of the linear $< n, f, 2 >$ structure for a specified value of parameter *f*.

Proposition 3. (i) The coordinates $s_i(n, f)$ of the signature vector of the linear $< n, f, 2 >$ $(f > 2)$ structure consisting of *n i.i.d.* components serves the next recursive scheme

$$\sum_{j=0}^{f}\binom{f}{j}(-1)^j\left(-i\binom{n-j-1}{i}s_i(n-j-1,f)+(i-1)\binom{n-j-2}{i-1}s_{i-1}(n-j-2,f)+3i\binom{n-j-2}{i}s_i(n-j-2,f)\right.$$
$$-3i\binom{n-j-3}{i}s_i(n-j-3,f)+2(i-2)\binom{n-j-4}{i-2}s_{i-2}(n-j-4,f)$$
$$-3(i-1)\binom{n-j-4}{i-1}s_{i-1}(n-j-4,f)+i\binom{n-j-4}{i}s_i(n-j-4,f)$$
$$+(i-2)\binom{n-j-5}{i-2}s_{i-2}(n-j-5,f)+2(i-1)\binom{n-j-5}{i-1}s_{i-1}(n-j-5,f)$$
$$\left.+(i-2)\binom{n-j-6}{i-2}s_{i-2}(n-j-6,f)+(i-3)\binom{n-j-6}{i-3}s_{i-3}(n-j-6,f)\right)=0, \quad (13)$$

where, $\binom{a}{b}$ corresponds to the total amount of different ways of picking up *b* objects out of *a* distinct elements ($a \geq b \geq 0$).

(ii) The sufficient group for setting forth the recursive relation (13) is given as
$s_i(n, f) = 0,$ for $1 \leq i \leq n \leq f-1$

$$s_i(f,f)\begin{cases}0, & 1 \leq i \leq f-1\\ 1, & i = f\end{cases}, \qquad s_i(f+1,f)=\begin{cases}0, & 1 \leq i \leq f-1\\ 1, & i=f\\ 0, & i=f+1.\end{cases} \quad (14)$$

Proof. **(i)** It is evident that the double generating function of the coordinates of the signature vector multiplied by some binomial coefficients, namely of the quantities $i\binom{n}{i}s_i(n),\ 1 \leq i \leq n$ is related to $R(z; p)$ via the following equality (Triantafyllou and Koutras, 2008a)

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} i\binom{n}{i} s_i(n)\, t^i x^n = tx\frac{\partial R(x(1+t);\frac{1}{1+t})}{\partial x} - t(t+1)\frac{\partial R(x(1+t);\frac{1}{1+t})}{\partial t}. \tag{15}$$

Substituting the generating function $R(z;p)$ given by (3) in equation (15) we may write

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} i\binom{n}{i} s_i(n,f)\, t^i x^n = \frac{P}{x(1-x-tx)(1-x-tx^2)^2(1-x)^f}, \tag{16}$$

where, $s_i(n,f)$ corresponds to the *i-th* coordinate of the signature vector of the linear $<n,f,2>$ structure consisting of n *i.i.d.* components, while

$$\begin{aligned} P = (tx)^f (f(1-x-tx^2)((1-t-x)(1-tx)x^f + x(1-2x)(1-x-tx)) \\ - tx^2(x(-2(1+t)x^2 + t + x + 3) - 2) \\ - tx^f(x((tx)^2(x-2) + 2x(x-2) + t + 3) - 1). \end{aligned}$$

We next denote by

$$c_{n,f}(t) = \sum_{i=1}^{n} i\binom{n}{i} s_i(n,f) t^i, \tag{17}$$

and equation (16) takes on the following form

$$x(1-x-tx)(1-x-tx^2)^2(1-x)^f \sum_{n=1}^{\infty} c_{n,f}(t) t^i = P$$

or equivalently

$$\left(x-(t+3)x^2+3x^3+(2t^2+3t-1)x^4-(t^2+2t)x^5-(t^2+t^3)x^6\right)\left(\sum_{j=0}^{f}\binom{f}{j}(-x)^j\right)\sum_{n=1}^{\infty} c_{n,f}(t)t^i = P. \tag{18}$$

We next rewrite the right-hand side of the above equation as

$$\begin{aligned} P = {} & ft^f x^{f+1} + (-4ft^f + (2-f)t^{f+1})x^{f+2} + (5ft^f - (2f-3)t^{f+1} - t^{f+2})x^{f+3} \\ & + (-2ft^f + (f-1)t^{f+1} + ft^{f+2})x^{f+4} + ((2-2f)(t^{f+1} + t^{f+2}))x^{f+5} \\ & + (ft^f - (f-1)t^{f+1})x^{2f} + (-2ft^f - 3t^{f+1} + (f-1)t^{f+2})x^{2f+1} \\ & + (ft^f + (f+4)t^{f+1})x^{2f+2} + (-2t^{f+1} + ft^{f+2} - (f-2)t^{f+3})x^{2f+3} \\ & - (ft^{f+2} + t^{f+3})x^{2f+4}. \end{aligned}$$

By observing that the coefficients of x^n of the left hand side of equation (18) should not appear for $n > 2f+4$, we may readily verify the following recursive scheme for $c_{n,f}(t)$

$$\sum_{j=0}^{f}\binom{f}{j}(-1)^j\Big(-c_{n-j-1,f}(t)+(t+3)c_{n-j-2,f}(t)-3c_{n-j-3,f}(t)+(2t^2-3t+1)c_{n-j-4,f}(t)$$
$$+(t^2+2t)c_{n-j-5,f}(t)+(t^2+t^3)\,c_{n-j-6,f}(t)\Big)=0,\qquad n\geq 2f+5.$$

We next substitute $c_{n,f}(t)$ by (17) in the above expression and select the coefficients of t^i, $i=1,2,\ldots,n$, on both sides; the recursive scheme we are looking for is readily deduced after some simple algebraic operations.

(ii) In order to apply the recurrence for determining the signature vector of the linear $<n, f, 2>$ structure (f > 2) provided in (13), a suitable set of initial conditions is needed. It is straightforward that in case of $f=0$ or $n=0$, the signature of the structure cannot be defined. Moreover, if the total number of components is no greater than the parameter *f*, then we readily deduce that the resulting structure never fails. In other words, it is evident that if $n\leq f-1$, then all coordinates of the signature vector of the linear $<n, f, 2>$ structure are assumed to be equal to zero. On the other hand, if the total number of components are equal to the parameter *f* (*f* >2), namely in case of *n* = *f*, then the underlying structure behaves actually as the traditional parallel system consisting of *f* components. Therefore, for such cases the only non-zero coordinate of the signature vector of the linear $<n, f, 2>$ structure appears at the last position and equals to 1. Finally, one may easily observe that for *n* = *f* +1, the linear $<n, f, 2>$ structure (*f* >2) operates with probability equal to 1 as long as at least two component has not failed and the same time it cannot operate after *f* failures have occurred. Based on the abovementioned argumentation, the set of initial conditions pointed out in (14) is readily reached.

Please mention that Eryilmaz and Zuo (2010) provided an alternative, non-recurrence way for determining the signature vector of reliability structures which operate under two common failure criteria. By the aid of their proposed method, the signature vector of the linear $<n, f, 2>$ system can be determined as

$$\mathbf{s}=\left(\underbrace{0,0,\ldots,0}_{f-1},\sum_{i=1}^{f}p_i(n),\,p_{f+1}(n),\ldots,\,p_n(n)\right),$$

where, $p_i(n)$ denotes the $i-th$ coordinate of the signature vector of the traditional consecutive−2−out−of−*n*: *F* structure. It is evident that if we focus on evaluating the signature of the linear $<n, f, 2>$ for specified design parameters (and not for a more general group of structures with different values of the parameters) the technique established by Eryilmaz and Zuo (2010) seems to be faster compared to the recurrence relation proved previously.

It is noteworthy that all numerical results and figures displayed throughout the lines of the present manuscript have been produced by the aim of the theoretical outcomes which are presented and proved in Sections 2 and 3. Table 2 displays the signature vectors of linear $<n, f, 2>$ structures for different values of its design parameters.

Table 2. Signature vector of <*n*, *f*, 2> systems for different values of *n*, *f*.

n	*f*	**Signature vector**	*n*	*f*	**Signature vector**
3	3	$(0,0,1)$	6	5	$(0,0,0,0,1,0)$
4	3	$(0,0,1,0)$	7	5	$(0,0,0,0,1,0,0)$
5	3	$(0,0,\frac{9}{10},\frac{1}{10},0)$	8	5	$(0,0,0,0,1,0,0,0)$
6	3	$(0,0,\frac{12}{15},\frac{3}{15},0,0)$	9	5	$(0,0,0,0,\frac{125}{126},\frac{1}{126},0,0,0)$
7	3	$(0,0,\frac{25}{35},\frac{9}{35},\frac{1}{35},0,0)$	10	5	$(0,0,0,0,\frac{41}{42},\frac{1}{42},0,0,0,0)$
8	3	$(0,0,\frac{18}{28},\frac{8}{28},\frac{2}{28},0,0,0)$	11	5	$(0,0,0,0,\frac{21}{22},\frac{10}{231},\frac{1}{462},0,0,0,0)$
9	3	$(0,0,\frac{7}{12},\frac{25}{84},\frac{1}{9},\frac{1}{126},0,0,0)$	12	5	$(0,0,0,0,\frac{92}{99},\frac{25}{396},\frac{1}{132},0,0,0,0,0)$
10	3	$(0,0,\frac{8}{15},\frac{3}{10},\frac{1}{7},\frac{1}{42},0,0,0,0)$	6	6	$(0,0,0,0,0,1)$
11	3	$(0,0,\frac{27}{55},\frac{49}{165},\frac{1}{6},\frac{10}{231},\frac{1}{462},0,0,0,0)$	7	6	$(0,0,0,0,0,1,0)$
12	3	$(0,0,\frac{5}{11},\frac{16}{55},\frac{91}{495},\frac{25}{396},\frac{1}{132},0,0,0,0,0)$	8	6	$(0,0,0,0,0,1,0,0)$
4	4	$(0,0,0,1)$	9	6	$(0,0,0,0,0,1,0,0,0)$
5	4	$(0,0,0,1,0)$	10	6	$(0,0,0,0,0,1,0,0,0,0)$
6	4	$(0,0,0,1,0,0)$	11	6	$(0,0,0,0,0,\frac{461}{462},\frac{1}{462}0,0,0,0)$
7	4	$(0,0,0,\frac{34}{35},\frac{1}{35},0,0)$	12	6	$(0,0,0,0,0,\frac{131}{132},\frac{1}{132},0,0,0,0,0)$
8	4	$(0,0,0,\frac{13}{14},\frac{1}{14},0,0,0)$	7	7	$(0,0,0,0,0,0,1)$
9	4	$(0,0,0,\frac{37}{42},\frac{1}{9},\frac{1}{126},0,0,0)$	8	7	$(0,0,0,0,0,0,1,0)$
10	4	$(0,0,0,\frac{5}{6},\frac{1}{7},\frac{1}{42},0,0,0,0)$	9	7	$(0,0,0,0,0,0,1,0,0)$
11	4	$(0,0,0,\frac{26}{33},\frac{1}{6},\frac{10}{231},\frac{1}{462}0,0,0,0)$	10	7	$(0,0,0,0,0,0,1,0,0,0)$
12	4	$(0,0,0,\frac{41}{55},\frac{91}{495},\frac{25}{396},\frac{1}{132}0,0,0,0,0)$	11	7	$(0,0,0,0,0,0,1,0,0,0,0)$
5	5	$(0,0,0,0,1)$	12	7	$(0,0,0,0,0,0,1,0,0,0,0,0)$

4. Conclusion

In the present manuscript, the linear <*n*, *f*, 2> (*f* > 2) structure with independent and identically distributed components ordered in a line has been investigated. The main theoretical result of the paper refers to the determination of the generating function of system's reliability. Based on that, recursive schemes for calculating the reliability function and the signature of the linear <*n*, *f*, 2> (*f* > 2) structure are provided. The figures and numerical results which are illustrated reveal the impact of the design parameters *n*, *f* on the reliability polynomial and the signature vector of the underlying structure. It is evident that when the parameter *f* decreases and/or the total number of components of the system increases, the corresponding reliability of the resulting linear <*n*, *f*, 2> (*f* > 2) structure worsens. The reliability study of the linear <*n*, *f*, *k*> system for *k* > 2 could be an interesting topic for future research.

Conflict of Interest
The author confirms that there is no conflict of interest to declare for this publication.

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