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On the Lifetime and Signature of Constrained (*k*, *d*)-out-of-*n*: *F* Reliability Systems

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Abstract
In the present paper we carry out a reliability study of the constrained (*k*, *d*)-out-of-*n*: *F* systems with exchangeable components. The signature vector is computed by the aid of the proposed algorithm. In addition, explicit signature-based expressions for the corresponding mean residual lifetime and the conditional mean residual lifetime of the aforementioned reliability system are also provided. For illustration purposes, a well-known multivariate distribution for modelling the lifetimes of the components of the constrained (*k*, *d*)-out-of-*n*: *F* structure is considered.

Keywords- Constrained (*k*,*d*)-out-of-*n*: *F* structures, Samaniego's signature, Monte-Carlo simulation, Mean residual lifetimes, Conditional mean residual lifetimes, Multivariate Pareto model.

1. Introduction

In the field of Reliability Engineering, an extensive activity is recorded referring to the consecutive-type reliability systems, mainly due to their applications in optimization of telecommunication networks, complex infrared detecting systems.

The constrained (*k*, *d*)–out–of–*n*: *F* system, which was established by Eryilmaz and Zuo (2010), fails if and only if there exist at least *k* failed components or less than *d* consecutive working components between any of two successive failures. It is straightforward that for *d*=0, the aforementioned reliability model behaves as the traditional *k*-out-of-*n: F* structure. Additional intriguing generalizations of the classical *k*–out–of–*n*: *F* or consecutive–*k*–out–of–*n*: *F* systems include the *m*–consecutive–*k*– out–of–*n*: *F* structure ($n \geq mk$) (see, e.g. Makri and Philippou, 1996; Eryilmaz et al., 2011), the sparsely connected consecutive-*k* structures (see, e.g. Zhao et al., 2007; Shen and Cui, 2015), the *m*-consecutive-*k*-out-of-*n:F-r-S-interrupted* systems (see Dafnis et al., 2019), the *r*–within–consecutive *k*–out–of–*n*: *F* model (see, e.g. Tong, 1985; Griffith, 1986; Triantafyllou and Koutras, 2011), the (*n*, *f*¸ *k*) and the < *n*, *f*¸ *k* > systems (see, e.g. Chang et al., 1999; Cui et al., 2006; Triantafyllou and Koutras, 2014) or the *m*-consecutive-*k*, *l*-out-of-*n* systems (see Cui et al., 2015).

For a detailed review of the consecutive–*k*–out–of–*n*: *F* structures and their extensions the works of Chao et al. (1995), Eryilmaz (2010) and Triantafyllou (2015) seem to be useful, while the surveys provided by Kuo and Zuo (2003), Ram (2013) and Chang et al. (2000) are suitable too.

Throughout the lines of the present manuscript, the reliability properties of the constrained (*k*, *d*)–out–of–*n*: *F* systems are studied. In Section 2, the proposed algorithm for determining the signature vector of the aforementioned reliability structures is presented, while several numerical results are displayed and appropriately commented. In Section 3, we deliver some signature-based

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expressions for the mean residual lifetime (*MRL*) and the conditional mean residual lifetime (*CMRL*) of the constrained (*k*, *d*)–out–of–*n*: *F* structure, while an illustrative example is discussed in some detail.

2. The Proposed Algorithmic Procedure for Computing the Signature of the Constrained (*k*, *d*)–out–of–*n*: *F* Systems

In the present section, we present the step-by-step procedure which can be followed in order to determine the coordinates of the signature of the constrained (*k*, *d*)–out–of–*n*: *F* models with exchangeable components. We first express by *T* the lifetime of a reliability structure consisting of *n* components and by $T_1, T_2, ..., T_n$ the corresponding components' lifetimes. Assuming that lifetimes $T_1, T_2, ..., T_n$ are exchangeable, the signature vector can be expressed as $(s_1(n), s_2(n), ..., s_n(n))$ with

$$s_i(n) = P(T = T_{i:n}), i = 1,2,...,n, \tag{1}$$

where, $T_{1:n} \le T_{2:n} \le ... \le T_{n:n}$ denote the corresponding order statistics of the random lifetimes $T_1, T_2, ..., T_n$. In practical terms, the coordinate $s_i(n), i = 1, 2, ..., n$ can be viewed as the ratio $A_i / n!$, where A_i denotes the number of those permutations of the components lifetimes of the system, for which the *i–th* component failure results in the breakdown of the reliability structure. Since the signature vector of a coherent system is promptly connected to its reliability performance, several well-known characteristics, such as the reliability polynomial of a structure or stochastic orderings among different reliability systems, can be approached by the aid of their signature (see, Samaniego, 1985; Kochar et al., 1999; Triantafyllou and Koutras, 2008a, 2008b). For some recent advances on the evaluation of the signature vector of a reliability structure, one may refer to Kumar and Ram (2018, 2019), Yi and Cui (2018) and Kumar et al. (2019).

We next illustrate the detailed procedure for determining the signature vector $(s_1(d,k,n), s_2(d,k,n), ..., s_n(d,k,n))$ of the underlying constrained (*k*, *d*)–out–of–*n*: *F* system (*C*(*k*,*d*,*n*), hereafter).

Step 1. Define the vector $\mathbf{a} = (a_1, a_2, ..., a_n)$ with initial values $a_i = 0, i = 1, 2, ..., n$.

Step 2. Generate a random sample of size *n* from an arbitrary continuous distribution *F*, which correspond to the lifetimes of the *n* components of the underlying constrained (*k*, *d*)–out–of–*n*: *F* structure.

Step 3. Determine the input parameters of the algorithm, namely the design parameters d, k, where $0 \le d, k \le n$.

Step 4. Define the random variable *M* as the chronologically ordered lifetime that leads to the breakdown of the *C*(*k*,*d*,*n*) model. The random variable *M* takes on values from 1 to *n* depending on which component proved to be the fatal one for the operation of the reliability structure.

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Step 5. Determine the value of variable M for the specific random sample of n observations which has been generated in Step 2. Each time the random variable M takes on the value i, where $i = 1, 2, ..., n$, the corresponding coordinate of vector $\mathbf{a}$ increases accordingly, namely the value a_i becomes $a_i + 1$.

All steps 2-5 are repeated k_1 times and the probability that the $C(k,d,n)$ model fails at the chronologically *i-th* ordered component failure is estimated as the value a_i divided by k_1, namely $s_i(d,k,n) = a_i / k_1, i = 1, 2, ..., n$.

In order to ascertain the validity of the proposed simulation procedure described above, we shall first apply the algorithm for small values of the design parameters n, d, k and compare the simulation-based outcomes to the corresponding results produced by the aid of the theoretical approach based on the definition of the signature vector mentioned previously. The simulation study has been accomplished based on the Java software environment and involves 10.000 replications for each result. The purpose for constructing the following table is to confirm numerically the correctness of the algorithmic method proposed in the present manuscript. In other words, before applying the new procedure for determining several reliability characteristics of the underlying structure, such as the signature vector or the *MRL*, we first proceed to implement the proposed algorithm in cases where the numerical results we are chasing for were already known by using alternative methods. In such way, we have the opportunity to assure that our proposal seems to produce correct outcomes.

Table 1 displays the signature vector of the corresponding $C(k,d,n)$ model for several values of its design parameters.

Table 1. Exact and simulation-based signatures of the constrained (k, d)–out–of–n: F structure.

n	**(*k*, *d*)**	*i* =1	*i* = 2	*i* = 3	*i* = 4	*i* = 5
4	**(3, 0)**	0 0	0 0	1 1	0 0	
	(3, 1)	0 0	0.4939 0.5000	0.5061 0.5000	0 0	
	(3, 2)	0 0	0,8269 0.8300	0,1731 0.1700	0 0	
5	**(4, 0)**	0 0	0 0	0 0	1 1	0 0
	(4, 1)	0 0	0.3997 0.4000	0.5028 0.5000	0.0975 0.1000	0 0
	(4, 2)	0 0	0,7027 0.7000	0,2973 0.3000	0 0	0 0
	(4, 3)	0 0	0,8981 0.9000	0,1019 0.1000	0 0	0 0

Each cell contains the simulation-based signature (upper entry) and the exact signature (lower entry).

As it is easily observed, the simulation-based results seem to be close enough to the exact values in all cases considered. For example, let us assume that a constrained (4,1)–out–of–5: F system is implemented. Under the aforementioned design, the exact non-zero coordinates of the corresponding vector, namely the 2nd, 3rd and 4th coordinate equal to 40%, 50% and 10%

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respectively, while the simulation-based results by the aid of the proposed procedure are 39.97%, 50.28% and 9.75%.

We next apply the abovementioned algorithm for computing the signatures of several members of the class of constrained (k, d)–out–of–n: F systems. Table 2 displays the numerical results for the signature vector of the constrained (k, d)–out–of–n: F structure for different designs. For example, let us consider the case $(d,k,n)=(1,5,7)$. The constrained (5,1)–out–of–7: F structure fails at the 2nd ordered failure component with probability 27.97%, at the 3rd ordered component lifetime with probability 43.90%, while its 4th and 5th failed component leads to the breakdown of the whole system with probability 25.12% and 3.01% respectively.

Table 2. The signatures of the constrained (k, d)–out–of–n: F system under several designs.

n	**(*k*, *d*)**	i =1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8
6	**(3, 1)**	0	0.3325	0.6675	0	0	0		
	(3, 2)	0	0.5977	0.4023	0	0	0		
	(4, 1)	0	0.3401	0.4639	0.1960	0	0		
	(4, 2)	0	0.5995	0.4005	0	0	0		
	(5, 1)	0	0.3343	0.4709	0.1948	0	0		
	(5, 2)	0	0.5975	0.4025	0	0	0		
7	**(3, 1)**	0	0.2905	0.7095	0	0	0	0	
	(3, 2)	0	0.5335	0.4665	0	0	0	0	
	(5, 1)	0	0.2797	0.4390	0.2512	0.0301	0	0	
	(5, 2)	0	0.5262	0.4480	0.0258	0	0	0	
	(5, 3)	0	0.7128	0.2872	0	0	0	0	
	(5, 4)	0	0.8555	0.1445	0	0	0	0	
	(6, 1)	0	0.2866	0.4254	0.2598	0.0282	0	0	
	(6, 2)	0	0.5186	0.4553	0.0261	0	0	0	
	(6, 3)	0	0.7202	0.2798	0	0	0	0	
	(6, 4)	0	0.8590	0.1410	0	0	0	0	
	(6, 5)	0	0.9511	0.0489	0	0	0	0	
8	**(4, 1)**	0	0.2423	0.4029	0.3548	0	0	0	0
	(4, 2)	0	0.4646	0.4598	0.0756	0	0	0	0
	(4, 3)	0	0.6453	0.3547	0	0	0	0	0
	(5, 1)	0	0.2459	0.3957	0.2902	0.0682	0	0	0
	(5, 2)	0	0.4657	0.4657	0.0686	0	0	0	0
	(5, 3)	0	0.6438	0.3562	0	0	0	0	0
	(5, 4)	0	0.7773	0.2227	0	0	0	0	0
	(6, 1)	0	0.2474	0.3945	0.2893	0.0688	0	0	0
	(6, 2)	0	0.4666	0.4642	0.0692	0	0	0	0

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Table 2 continued …

	(6, 3)	0	0.6479	0.3521	0	0	0	0	0
	(6, 4)	0	0.7844	0.2156	0	0	0	0	0
	(6, 5)	0	0.8888	0.1112	0	0	0	0	0
	(7, 1)	0	0.2469	0.4010	0.2809	0.0712	0	0	0
	(7, 2)	0	0.4726	0.4561	0.0713	0	0	0	0
	(7, 3)	0	0.6461	0.3539	0	0	0	0	0
	(7, 4)	0	0.7874	0.2126	0	0	0	0	0
	(7, 5)	0	0.8936	0.1064	0	0	0	0	0
	(7, 6)	0	0.9659	0.0341	0	0	0	0	0

3. The Residual Lifetime of the Constrained (*k*, *d*)–out–of–*n*: *F* Systems

In this section, we shall discuss the residual lifetime of a constrained (*k*, *d*)–out–of–*n*: *F* system consisting of *n* exchangeable components. More precisely, some signature-based expressions for the *MRL* and the *CMRL* of the aforementioned reliability structure are delivered, while an illustrative example is presented.

Let us denote by $T_1, T_2, \ldots, T_n$ the lifetimes of the components of a constrained (*k*, *d*)–out–of–*n*: *F* system, while $T_{1:n} \le T_{2:n} \le \ldots \le T_{n:n}$ express the corresponding ordered observations. If *T* is the structure's lifetime, then its mean residual lifetime function can be viewed as

$$m_{k,d,n}(t) = E(T - t \,|\, T > t) = \frac{1}{P(T > t)} \int_t^{\infty} P(T > x)dx \ . \tag{2}$$

In practical terms, the *MRL* function coincides to the expected (additional) survival time of the underlying structure of age *t*. Since the representation for the reliability function of the *C*(*k*,*d*,*n*) model holds true

$$P(T > t) = \sum_{i=1}^{n} s_i(d,k,n) P(T_{i:n} > t) \tag{3}$$

(see, e.g. Samaniego, 1985), the expression (2) can be rewritten as

$$m_{d,k,n}(t) = \frac{\sum_{i=1}^{n} s_i(d,k,n) P(T_{i:n} > t) m_{i:n}(t)}{\sum_{i=1}^{n} s_i(d,k,n) P(T_{i:n} > t)} \ . \tag{4}$$

Note that the quantity $m_{i:n}(t)$ is defined as

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$$m_{i:n}(t) = \frac{1}{P(T_{i:n} > t)} \int_t^{\infty} P(T_{i:n} > x)dx \tag{5}$$

and denotes the *MRL* function of an *i*-out-of-*n*: *F* structure having components with exchangeable lifetimes $T_1, T_2, ..., T_i$ $(1 \le i \le n)$.

Given that the lifetimes of the components are exchangeable, the probabilities appearing in expressions (3)-(5), can be readily determined by the aid of the following general result (see, e.g. David and Nagaraja, 2003)

$$P(T_{i:n} > t) = 1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} P(T_{1:j} \le t),$$

while

$$P(T_{1:j} \le t) = 1 - P(T_1 > t, ..., T_j > t) = 1 - \overline{F}_j(t, ..., t),$$

where, $\overline{F}_j(t_1, ..., t_j) = P(T_1 > t_1, ..., T_j > t_j)$ corresponds to the joint survival function of lifetimes $T_1, T_2, ..., T_j$ picked out from the exchangeable random lifetimes $T_1, T_2, ..., T_n$. Therefore, under the exchangeability assumption we conclude that

$$P(T_{i:n} > t) = 1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} (1 - \overline{F}_j(t, ..., t)). \tag{6}$$

In addition, the so-called conditional mean residual lifetime (*CMRL*), which corresponds to the average value of lifetime $T - t$ under the condition that $T_{r:n} > t$ provides an appraisal of the expected remaining lifetime of a structure of age *t* given that, at least $n - r + 1$ components of the structure still operate at that time $(1 \le r \le n)$. Manifestly, the *CMRL* of a constrained (*k*, *d*)–out–of–*n*: *F* system may be expressed as

$$m_{d,k,n}(t; r) = E(T - t \,|\, T_{r:n} > t) = \int_0^{\infty} P(T > t + x \,|\, T_{r:n} > t)dx. \tag{7}$$

Since

$$P(T > s \,|\, T_{r:n} > t) = \sum_{i=1}^{n} s_i(d, k, n) P(T_{i:n} > s \,|\, T_{r:n} > t), \text{ for } t < s$$

the following ensues

$$m_{d,k,n}(t; r) = \sum_{i=1}^{n} s_i(d, k, n) \int_0^{\infty} P(T_{i:n} > t + x \,|\, T_{r:n} > t)dx$$

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$$= \frac{1}{P(T_{r:n} > t)} \sum_{i=1}^{n} s_i(d,k,n) \int_0^\infty P(T_{i:n} > t + x, T_{r:n} > t) dx. \tag{8}$$

Please note that a parallel argumentation for the determination of *MRL* and *CMRL* functions of some other reliability structures has been followed by Eryilmaz et al. (2011); Triantafyllou and Koutras (2014).

For illustration purposes, we next consider an example for implementing the aforementioned formulae. Let us assume that the random vector $(T_1, T_2, ..., T_n)$ follows a *multivariate Pareto* distribution, namely

$$\bar{F}_n(t_1,...,t_n) = \left(\sum_{i=1}^{n} t_i - n + 1 \right)^{-a}, \quad t_i > 1, \text{ for } i = 1,2,...,n,$$

where, a is a positive parameter. Under the aforementioned distribution, the following ensues

$$\bar{F}_j(t,t,...,t) = \left(j(t-1)+1 \right)^{-a}, \quad t > 1.$$

Moreover, $m_{i:n}(t)$ could be determined as

$$m_{i:n}(t) = \frac{\int_t^\infty [1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} (1 - (j(x-1)+1)^{-a})] dx}{1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} (1 - (j(t-1)+1)^{-a})}.$$

Consequently, the *MRL* and the *CMRL* function of the *C*(*k*,*d*,*n*) model are given as

$$m_{d,k,n}(t) = \frac{\sum_{i=1}^{n} s_i(d,k,n)(1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \left(1 - (j(t-1)+1)^{-a}\right)) m_{i:n}(t)}{\sum_{i=1}^{n} s_i(d,k,n)(1 - \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \left(1 - (j(t-1)+1)^{-a}\right))},$$

$$E(T_{d,k,n} - t \,|\, T_{1:n} > t) = \frac{1 + n(t-1)}{a-1} \sum_{i=1}^{n} s_i(d,k,n) \sum_{j=0}^{i-1} \sum_{l=0}^{j} (-1)^l \binom{n}{j} \binom{j}{l} \frac{1}{n-j+l}, \quad a > 1.$$

Table 3 displays the *MRL* and *CMRL* of the *C*(*k*,*d*,*n*) model for several values of the parameters d, n, k, a and $t > 0$.

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Table 3. The *MRL* and *CMRL* of the *C*(*k*,*d*,*n*) model under the multivariate Pareto model.

			***a* = 1.5**		***a* = 2**		***a* = 2.5**		***a* = 3**	
n	**(*k*, *d*)**	***t***	***MRL***	***CMRL***	***MRL***	***CMRL***	***MRL***	***CMRL***	***MRL***	***CMRL***
5	(4, 1)	2	3.04432	8.3862	1.5011	4.1931	0.9913	2.7954	0.7386	2.0966
		3	5.02506	15.3747	2.4930	7.68735	1.6536	5.1249	1.2363	3.8437
		4	7.01837	22.3632	3.4903	11.1816	2.3190	7.4544	1.7356	5.5908
		5	9.01498	29.3517	4.4889	14.6759	2.9850	9.7839	2.2354	7.3379
	(4, 2)	2	2.8174	6.5892	1.3916	3.2946	0.9196	2.1964	0.6852	1.6473
		3	4.8038	12.0802	2.3853	6.0401	1.5825	4.0267	1.1827	3.0201
		4	6.7989	17.5712	3.3831	8.7856	2.2478	5.8571	1.6818	4.3928
		5	8.7965	23.0622	4.3820	11.5311	2.9139	7.6874	2.1814	5.7656
6	(4, 1)	2	2.8648	8.3577	1.4112	4.1788	0.9305	2.7859	0.6921	2.0894
		3	4.8469	15.5214	2.4028	7.7607	1.5921	5.1738	1.1887	3.8803
		4	6.8405	22.6851	3.3999	11.3425	2.2570	7.5617	1.6875	5.6713
		5	8.8373	29.8488	4.3984	14.5244	2.9228	9.9496	2.1869	7.4622
	(4, 2)	2	2.6746	6.5351	1.3208	3.2675	0.8725	2.1784	0.6498	1.6338
		3	4.6631	12.1366	2.3154	6.0683	1.5358	4.0455	1.1475	3.0342
		4	6.6590	17.7381	3.3135	8.8690	2.2013	5.9127	1.6467	4.4345
		5	8.6570	23.3396	4.3125	11.6698	2.8674	7.7799	2.1463	5.8349
	(5, 1)	2	2.8648	8.3724	1.4110	4.1862	0.9303	2.7908	0.6919	2.0931
		3	4.8468	15.5487	2.4026	7.7743	1.5919	5.1829	1.1885	3.8872
		4	6.8404	22.725	3.3997	11.3625	2.2568	7.4750	1.6873	5.6812
		5	8.8371	29.9013	4.3982	14.9506	2.9226	9.9671	2.1867	7.4753
	(5, 2)	2	2.6752	6.5421	1.3211	3.2710	0.8726	2.1807	0.6499	1.6355
		3	4.6636	12.1496	2.3156	6.0748	1.5359	4.0499	1.1475	3.0374
		4	6.6595	17.7571	3.3137	8.8785	2.2014	5.9190	1.6467	4.4393
		5	8.6575	23.3646	4.3127	11.6823	2.8675	7.7882	2.1463	5.8412
7	(5, 1)	2	2.7609	8.5431	1.3610	4.2715	0.8981	2.8477	0.6684	2.1358
		3	4.7463	16.0183	2.3543	8.0091	1.5608	5.3394	1.1659	4.0046
		4	6.7411	23.4395	3.3520	11.7467	2.2261	7.8394	1.6650	5.8734
		5	8.7385	30.9686	4.3509	15.4843	2.8921	10.3229	2.1646	7.7422
	(5, 2)	2	2.5861	6.5717	1.2783	3.2859	0.8451	2.1906	0.6298	1.6429
		3	4.5769	12.3220	2.2740	6.1610	1.5091	4.1073	1.1280	3.0805
		4	6.5737	18.0723	3.2726	9.0361	2.1749	6.0241	1.6274	4.5181
		5	8.5721	23.8226	4.2718	11.9113	2.8411	7.9409	2.1271	5.9556
	(6, 1)	2	2.7611	8.5377	1.3610	4.2688	0.8980	2.8459	0.6683	2.1344
		3	4.7463	16.0081	2.3542	8.0041	1.5606	5.3360	1.1656	4.0020
		4	6.7411	23.4786	3.3519	11.7393	2.2259	7.8262	1.6648	5.8696
		5	8.7384	30.9490	4.3507	15.4745	2.8919	10.3163	2.1643	7.7373
	(6, 2)	2	2.6199	6.9361	1.2945	3.4680	0.8556	2.3120	0.6376	1.7340
		3	4.6099	13.0051	2.2899	6.5026	1.5195	4.3350	1.1357	3.2513
		4	6.6064	19.0742	3.2883	9.5371	2.1852	6.3581	1.6350	4.7685
		5	8.6047	25.1432	4.2875	12.5716	2.8514	8.3811	2.1347	6.2858
8	(4, 1)	2	2.6330	8.3718	1.2965	4.1859	0.8544	2.7906	0.6350	2.0930
		3	4.6195	15.8134	2.2901	7.9067	1.5171	5.2711	1.1322	3.9534

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Table 3 continued…

		4	6.6148	23.255	3.2879	11.6275	2.1824	7.7517	1.6313	5.8138
		5	8.6124	30.6966	4.2868	15.3483	2.8483	10.2322	2.1308	7.6742
	(4, 2)	2	2.5262	6.6998	1.2495	3.3499	0.8264	2.2333	0.6160	1.6750
		3	4.5185	12.6552	2.2459	6.3276	1.4908	4.2184	1.1145	3.1638
		4	6.5185	18.6105	3.2446	9.3053	2.1567	6.2035	1.6140	4.6526
		5	8.5144	24.5659	4.2440	12.2829	2.8230	8.1886	2.1138	6.1415
	(5, 1)	2	2.6751	8.6809	1.3188	4.3404	0.8700	2.8936	0.6474	2.1702
		3	4.6620	16.3972	2.3127	8.1986	1.5330	5.4657	1.1448	4.0993
		4	6.5474	24.1135	3.3105	12.0568	2.1984	8.0378	1.6440	6.0284
		5	8.6550	31.8299	4.3095	15.9149	2.8644	10.610	2.1435	7.9575
	(5, 2)	2	2.5233	6.6713	1.2460	3.3356	0.8254	2.2238	0.6153	1.6678
		3	4.5156	12.6013	2.2444	6.3007	1.4899	4.2004	1.1138	3.1503
		4	6.5129	18.5314	3.2432	9.2657	2.1558	6.1771	1.6133	4.6328
		5	8.5115	24.6414	4.2425	12.2307	2.8221	8.1538	2.1131	6.1154
	(6, 1)	2	2.6753	8.6779	1.3189	4.3390	0.8702	2.8927	0.6475	2.1695
		3	4.6622	16.3918	2.3128	8.1959	1.5331	5.4639	1.1449	4.0979
		4	6.6576	24.1055	3.3107	12.0528	2.1985	8.0352	1.6441	6.0264
		5	8.6553	31.8193	4.3096	15.9096	2.8645	10.6064	2.1436	7.9548
	(6, 2)	2	2.5234	6.6708	1.2481	3.3354	0.8255	2.2236	0.6154	1.6677
		3	4.5157	12.6003	2.2445	6.3002	1.4899	4.2001	1.1139	3.1501
		4	6.5130	18.5299	3.2433	9.2649	2.1559	6.1766	1.6134	4.6325
		5	8.5117	24.4594	4.2426	12.2297	2.8221	8.1531	2.1131	6.1149
	(7, 1)	2	2.6754	8.6687	1.3191	4.3343	0.8704	2.8896	0.6477	2.1672
		3	4.6624	16.3742	2.3131	8.1871	1.5334	5.4581	1.1452	4.0936
		4	6.6579	24.0797	3.3110	12.0398	2.1988	8.0266	1.6443	6.0199
		5	8.6555	31.7852	4.3099	15.8926	2.8648	10.5951	2.1439	7.9463
	(7, 2)	2	2.5235	6.6603	1.2482	3.3302	0.8256	2.2201	0.6155	1.6651
		3	4.5159	12.5806	2.2447	6.2903	1.4901	4.1935	1.1140	3.1452
		4	6.5132	18.5009	3.2434	9.2504	2.1560	6.1669	1.6135	4.6252
		5	8.5119	24.2411	4.2428	12.2106	2.8223	8.1404	2.1133	6.1053

Finally, we shall construct several illustrative figures in order to depict more clearly the connection between the *MRL* (*CMRL*) function and the design parameters of the constrained (*k*, *d*)–out–of–*n*: *F* system under the multivariate Pareto. model.

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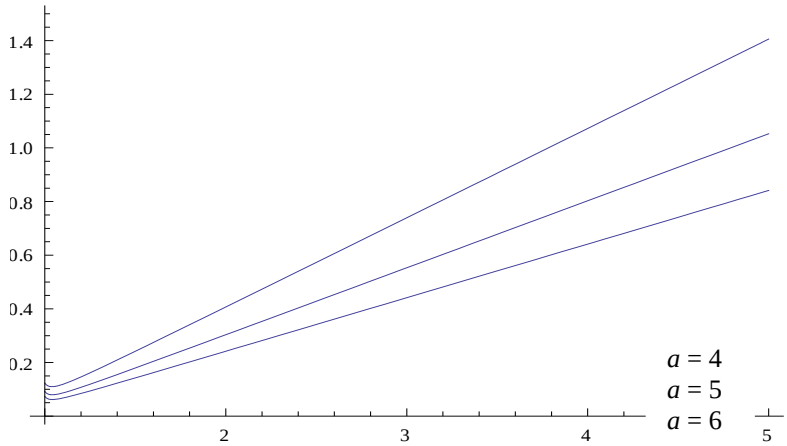

Figure 1. The impact of parameter *a* on the *MRL* of the constrained (*k*, *d*)-out-of-*n*: *F* system.

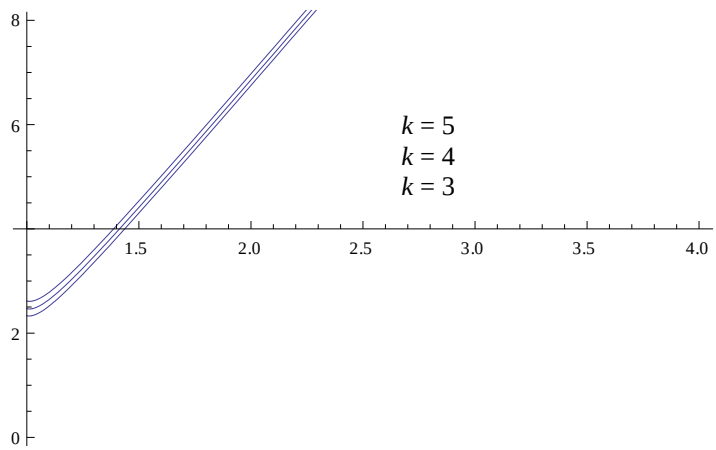

Figure 2. The impact of parameter *k* on the *MRL* of the constrained (*k*, *d*)-out-of-*n*: *F* system.

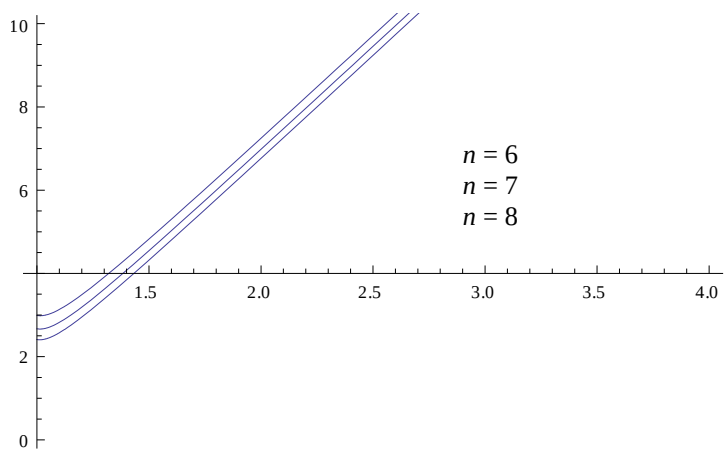

Figure 3. The impact of parameter *n* on the *MRL* of the *C*(*k*,*d*,*n*) model.

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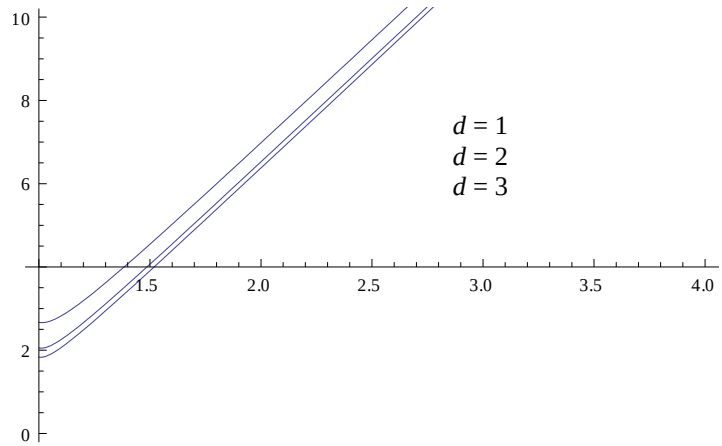

Figure 4. The impact of parameter *d* on the *MRL* of *C*(*k*,*d*,*n*) model.

It is noteworthy that, as it can be readily confirmed, the *MRL* function of the *C*(*k*,*d*,*n*) model under the multivariate Pareto model satisfies the following monotonicity attributes:

- is dropping off in regard to *n* (for a specified group of t, d, k, a) (see Figure 3).
- is dropping off in regard to *d* (for a specified group of t, n, k, a) (see Figure 4).
- is dropping off in regard to *t* (for a specified group of n, d, k, a).
- is dropping off in regard to *k* (for a specified group of t, d, n, a) (see Figure 2).
- is dropping off in regard to *a* (for a specified group of t, d, k, n) (see Figure 1).

4. Conclusion

In the present manuscript, the constrained (*k*, *d*)–out–of–*n*: *F* structure consisting of exchangeable components has been investigated. The main contribution of the paper refers to the development of a new algorithmic procedure for determining several reliability characteristics of the underlying system, such as its signature, mean residual lifetime or conditional mean residual lifetime. The figures and numerical results which are illustrated reveal the impact of the design parameters *n*, *k* and *d* on the performance of the constrained (*k*, *d*)–out–of–*n*: *F* system. The reliability study of alternative consecutive-type structures based on the proposed algorithm could be an interesting topic for future research.

Conflict of Interest

The author confirms that there is no conflict of interest to declare for this publication.

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