

## Particle Swarm Optimization and Maximum Entropy Results for $M^X/G/1$ Retrial G-Queue with Delayed Repair

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### Abstract

This paper examines an  $M^X/G/1$  retrial model with negative customers including the concepts of working vacation, Bernoulli feedback, delayed repair, state-dependent and multi-optional services. Such a queue is quite relevant in real world, for instance in computer systems, manufacturing organisations, packet-switching networks, telecommunication systems, etc. The arrival pattern of customers is according to Poisson distribution. The service is such that first essential service (FES) is provided to every customer and second optional service (SOS) is provided in  $k$  phases to those who wants to opt for the same. The negative customers may arrive during the time when the server is busy in serving a positive customer. This leads to breakdown of the server and thus the server has to be restored (repair) by the repair man. Some moments may be taken by the repair man to initiate the repair process, leading to delay in repair. In our work, firstly we have calculated performance measures like long run probabilities and orbit size along with some reliability indices. Then a relative study between the exact expected waiting time and approximate expected waiting time of the system is presented via maximum entropy approach. Also we perform cost optimization using particle swarm optimization (PSO method). Few numerical results are also provided.

**Keywords-** Retrial, Multi-optional service, Queue length, Maximum entropy principle, Particle swarm optimisation.

### 1. Introduction

This study deals with bulk arrival retrial G-queue where the server is subjected to state dependent and multi-optional services. In G-queues, a negative customer arrives only when server is occupied with a positive customer and forces it to leave the system thereby causing an interruption in the service. A very common example is virus attacking a system. Banks, hospitals and shopping malls are some of the places where such queues can be observed. The concept was first given by Gelenbe (1991). Yang et al. (2013) have provided reliability measures and queueing measures for a batch arrival G-queue under general retrial times, breakdown and single vacation policy. Recently, Upadhyaya (2020b) investigated a single server retrial G-queue with second phase service by applying supplementary variable technique and also discussed its application to local area networks.

Our research also involves *recurrent (retrial) queues* wherein if the service provider is unavailable at the time of incoming customer then the latter enters a pond of waiting customers called ‘orbit’ and retries repeatedly to obtain that service. Such a situation is quite evident in our day to day life where blocked customers temporarily departs from the service zone and retries for the same service after an arbitrary interval of time. For instance, in computer networking if a message or packet is lost then later on it is retransmitted via Transmission Control Protocol (TCP). From past few decades, a lot of research has been done and is still going on in retrial queues. Authors like Artelajo (1994) and many more have contributed a lot in this field. Upadhyaya (2014) and Rajadurai et al. (2018) both have calculated all the beneficial performance indices for single server retrial system by applying supplementary variable technique. The former has considered a batch arrival pattern following modified vacation policy while the latter has included the concept of server breakdown, repair and multiple vacation policy. Later on, Upadhyaya (2018) has calculated all the necessary results for a non-continuous retrial model with geometric arrival following J-vacation policy.

In this model, the service is provided in two stages which include *first essential service (FES)* and *second optional service (SOS)*. The SOS is in  $k$  stages which depend on the customer to demand for it or not. Further, whenever the server is free, it goes on working vacation. In working vacation state, server does not stop service completely rather it provides service at a slower rate. However, in this working vacation phase if customers are present at any service fulfilment, then it may immediately stop that phase and return to normal busy state i.e. there can be a possibility of vacation interruption. Sethi et al. (2019) have worked on an M/M/1 model under working vacation and further performed a comparative analysis of their results via Runge-Kutta method and ANFIS (Adaptive Neuro Fuzzy Interface System). The research done by Jain et al. (2019) and Jain and Meena (2020) on systems with vacation and server breakdown is worth mentioning.

The main motive of our investigation is to obtain maximum entropy results and present a relative study between the exact expected waiting time and approximate expected waiting time of the system. Jain and Upadhyaya (2012) have analysed a bulk retrial queue under N-policy using maximum entropy principle (MEP). Pandey and Tripathi (2014) derived approximate waiting time of a model with multi-optional services and made a comparison of maximum entropy results with the exact results. In recent years, Parkash and Mukesh (2016) have depicted the relation between this method and queueing theory in a very detailed and explanatory manner. A comparative study by applying MEP on the results obtained and exact results of  $M^X/(G_1, G_2)/1$  system with two stages of services has been presented by Chauhan (2018).

As per the literature survey, no work has been done on finding maximum entropy results and optimal parameters for an  $M^X/G/1$  retrial G-queue with multi-optional and state dependent service under Bernoulli feedback, working vacations as well as delayed repair. This work is very much relevant and helpful in the areas with high congestion and traffic intensity. We have also carried out cost analysis of the considered model by implementing PSO. This technique has become quite renowned in last few years due to the fact that the candidate solution gets better iteratively and thus the system provides us the best solution available. Whether we have to maximize performance indices or minimize the total cost, this method can be applicable in queueing analysis for obtaining fruitful results. Work done by Zhou et al. (2018) and Wang et al. (2019) in applying particle swarm optimization is worthy of praise. Former has investigated equilibrium strategies for constant retrial queue while the latter has worked on machine repair problem with retrial phenomenon and working breakdown. Recently authors like Sengupta et al. (2019) and Upadhyaya (2020a) have also worked on this technique.

This paper is divided in the following manner. Details of model and all symbols used are described in Section 2. Then the results obtained are given in Section 3. Following this Section 4 provides detailed analysis of ‘maximum entropy principle’. Section 5 refers to analyze the system and obtain the minimum cost. All numerical results obtained are given in Section 6. Finally, conclusion and future possibility has been highlighted in Section 7.

## 2. Explanation of the Model

This work involves a single server system where customers come in batches and server provides multi-optional service and undergo delayed repair. The total customers present at times  $t$  is denoted by  $N(t)$ . The detailed description is as follows:

Arrival pattern- Arrival occurs in batches with respect to compound Poisson process and it is dependent upon the server as given below

- $\lambda_0$  - for server being idle either during working vacation or during regular service mode.
- $\lambda_1$  - for server being occupied with FES or SOS at time  $t$ .
- $\lambda_2$  - for server being occupied and on working vacation at time  $t$ .
- $\lambda_3$  - for server waiting for repair or under repair.

A random variable  $X$  is defined to represent the batch size and  $P(X=k) = C_k$  is the probability distribution of batch size with  $\sum_{k=1}^{\infty} C_k = 1$ . Also we consider the random variable  $\delta(t)$  defined as

$$\delta(t) = \begin{cases} 0, & \text{when server is free and on working vacation} \\ 1, & \text{when server is free and in regular service} \\ 2, & \text{when server is occupied with first essential service at time } t \\ 3, & \text{when server is busy with second optional service at time } t \\ 4, & \text{when server is busy and on working vacation} \\ 5, & \text{when server is waiting for repair} \\ 6, & \text{when server is undergoing repair} \end{cases}$$

Retrial Process – When server is busy or breakdown or under repair/delayed repair state, the customer joins a virtual pond known as *orbit*. From the orbit, the customers keep trying for service until they get it. We assume that only the customer at the head of the queue is permitted to access the server. Further, let the inter-retrial time follows a general random variable  $S$  with distribution  $S(t)$  and its Laplace Stieltjes Transform (LST)  $S^*(\theta)$ .

Service Process - Service is assumed to be multi-optional. The service time for first essential service (FES) obeys a general distribution with random variable  $B_0$  and distribution function  $B_0(t)$ . Also, the service time for second optional service (SOS) obeys a general distribution with random variable  $B_i$  and distribution function  $B_i(t)$  ( $1 \leq i \leq k$ ). The corresponding LST for FES and SOS are  $B_0^*(\theta)$  and  $B_i^*(\theta)$ ,  $1 \leq i \leq k$  respectively. Also, we have taken  $m_i$  ( $0 \leq i \leq k$ ) as the probability that the customer opts for  $i^{th}$  service.

Working vacation process - If the system is free of customers the server leaves for a working vacation. In such type of vacation, the server does not terminate the service completely rather it serves with a comparatively lower service rate. This period obeys an exponential distribution with

parameter  $\eta$ . On completion of a working vacation period if server observes that the system is vacant then either it remains idle for a new customer with probability  $p$  or it continues its vacation with probability  $q(=1-p)$ . But if customers are there at the end of a working vacation then vacation interruption may take place wherein the server immediately returns to normal busy mode. While being on working vacation, the server obeys a general distribution with random variable  $B_v$  and distribution function  $B_v(t)$ . The corresponding LST is  $B_v^*(\vartheta)$  and  $b_w$  and  $b_w^2$  are the first and second moments respectively.

**Repair and Delayed Repair Process** - The entrance of a negative customer during the service of a positive customer leads to a server breakdown. The arrival rate of such customers is  $\theta$ . After a breakdown the server needs to get repaired. It may be possible that repairman takes time to initiate the repair. This is known as delayed repair time. Let the repair time obeys a general distribution with random variable  $G_0$  for FES and  $G_i$  for SOS. The corresponding distribution function is  $G_0(t)$  and  $G_i(t)$ . So the LST will be  $G_0^*(\vartheta)$  and  $G_i^*(\vartheta)$ . On similar pattern the delayed repair time obeys a general distribution  $H_0(t)$  and  $H_i(t)$  with random variable  $H_0$  and  $H_i$ . Thus, the LST for delayed repair function becomes  $H_0^*(\vartheta)$  and  $H_i^*(\vartheta)$ , where  $1 \leq i \leq k$ . The  $r^{th}$  moment for delayed repair at FES is  $h_0^r$  and for SOS it is  $h_i^r$ . Similarly, the  $r^{th}$  moment for repair state at FES is  $g_0^r$  and for SOS it is  $g_i^r$ ,  $1 \leq i \leq k$

**Bernoulli Feedback** - We have also considered the concept of Bernoulli Feedback. If the customer is dissatisfied by the service due to some issue, then it may join the queue again for availing the same service. Let the probability that a customer goes for feedback be  $s$  and will exit the system once it is served be  $r$ , where  $s=1-r$ .

The hazard rates for retrial, busy (with FES and SOS), working vacation, repair and delayed repair state of the server respectively are as follows:

$$\alpha(x) = \frac{dS(x)}{1-S(x)}; \quad \beta_0(x) = \frac{dB_0(x)}{1-B_0(x)}; \quad \beta_i(x) = \frac{dB_i(x)}{1-B_i(x)}, 1 \leq i \leq k$$

$$\beta_v(x) = \frac{dB_v(x)}{1-B_v(x)}; \quad \xi_0(x) = \frac{dG_0(x)}{1-G_0(x)}; \quad \xi_i(x) = \frac{dG_i(x)}{1-G_i(x)}, 1 \leq i \leq k$$

$$\zeta_0(x) = \frac{dH_0(x)}{1-H_0(x)}; \quad \zeta_i(x) = \frac{dH_i(x)}{1-H_i(x)}, 1 \leq i \leq k.$$

Let us denote the elapsed retrial time, elapsed service time (for FES and SOS), elapsed working vacation time, elapsed repair time (during FES and SOS) and elapsed delayed repair time (during FES and SOS) respectively by  $S^o(t)$ ,  $B^o(t)$ ,  $B_i^o(t)$ ,  $B_v^o(t)$ ,  $G^o(t)$ ,  $G_i^o(t)$ ,  $H^o(t)$  and  $H_i^o(t)$ . These supplementary variables are introduced to achieve a *bivariate Markov process*  $\{\hat{o}(t), N(t); t \geq 0\}$ , where  $\hat{o}(t)$  is the state of the server  $\{0, 1, 2, 3, 4, 5, 6\}$  as described above.

Some other assumptions of the model are

$$C'(1) = E[X] = C_1; \quad C''(1) = E[X^2] = C_2; \quad C'''(1) = E[X^3] = C_3.$$

The *limiting probabilities* of different state of the server are described by

$Q_0(t) = P\{\delta(t) = 0, N(t) = 0\}$  and  $P_0(t) = P\{\delta(t) = 1, N(t) = 0\}$ . Let  $Q_0 = \lim_{t \rightarrow \infty} Q_0(t)$  and  $P_0 = \lim_{t \rightarrow \infty} P_0(t)$ .

## 2.1 Practical Application

This study is quite applicable in telecommunication sector, for instance, we consider a communication system which is built for making restaurant reservations. Let us assume that a restaurant takes its bookings and offer various other services via telephone system, where a customer can call and book a table for itself (positive customer). This telephone system is staffed with a manager (single server) who answers all the phone calls. After booking a table (completion of first essential service) the customer may hang up the phone call (exit the system) or it may enquire about some party bookings, book tickets for an upcoming musical concert, etc. (optional services). There may be a miscommunication due to the lack of clarity in network or any other such issues and a person has to call again to confirm his bookings (Bernoulli feedback). Whenever the manager is busy in supervising other parts of the restaurant then he is unable to answer the calls (on vacation mode) and in such situation the assistant manager (substitute server) serves usually but comparatively at a slower pace (working vacation). During this phase, once phone call is completed (at the service completion) if there are calls in the system then the manager returns immediately (i.e. vacation interruption occurs). However, if there are no calls once the manager has finished his secondary job (vacation is over) then manager continues to handle other issues of the restaurant (vacation mode). Chances are there that the line is busy when a customer makes a call, then customer calls back after a certain period of time (retrial). There is a possibility that during a phone call, signal may not be appropriate or low network coverage or virus attack (negative customers) due to which customer loses his service. The telephone line then has to get repaired and the repairman may take time to initiate the repair process (delayed repair). Once the signal repair is completed, the communication system works as good as new. There is also a chance that enormous number of calls arrives on a weekend then either manager works more efficiently or the line may get jam (state dependent condition). This scenario completely matches with our model in consideration. Thus, all the results and analysis presented in our work will be very profitable for the system manager. Numerical illustration for the discussed example is given in section 6.

This model is also applicable in the field of packet switching network, Simple Mail Transfer Protocol (SMTP), call centres, etc.

## 3. Performance Measures

For completely analysing a queueing system we must know how well it can perform. This can be done by obtaining the performance indices of our interest. A queueing system with no waiting line or queue length, is the most ideal one, however it is not practical to form such model. Therefore, it is important to predict average number of customers and their delay time which has a great effect on whether a customer will join a queue or not. Also, the probability of the server in idle state, busy state or any other different state is another essential performance measure. If all these fruitful results are available, then it is beneficial for the system designers and for the arriving customers as well. On one hand, the system managers/designers can make the system better by predicting the service time, repair time, etc. whereas on the other hand, customers are at an ease if they are aware of the average waiting time in the queue. This particular section is devoted to all the performance indices calculated. Very firstly, we have provided the *marginal generating functions* of different states of the server.

**Lemma 1**

The *marginal probability generating functions* for retrial period, busy period with FES and SOS, working vacation period, delayed repair period under FES and SOS and repair period under FES and SOS are respectively given by

$$\begin{aligned}
 P_R(z) &= \frac{Nr(z)}{Dr(z)} \frac{(1-S^*(\lambda_0))}{\lambda_0} ; \pi_o(z) = \frac{Nr_1(z)}{Dr(z)} \frac{(1-B_o^*(A_1(z)))}{A_1(z)} ; \\
 \pi_i(z) &= \frac{Nr_1(z)}{Dr(z)} m_i B_o^*(A_1(z)) \frac{(1-B_i^*(A_1(z)))}{A_1(z)} , 1 \leq i \leq k ; Q_v(z) = \frac{\lambda_0 Q_0 V(z)}{\eta} \\
 D_0(z) &= \frac{Nr_1(z)}{Dr(z)} \frac{\theta}{z} \frac{(1-B_0^*(A_1(z))) (1-H_0^*(A_3(z)))}{A_1(z) A_3(z)} \\
 D_i(z) &= \frac{Nr_1(z)}{Dr(z)} m_i B_o^*(A_1(z)) \frac{\theta}{z} \frac{(1-B_i^*(A_1(z))) (1-H_i^*(A_3(z)))}{A_1(z) A_3(z)} , 1 \leq i \leq k \\
 R_0(z) &= \frac{Nr_1(z)}{Dr(z)} \frac{\theta}{z} H_0^*(A_3(z)) \frac{(1-B_0^*(A_1(z))) (1-G_0^*(A_3(z)))}{A_1(z) A_3(z)} \\
 R_i(z) &= \frac{Nr_1(z)}{Dr(z)} m_i B_o^*(A_1(z)) H_i^*(A_3(z)) \frac{\theta}{z} \frac{(1-B_i^*(A_1(z))) (1-G_i^*(A_3(z)))}{A_1(z) A_3(z)} , 1 \leq i \leq k
 \end{aligned}$$

where,

$$\begin{aligned}
 Nr(z) &= zQ_0 \{ \lambda_0 (B_v^*(A_2(z)) - 1) - \eta p \} + (\lambda_0 V(z) + \eta p) [(s + zr) B_0^*(A_1(z)) (m_0 + \sum_{i=1}^k m_i B_i^*(A_1(z))) \\
 &\quad + \frac{\theta}{z} (F_0(z) H_0^*(A_3(z)) G_0^*(A_3(z)) + B_0^*(A_1(z)) \sum_{i=1}^k m_i F_i(z) H_i^*(A_3(z)) G_i^*(A_3(z)))] ;
 \end{aligned}$$

$$\begin{aligned}
 Dr(z) &= \{ z - (S^*(\lambda_0) + z(1 - S^*(\lambda_0))) ((s + zr) B_0^*(A_1(z)) (m_0 + \sum_{i=1}^k m_i B_i^*(A_1(z))) \\
 &\quad + \frac{\theta}{z} (F_0(z) H_0^*(A_3(z)) G_0^*(A_3(z)) + B_0^*(A_1(z)) \sum_{i=1}^k m_i F_i(z) H_i^*(A_3(z)) G_i^*(A_3(z)))) \} ;
 \end{aligned}$$

$$Nr_1(z) = Q_0 \{ (\lambda_0 (B_v^*(A_2(z)) - 1) - \eta p) (S^*(\lambda_0) + z(1 - S^*(\lambda_0))) + z(\lambda_0 V(z) + \eta p) \} ;$$

$$F_i(z) = \frac{1 - B_i^*(A_1(z))}{A_1(z)} , 0 \leq i \leq k \quad \text{and} \quad V(z) = \frac{(1 - B_v^*(A_2(z))) \eta}{\eta + \lambda_2 (1 - C(z))} .$$

$$A_1(z) = \lambda_1 (1 - C(z)) + \theta ; \quad A_2(z) = \lambda_2 (1 - C(z)) + \eta ; \quad A_3(z) = \lambda_3 (1 - C(z)) .$$

### 3.1 Queuing Measures

In this subsection, we aim to find the long run probabilities of the state of the server as well as the queue size and waiting time of the customer which is a great source of information for making a model more sufficient and economically friendly. The results are presented in the following two theorems as follows:

#### Theorem 1

Chances that server is idle during retrial time  $P(P_R)$ , server is occupied with FES  $P(\pi_0)$ , with SOS  $P(\pi_i)$ , server is on working vacation  $P(Q_v)$ , server is under delayed repair while providing FES  $P(D_0)$ , while providing SOS  $P(D_i)$  and server is under repair while providing FES  $P(R_0)$  and while providing SOS  $P(R_i)$  (for  $1 \leq i \leq k$ ) are respectively given by

$$P(P_R) = \lim_{z \rightarrow 1} P_R(z) = \frac{N}{D} \quad (1)$$

$$P(\pi_0) = \lim_{z \rightarrow 1} \pi_0(z) = Q_0 \frac{(1 - B_0^*(\theta)) N_1}{\theta D} \quad (2a)$$

$$P(\pi_i) = \lim_{z \rightarrow 1} \pi_i(z) = Q_0 m_i B_0^*(\theta) \frac{(1 - B_i^*(\theta)) N_1}{\theta D}, \quad 1 \leq i \leq k \quad (2b)$$

$$P(Q_v) = \lim_{z \rightarrow 1} Q_v(z) = \frac{\lambda_0 Q_0}{\eta} (1 - B_v^*(\eta)) \quad (3)$$

$$P(D_0) = \lim_{z \rightarrow 1} D_0(z) = Q_0 h_0^{-1} (1 - B_0^*(\theta)) \frac{N_1}{D} \quad (4a)$$

$$P(D_i) = \lim_{z \rightarrow 1} D_i(z) = Q_0 m_i h_i^{-1} (1 - B_i^*(\theta)) B_0^*(\theta) \frac{N_1}{D}, \quad 1 \leq i \leq k \quad (4b)$$

$$P(R_0) = \lim_{z \rightarrow 1} R_0(z) = Q_0 g_0^{-1} (1 - B_0^*(\theta)) \frac{N_1}{D} \quad (5a)$$

$$P(R_i) = \lim_{z \rightarrow 1} R_i(z) = Q_0 m_i g_i^{-1} (1 - B_i^*(\theta)) B_0^*(\theta) \frac{N_1}{D}, \quad 1 \leq i \leq k \quad (5b)$$

where,

$$N = Q_0 \frac{(1 - S^*(\lambda_0))}{\lambda_0} \left[ \frac{\lambda_0 \lambda_2 C_1}{\theta} (1 - B_v^*(\eta)) + (\lambda_0 (1 - B_v^*(\eta))) \{ r B_0^*(\theta) (m_0 + \sum_{i=1}^k m_i B_i^*(\theta)) \right. \\ \left. - (1 - B_0^*(\theta)) (1 - \frac{\lambda_1 C_1}{\theta} + \lambda_3 C_1 (h_0^{-1} + g_0^{-1})) - B_0^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta)) (1 - \frac{\lambda_1 C_1}{\theta} + \lambda_3 C_1 (h_i^{-1} + g_i^{-1})) \} \right];$$

$$N_1 = \frac{\lambda_0 \lambda_2 C_1}{\theta} (1 - B_v^*(\theta)) + S^*(\lambda_0) (\lambda_0 (1 - B_v^*(\eta)) + \eta p); \quad Q_0 = \frac{\lambda_0 \theta D}{D_1};$$



$$D = S^*(\lambda_0) - rB_0^*(\theta)(m_0 + \sum_{i=1}^k m_i S_i^*(\theta)) + (1 - B_0^*(\theta))(1 - \frac{\lambda_1 C_1}{\theta} + \lambda_3 C_1 (h_0^1 + g_0^1) + B_0^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta))(1 - \frac{\lambda_1 C_1}{\theta} + \lambda_3 C_1 (h_i^1 + g_i^1)));$$

$$D_1 = \lambda_0 \eta N_1 \left( 1 + (1 - B_0^*(\theta)(h_0^1 + g_0^1)) + B_0^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta)) \left( \frac{1}{\theta} + h_i^1 + g_i^1 \right) + \theta S^*(\lambda_0)(\eta p + \lambda_0) - \lambda_0 (B_v^*(\eta) - 1)(\lambda_2 C_1 (1 - S^*(\lambda_0)) + \lambda_0 D) - \eta (D - S^*(\lambda_0))(\eta p S^*(\lambda_0) + \lambda_0 S^*(\lambda_0)(B_v^*(\eta) - 1) - B_v^*(\eta)) \right).$$

**Theorem 2**

The average queue length  $L_q$  i.e. the total number of customers waiting in the orbit and the *exact expected waiting time* is respectively given by

$$L_q = Q_0 \left[ \frac{Num'''(1)Dem''(1) - Dem'''(1)Num''(1)}{3(Dem''(1))^2} \right] + Q_0 \left[ \frac{Num_1'''(1)Dem_1''(1) - Dem_1'''(1)Num_1''(1)}{3(Dem_1''(1))^2} \right] + Q_0 \left[ \frac{Num_2'''(1)Dem_2''(1) - Dem_2'''(1)Num_2''(1)}{3(Dem_2''(1))^2} \right] + \frac{Q_0 \lambda_0 V'(1)}{\eta} + Q_0 \left[ \frac{Num_3'''(1)Dem_3''(1) - Dem_3'''(1)Num_3''(1)}{3(Dem_3''(1))^2} \right] + Q_0 \left[ \frac{Num_4'''(1)Dem_4''(1) - Dem_4'''(1)Num_4''(1)}{3(Dem_4''(1))^2} \right] + Q_0 \left[ \frac{Num_5'''(1)Dem_5''(1) - Dem_5'''(1)Num_5''(1)}{3(Dem_5''(1))^2} \right] + Q_0 \left[ \frac{Num_6'''(1)Dem_6''(1) - Dem_6'''(1)Num_6''(1)}{3(Dem_6''(1))^2} \right]$$

$$W_s = \frac{L_q}{\lambda_{eff}} \tag{6}$$

where,

$$Num''(1) = e'' + 2f'\omega + f'' + \omega_1 ; \quad Num'''(1) = e''' + 3f'\omega + 3f''\omega_1 + f''' + f\psi ;$$

$$Dem''(1) = -2\omega(1 - S^*(\lambda_0)) - \omega_1 ; \quad Dem'''(1) = -3\omega_1(1 - S^*(\lambda_0)) - \psi ;$$

$$Num_1''(1) = j_0''\rho + 2j_0'\rho_1 + j_0\rho_2 ; \quad Num_1'''(1) = j_0'''\rho + 3j_0''\rho_1 + 3j_0'\rho_2 + j_0\rho_3 ;$$

$$Dem_1''(1) = -2\lambda_1 C_1 Dem''(1) + \theta Dem''(1) ;$$

$$Dem_1'''(1) = -3\lambda_1 C_2 Dem''(1) + -3\lambda_1 C_1 Dem'''(1) + \delta Dem'''(1) ;$$

$$Num_2''(1) = m_i [B_0^*(\theta) \{ j_i''\rho + 2j_i'\rho_1 + j_i\rho_2 \} - B_0^*(\theta) \{ 2\lambda_1 C_1 j_i'\rho + \lambda_1 C_2 j_i\rho_1 + 2\lambda_1 C_1 j_i\rho_1 \} + \lambda_1^2 C_1^2 j_i\rho B_0^*(\theta)], 1 \leq i \leq k;$$



$$\begin{aligned} Num_2'''(1) = & m_i [B_0^*(\theta) \{j_i'''\rho + 3j_i''\rho_1 + 3j_i'\rho_2 + j_i\rho_3\} - B_0^{**}(\theta) \{3\lambda_1 C_1 j_i''\rho + 3\lambda_1 C_2 j_i'\rho \\ & + 6\lambda_1 C_1 j_i'\rho_1 + \lambda_1 C_2 j_i\rho_1 + 3\lambda_1 C_1 j_i\rho_2\} + B_0^{***}(\theta) \{3\lambda_1^2 C_1^2 j_i''\rho + 2\lambda_1^2 C_1 C_2 j_i\rho \\ & + 3\lambda_1^2 C_1^2 j_i\rho - \lambda_1 C_2 j_i\rho - \lambda_1 C_3 j_i\rho\} + \lambda_1^2 C_1^2 j_i\rho B_0^{****}(\theta)], 1 \leq i \leq k; \end{aligned}$$

$$Num_3''(1) = \theta(\varphi_0'' Num_1(1) + 2\varphi_0' Num_1'(1));$$

$$Num_3'''(1) = \theta(\varphi_0''' Num_1(1) + 3\varphi_0'' Num_1'(1) + 3\varphi_0' Num_1''(1));$$

$$Dem_2''(1) = -2\lambda_3 C_1 Dem_1'(1);$$

$$Dem_2'''(1) = -6\lambda_3 C_1 Dem_1'(1) - 3\lambda_3 C_2 Dem_1'(1) - 3\lambda_3 C_1 Dem_1''(1);$$

$$Num_4''(1) = \theta(\varphi_i'' Num_2(1) + 2\varphi_i' Num_2'(1)), 1 \leq i \leq k ;$$

$$Num_4'''(1) = \theta(\varphi_i''' Num_2(1) + 3\varphi_i'' Num_2'(1) + 3\varphi_i' Num_2''(1)), 1 \leq i \leq k ;$$

$$Num_5''(1) = \theta(2\lambda_3 C_1 h_0^1 \tau_0' Num_1(1) + \tau_0'' Num_1(1) + 2\tau_0' Num_1'(1)) ;$$

$$\begin{aligned} Num_5'''(1) = & \theta(3(-\lambda_3^2 C_1^2 h_0^2 + \lambda_3 C_2 h_0^1) \tau_0' Num_1(1) + 3\lambda_3 C_1 h_0^1 \tau_0'' Num_1(1) + 6\lambda_3 C_1 h_0^1 \tau_0' Num_1'(1) \\ & + \tau_0''' Num_1(1) + 3\tau_0'' Num_1'(1) + 3\tau_0' Num_1''(1)); \end{aligned}$$

$$Num_6''(1) = \theta(2\lambda_3 C_1 h_i^1 \tau_i' Num_3(1) + \tau_i'' Num_3(1) + 2\tau_i' Num_3'(1)), 1 \leq i \leq k ;$$

$$\begin{aligned} Num_6'''(1) = & \theta(3(-\lambda_3^2 C_1^2 h_i^2 + \lambda_3 C_2 h_i^1) \tau_i' Num_3(1) + 3\lambda_3 C_1 h_i^1 \tau_i'' Num_3(1) \\ & + 6\lambda_3 C_1 h_i^1 \tau_i' Num_3'(1) + \tau_i''' Num_3(1) + 3\tau_i'' Num_3'(1) + 3\tau_i' Num_3''(1)), 1 \leq i \leq k; \end{aligned}$$

$$\lambda_{\text{eff}} = \lambda_o (P_o + Q_o + P(P_R)) + \lambda_1 (P(\pi_o) + \sum_{i=1}^k P(\pi_i)) + \lambda_2 P(Q_v) + \lambda_3 ((P(R_o) + \sum_{i=1}^k P(R_i) + P(D_o) + \sum_{i=1}^k P(D_i))).$$

All the above symbols used are given in appendix A.

Proof: The *PGF* of the total customers present in the orbit is given by

$$K_o(z) = P_o + Q_o + P_R(z) + \pi_o(z) + \sum_{i=1}^k \pi_i(z) + Q_v(z) + R_o(z) + \sum_{i=1}^k R_i(z) + D_o(z) + \sum_{i=1}^k D_i(z);$$

$$L_q = \lim_{z \rightarrow 1} K_o'(z).$$

Solving the above equation by using *L' Hospital's* rule we obtain our desired results.

### 3.2 Reliability Measures

In this system, the server is unreliable so reliability measures provide the necessary information that is required to improve the model. We have obtained the availability measure (AV) which is the probability that the server is either rendering service to a positive customer or it is in idle period. We have also derived the expression for the failure frequency (FF) of the server.

**Lemma 2**

The steady state availability of the server (AV) is

$$AV = 1 - (P(R_o) + \sum_{i=1}^k P(R_i))$$

$$= 1 - \left[ Q_0 g_0^{-1} (1 - B_0^*(\theta)) \frac{N_1}{D} + \sum_{i=1}^k Q_0 m_i g_i^{-1} (1 - B_i^*(\theta)) B_0^*(\theta) \frac{N_1}{D} \right].$$

The failure frequency, (FF) is

$$FF = \theta (P(\pi_o) + \sum_{i=1}^k P(\pi_i))$$

$$= \theta \left[ Q_0 \frac{(1 - B_0^*(\theta)) N_1}{\theta D} + \sum_{i=1}^k Q_0 m_i B_0^*(\theta) \frac{(1 - B_i^*(\theta)) N_1}{\theta D} \right].$$

**4. Maximum Entropy Approach**

Maximum entropy approach was originated by Jaynes (1957). ‘Entropy’ basically refers to the variation of a system or probabilistic distribution. Many a times in probabilistic analysis it happens that we are unable to evaluate an appropriate distribution. In queueing systems, the information available for the queue size, steady state probabilities, etc. can be considered as constraints. However, there is a large variety of distributions satisfying them. In this method, we obtain the best distribution that suits them using the available information. The probability distribution that maximizes the entropy function is the optimal one.

**4.1 Construction of Entropy Function**

Let  $P_1(n)$ ,  $P_2(n)$ ,  $P_{2i}(n)$ ,  $P_3(n)$ ,  $P_4(n)$ ,  $P_{4i}(n)$ ,  $P_5(n)$  and  $P_{5i}(n)$  ( $1 \leq i \leq k$ ),  $n$  being the total number of customers there in the system, be the probabilities that server is in retrial period, busy period, working vacation period, delayed repair and repair period respectively.

Using the work done by El-Affendi and Kouvatso (1983), we express the *entropy function*  $Y$  as

$$Y = - \sum_{n=1}^{\infty} P_1(n) \log P_1(n) - \sum_{n=1}^{\infty} P_2(n) \log P_2(n) - \sum_{n=1}^{\infty} \sum_{i=1}^k P_{2i}(n) \log P_{2i}(n) - \sum_{n=1}^{\infty} P_3(n) \log P_3(n)$$

$$- \sum_{n=1}^{\infty} P_4(n) \log P_4(n) - \sum_{n=1}^{\infty} \sum_{i=1}^k P_{4i}(n) \log P_{4i}(n) - \sum_{n=1}^{\infty} P_5(n) \log P_5(n) - \sum_{n=1}^{\infty} \sum_{i=1}^k P_{5i}(n) \log P_{5i}(n)$$
(7)

We will find the steady state probabilities using the following set of constraints:

$$\sum_{n=1}^{\infty} P_1(n) = E_1$$
(8)

$$\sum_{n=1}^{\infty} P_2(n) = E_2; \sum_{n=1}^{\infty} P_{2i}(n) = E_{2i}, 1 \leq i \leq k \quad (9)$$

$$\sum_{n=1}^{\infty} P_3(n) = E_3 \quad (10)$$

$$\sum_{n=1}^{\infty} P_4(n) = E_4; \sum_{n=1}^{\infty} P_{4i}(n) = E_{4i}, 1 \leq i \leq k \quad (11)$$

$$\sum_{n=1}^{\infty} P_5(n) = E_5; \sum_{n=1}^{\infty} P_{5i}(n) = E_{5i}, 1 \leq i \leq k \quad (12)$$

$$\sum_{n=1}^{\infty} n(P_1(n) + P_2(n) + \sum_{i=1}^k P_{2i}(n) + P_3(n) + P_4(n) + \sum_{i=1}^k P_{4i}(n) + P_5(n) + \sum_{i=1}^k P_{5i}(n)) = E_6 \quad (13)$$

where,

$$E_1 = P(P_R), E_2 = P(\pi_o), E_{2i} = P(\pi_i), E_3 = P(Q_v), E_4 = P(D_o), E_{4i} = P(D_i), E_5 = P(R_o), \\ E_{5i} = P(R_i), E_6 = L_q, 1 \leq i \leq k.$$

Now, we form Lagrange's function,  $J(P_1(n), P_{2i}(n), P_3(n), P_{4i}(n), P_{5i}(n))$  to maximize the entropy function subject to constraints (8) - (13). For this let us introduce Lagrange's multipliers  $\theta_1, \theta_2, \theta_{2i}, \theta_3, \theta_4, \theta_{4i}, \theta_5, \theta_{5i}$  and  $\theta_6$  where  $1 \leq i \leq k$ .

$$J(P_1(n), P_2(n), P_{2i}(n), P_3(n), P_4(n), P_{4i}(n), P_5(n), P_{5i}(n)) = -\sum_{n=1}^{\infty} P_1(n) \log P_1(n) - \sum_{n=1}^{\infty} P_2(n) \log P_2(n) \\ - \sum_{n=1}^{\infty} \sum_{i=1}^k P_{2i}(n) \log P_{2i}(n) - \sum_{n=1}^{\infty} P_3(n) \log P_3(n) - \sum_{n=1}^{\infty} P_4(n) \log P_4(n) - \sum_{n=1}^{\infty} \sum_{i=1}^k P_{4i}(n) \log P_{4i}(n) \\ - \sum_{n=1}^{\infty} P_5(n) \log P_5(n) - \sum_{n=1}^{\infty} \sum_{i=1}^k P_{5i}(n) \log P_{5i}(n) - \theta_1 (\sum_{n=1}^{\infty} P_1(n) - E_1) - \theta_2 (\sum_{n=1}^{\infty} P_2(n) - E_2) \\ \sum_{i=1}^k \theta_{2i} (\sum_{n=1}^{\infty} P_{2i}(n) - E_{2i}) - \theta_3 (\sum_{n=1}^{\infty} P_3(n) - Q_v(1)) - \theta_4 (\sum_{n=1}^{\infty} P_4(n) - E_4) - \sum_{i=0}^k \theta_{4i} (\sum_{n=1}^{\infty} P_{4i}(n) - E_{4i}) \\ - \theta_5 (\sum_{n=1}^{\infty} P_5(n) - E_5) - \sum_{i=0}^k \theta_{5i} (\sum_{n=1}^{\infty} P_{5i}(n) - E_{5i}) - \theta_6 (\sum_{n=1}^{\infty} n(P_1(n) + P_2(n) + \sum_{i=1}^k P_{2i}(n) + P_3(n) \\ + P_4(n) + \sum_{i=1}^k P_{4i}(n) + P_5(n) + \sum_{i=1}^k P_{5i}(n)) - E_6) \quad (14)$$

**Theorem 3**

The maximum entropy solutions for approximate values of the steady state probabilities, subject to constraints are given by

$$\begin{aligned}
 P_1(n) &= \frac{\sigma E_1 (E_6 - \sigma)}{E_6^n}; P_2(n) = \frac{\sigma E_2 (E_6 - \sigma)}{E_6^n}; P_{2i}(n) = \frac{\sigma E_{2i} (E_6 - \sigma)}{E_6^n}, 1 \leq i \leq k; \\
 P_3(n) &= \frac{\sigma E_3 (E_6 - \sigma)}{E_6^n}; P_4(n) = \frac{\sigma E_4 (E_6 - \sigma)}{E_6^n}; P_{4i}(n) = \frac{\sigma E_{4i} (E_6 - \sigma)}{E_6^n}, 1 \leq i \leq k; \\
 P_5(n) &= \frac{\sigma E_5 (E_6 - \sigma)}{E_6^n}; P_{5i}(n) = \frac{\sigma E_{5i} (E_6 - \sigma)}{E_6^n}, 1 \leq i \leq k.
 \end{aligned}$$

where,  $\sigma = E_1 + E_2 + E_{2i} + E_3 + E_4 + E_{4i} + E_5 + E_{5i}$ .

Proof: Partially differentiating *Lagrange’s function J* with respect to  $P_1(n)$ ,  $P_2(n)$ ,  $P_{2i}(n)$ ,  $P_3(n)$ ,  $P_4(n)$ ,  $P_{4i}(n)$ ,  $P_5(n)$ , and  $P_{5i}(n)$  respectively ( $1 \leq i \leq k$ ) and then equating the result to ‘0’ and further manipulating the equations, we get the desired results.

**Theorem 4**

The approximate expected waiting time of the customers present in the queue (via *maximum entropy approach*) is

$$\begin{aligned}
 W_a &= \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right\} P_1(n) + \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \left( \frac{n}{\beta_i} + \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right) \right\} P_{2i}(n) \\
 &+ \sum_{n=1}^{\infty} \left\{ \frac{b_w^2}{2b_w} + \sum_{i=0}^k \left( \frac{n}{\beta_i} + \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right) \right\} P_3(n) \\
 &+ \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \frac{h_i^2}{2h_i} + \frac{1}{\xi_i} + \sum_{i=0}^k \left( \frac{n}{\beta_i} + \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right) \right\} P_{4i}(n) \\
 &+ \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \frac{g_i^2}{2g_i} + \sum_{i=0}^k \left( \frac{n}{\beta_i} + \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right) \right\} P_{5i}(n).
 \end{aligned}$$

Proof: Consider a particular customer, say M who has  $n$  customers preceding him in the queue. At this instant server may be idle, busy, under working vacation, repair or delayed repair. Therefore, five cases arise depending upon the state of the server. An analysis of each case is given below in detail

(a). Idle state – This is the most convenient case for the customer. The server is free to serve. So, the batch of incoming customers is served instantly. This implies that service time of the customers preceding the customer M will be the approximate expected waiting time.

$$W_1 = \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right\}.$$

(b). Busy state – In this case server is occupied, so the batch of arriving customers joins the orbit. Firstly,  $n$  customers who are ahead in the queue get the service then those who are ahead of the arbitrary customer  $M$  will receive the service. So the approximate expected waiting time becomes

$$W_2 = \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \frac{n}{\beta_i} + \sum_{i=0}^k \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right\}.$$

(c). Working vacation state – The customer  $M$  waits for the fulfilment of working vacation period. When the server returns to normal busy state, those customers which are present in the orbit are served first. After that the customers before  $M$  get their service.

$$W_3 = \sum_{n=1}^{\infty} \left\{ \frac{b_w^2}{2b_w} \sum_{i=0}^k \frac{n}{\beta_i} + \sum_{i=0}^k \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right\}.$$

(d). Delayed repair state – Now the waiting time includes the time taken for the start-up of repair plus the time taken for the completion of repair. Once the server comes back to serve the customers, the ones there in the orbit are served first then those who are in front of  $M$  in the queue.

$$W_4 = \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \frac{h_i^2}{2h_i} + \frac{1}{\xi_i} + \sum_{i=0}^k \frac{n}{\beta_i} + \sum_{i=0}^k \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right\}.$$

(e). Repair state – It is quite similar to the above case. Only the start-up time is not included.

$$W_5 = \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^k \frac{g_i^2}{2g_i} + \sum_{i=0}^k \frac{n}{\beta_i} + \sum_{i=0}^k \frac{1}{2\beta_i} \left( \frac{E(X^2)}{E(X)} - 1 \right) \right\}.$$

By adding the waiting time of all the five cases discussed above, we obtain the required result.

## 5. Particle Swarm Optimization

With a rapid growth of telecommunication and computer networking an appropriate analysis of queueing model is highly required to provide suitable service and reduce congestion. This is possible if maximum number of people reach to a cost effective system. Therefore, systems including cost optimization are very much helpful and recommended. Recently, Jain et al. (2020) and Jain and Meena (2020) have performed a cost analysis for a queueing model with vacation and unreliable server. We have made an attempt to solve the financial restrictions in networking systems using the technique of *particle swarm optimization (PSO)*. This method is capable of finding the best suitable solution from a very large space of solutions. Another advantage of using PSO is that

the objective function is not required to be differentiable unlike many other optimization techniques. There are numerous applications of this method. Some of them include the fields of signal processing, telecommunications, designing and management, etc. In this method we first initialize a given set of population comprising of numerous candidates/particles. Then these candidates are made to move in the search space with respect to the given constraints and objective function over the candidate's location and velocity. The fitness value of each particle is calculated to further evaluate personal best (pbest) and global best (gbest) values. If any particle's pbest value is better than gbest then that becomes the new gbest value. This process continues till the number of iterations is completed.

The cost elements ( $cs_1, cs_2, cs_3, cs_4$ ) and the cost function are given by  
 $cs_1$  – holding cost per unit time for every customer  
 $cs_2$  – cost per unit time required for keeping the server in and on operations  
 $cs_3$  – cost for setup in every busy cycle  
 $cs_4$  – preparation cost of the server required to start up the service

$$CF(\theta, \eta) = cs_1 L_q + cs_2 (1 - P_o) + cs_3 \lambda_1 + C_1 + cs_4 m_o.$$

Our objective is to find optimal values ( $\theta^*, \eta^*$ ) such that the cost function can be minimized. We have taken five sets of cost elements ( $cs_1, cs_2, cs_3, cs_4$ ) as given below

Set1 = ( $cs_1 = \$4.2, cs_2 = \$13, cs_3 = \$18, cs_4 = \$20$ );

Set2 = ( $cs_1 = \$10, cs_2 = \$32, cs_3 = \$34, cs_4 = \$15$ );

Set3 = ( $cs_1 = \$2, cs_2 = \$7, cs_3 = \$5, cs_4 = \$9$ );

Set4 = ( $cs_1 = \$5, cs_2 = \$16, cs_3 = \$19, cs_4 = \$10$ );

Set5 = ( $cs_1 = \$8.9, cs_2 = \$29, cs_3 = \$28, cs_4 = \$3$ ).

PSO method is applied to the above defined cost elements using the MATLAB software for  $k=3$  optional services. To understand the concept better, let us describe an algorithm for this technique. For that purpose assume that  $\hat{h} : \mathfrak{R}^n \rightarrow \mathfrak{R}$  is the cost function which is to be minimized. Firstly,  $\hat{h}$  takes an individual solution as an argument and generates an output which becomes the objective function value of that individual solution. We are interested in finding a solution  $i$  for which  $\hat{h}(i) \leq \hat{h}(j)$  for each  $j$  in the search-space.

Let the number of individuals in the community be  $M$  and every individual has a location  $p_m$  and velocity  $vel_m$ . Let  $pos_m$  be the best location of candidate  $m$  and  $P$  the universal best location (best among the entire swarm). Also,  $lo$  and  $up$  are the lower and upper bounds of the search space respectively. The algorithm is then given by

for every candidate  $m=1,2,3 \dots M$  do

    Initialize the individual's location with a random vector:  $p_m \in (lo, up)$

    Initialize the individual's best known location to its initial location:  $p_m \rightarrow pos_m$

if  $\hat{h}(pos_m) < \hat{h}(P)$  then

Update the community's best location:  $pos_m \rightarrow P$

Initialize the candidate's velocity:  $vel_m \in (-|up - lo|, |up - lo|)$

while the termination criteria is not reached, do

For each candidate  $m=1, 2, \dots, M$

Update the candidate's velocity and location

if  $\hat{h}(p_m) < \hat{h}(pos_m)$  then

Update the community's best location:  $p_m \rightarrow pos_m$

if  $\hat{h}(pos_m) < \hat{h}(P)$  then

Update the community's best location:  $Pos_m \rightarrow P$ .

In our paper, number of candidates is 100, total number of iterations is 500 and lower and upper bound of the variables varies from 0.006 to 0.65. The results are given in Table 1.

The default parameters taken are

$r = 0.01, \lambda_0 = 0.28, \theta = 0.089, \lambda_2 = 5.5, \lambda_3 = 3, \beta_v = 10, \beta_1 = 25, \beta_2 = 15, \beta_3 = 20, \eta = 6,$   
 $p = 0.001, \xi_0 = 12, \zeta_0 = 14, \xi_i = \zeta_i = 8, m_i = 0.0055, \text{ for } 1 \leq i \leq 3, m_0 = 1 - m_1 - m_2 - m_3$

We have calculated the total cost of the system by varying some of the parameters and observe that the minimum cost obtained is \$2.5862 for  $\lambda_1 = 7.1$  and  $(\theta^*, \eta^*) = (0.008, 0.58)$ .

**Table 1.** Implementation of PSO method for finding minimum cost and  $(\theta^*, \eta^*)$  for various cost sets by varying  $\lambda_1, \lambda_2, \beta_0$  and  $\alpha$ .

$\lambda_1$	Set1	Set 2	Set 3	Set 4	Set 5
7.1	<b>\$2.5862</b>	\$4.7386	\$3.4077	\$4.3693	\$5.6494
7.2	\$3.9703	\$8.1388	\$4.1478	\$6.0694	\$8.7276
7.3	\$5.3543	\$11.5489	\$4.8878	\$7.7694	\$11.8056
$\lambda_2$	Set 1	Set 2	Set 3	Set 4	Set 5
5.2	\$17.9229	\$41.1497	\$10.63	\$22.5749	\$38.0033
5.3	\$12.552	\$28.3695	\$8.0739	\$16.1847	\$26.6288
5.4	\$6.9828	\$15.102	\$5.4204	\$9.551	\$14.8208
$\beta_0$	Set 1	Set 2	Set 3	Set 4	Set 5
8.5	\$7.2864	\$15.8247	\$5.565	\$9.9124	\$15.464
9	\$5.2123	\$10.8865	\$4.5773	\$7.4432	\$11.069
9.5	\$3.1852	\$6.0601	\$3.612	\$5.03	\$6.7735
$\alpha$	Set 1	Set 2	Set 3	Set 4	Set 5
0.0085	\$7.0625	\$15.206	\$5.4412	\$9.603	\$14.9134
0.009	\$5.0625	\$10.5298	\$4.506	\$7.2649	\$10.7515
0.0095	\$3.1212	\$5.9075	\$3.5815	\$4.9538	\$6.6377

## 6. Sensitivity Analysis

Numerical examples are very much mandatory so that we can predict the outcome of parameters considered in our model on the performance metrics. Performing numerical analysis of any model predicts its behaviour in real life scenario.



### 6.1 Numerical Simulation of Practical Example

In this particular section we have illustrated the example considered in Section 2.1, i.e. how a communication system of a reservation counter of a restaurant behaves in different situations by performing a numerical experiment. We assume that customers book the table via phone call with rate  $\lambda_o=0.28/\text{min}$ . The manager answers the phone call with rate  $\beta_o=19/\text{min}$ . Let us consider that server provides three optional services such as party lawn booking, home delivery and pick and drop service with rate  $\beta_1=25/\text{min}$ ,  $\beta_2=15/\text{min}$  and  $\beta_3=20/\text{min}$ . The probability that customer opts for these services is  $m_i=0.0055$  for  $i=1, 2, 3$  and arrival rate is  $\lambda_1=7.2/\text{min}$ ,  $\lambda_2=5.5/\text{min}$  and  $\lambda_3=3/\text{min}$  respectively. Whenever there is a miscommunication then customer calls again for feedback with probability  $s=0.9$ . There is also a possibility that line is busy and customer has to make the phone call again with rate  $\alpha=0.009/\text{min}$ . If the line has to get repaired due to this then the delayed repair and repair parameters are respectively  $\zeta_o=12$ ,  $\zeta_i=8$  and  $\zeta_o=14$  and  $\zeta_i=14$  for  $i=1, 2, 3$ . When the manager is off duty then assistant manager (substitute server) takes the charge with rate  $\beta_v=10/\text{min}$ . At the end of his vacation when there is no phone call then the manager returns to provide service with probability  $p=0.001$ . The results obtained are  $P(P_R)=0.1596$ ,  $P(\pi_i)=0.0014$  ( $0 \leq i \leq 3$ ),  $P(Q_v)=0.023$  and  $L_q=1.002$ .

### 6.2 Effect of System Parameters on Long Run Probabilities and Average Queue Size

We have also explored some other numerical results. These results convey how different variables can have repercussions on system performance measures. MATLAB software is used to illustrate the following results:

**Table 2.** Effect of  $\lambda_1$ ,  $\theta$  and  $p$  on long run probabilities for  $M^X/M/1$  Retrial G-queue.

$\lambda_1$	$P_o$	$Q_o$	$P(P_R)$	$P(\pi_{bi})$	$P(Q_v)$
5	0.0019	0.9046	0.1247	0.0011	0.023
6	0.002	0.9122	0.1358	0.0012	0.0232
7	0.002	0.9202	0.1474	0.0013	0.0234
$\theta$	$P_o$	$Q_o$	$P(P_R)$	$P(\pi_{bi})$	$P(Q_v)$
0.08	0.002	0.9358	0.1699	0.0014	0.0238
0.09	0.002	0.9186	0.145	0.0013	0.0234
0.1	0.0019	0.9031	0.1226	0.0012	0.023
$p$	$P_o$	$Q_o$	$P(P_R)$	$P(\pi_{bi})$	$P(Q_v)$
0.0001	0.002	0.9202	0.1474	0.0013	0.0234
0.0003	0.0059	0.9159	0.1467	0.0013	0.0233
0.0005	0.0098	0.9116	0.146	0.0013	0.0232

**Table 3.** Effect of  $\eta$ ,  $\beta_o$  and  $\alpha$  on long run probabilities for  $M^X/M/1$  Retrial G-queue.

$\eta$	$P_o$	$Q_o$	$P(P_R)$	$P(\pi_{bi})$	$P(Q_v)$
5	0.0017	0.9247	0.1495	0.0013	0.0259
6	0.002	0.9202	0.1474	0.0013	0.0234
7	0.0023	0.9162	0.1456	0.0013	0.0214
$\beta_o$	$P_o$	$Q_o$	$P(P_R)$	$P(\pi_{bi})$	$P(Q_v)$
8	0.0019	0.8996	0.1175	0.0013	0.0229
11	0.002	0.9258	0.1593	0.0013	0.0236
14	0.002	0.9477	0.1872	0.0013	0.0241
$\alpha$	$P_o$	$Q_o$	$P(P_R)$	$P(\pi_{bi})$	$P(Q_v)$
0.006	0.0019	0.8957	0.2647	0.0023	0.0228
0.008	0.002	0.9115	0.1894	0.0017	0.0232
0.01	0.002	0.9202	0.1474	0.0013	0.0234

The default parameters for Table 2 and Table 3 are same as that of Table 1.

Now we show the variation in average queue length with different parameters using Figures 1-4. The default parameters are same as for Table 1.

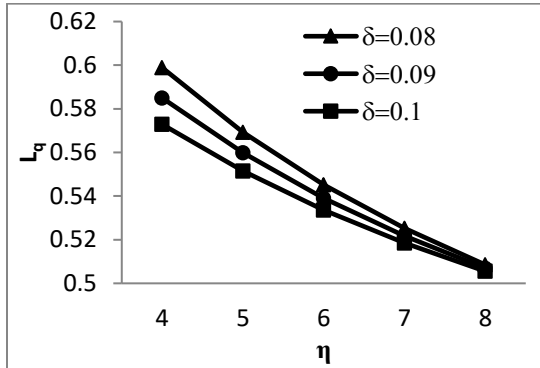


Figure 1.  $L_q$  versus  $\eta$ .

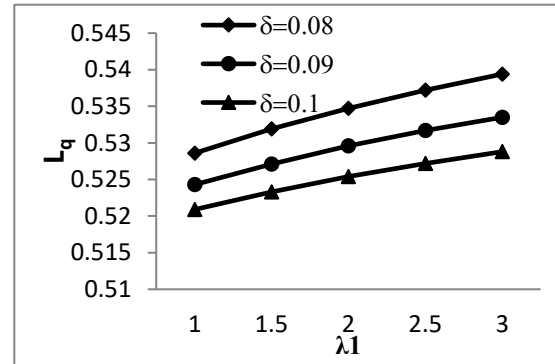


Figure 2.  $L_q$  versus  $\lambda_1$ .

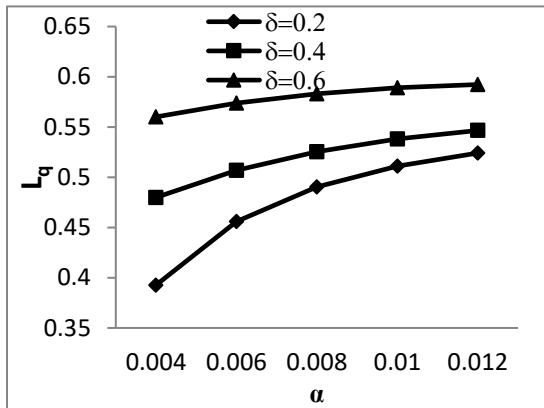


Figure 3.  $L_q$  versus  $\alpha$ .

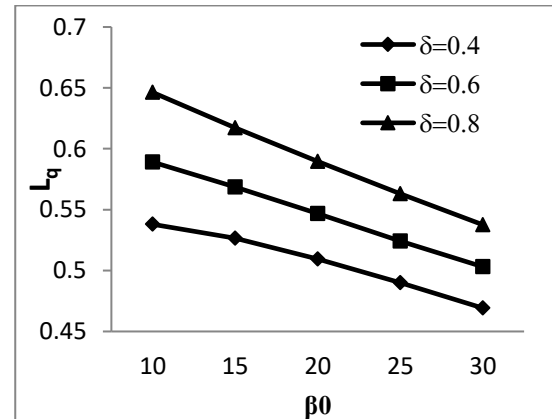


Figure 4.  $L_q$  versus  $\beta_0$ .

In Figures 1 and 4, we see a decrease in average queue length with an increase in  $\eta$  and  $\beta_0$ . While in rest of the two figures we notice that it is directly proportional to the values of  $\lambda_1$  and  $\alpha$ .

### 6.3 Comparison between Exact Waiting Time and Expected Waiting Time

In the next two tables we show a relative study comprising the exact expected waiting time ( $W_s$ ) and approximate expected waiting time ( $W_a$ ) of our model along with absolute percentage error

(APE) given by  $APE = \left[ \frac{W_s - W_a}{W_s} \right] \times 100$  for two sets of probabilities  $m_1, m_2$  and  $m_3$ . For Case 1,

we set  $m_1 = m_2 = 0.001, m_3 = 0.004$  and for Case 2, we set  $m_1 = 0.06, m_2 = 0.04, m_3 = 0.02$ . The

default parameters for Tables 4-5 are  $r = 0.001, \lambda_i = 5, \beta_v = 2.6, \beta_i = 5, p = 0.001, \xi_0 = 12, \zeta_0 = 14, \xi_i = \zeta_i = 8, 1 \leq i \leq 3$ .

**Table 4.** Effect of  $\theta, \lambda_1$  and  $\beta_0$  on  $W_s$  and  $W_a$  for different values of  $m_i, 1 \leq i \leq 3$ .

$\theta$	Case 1			Case 2		
	$W_s$	$W_a$	APE	$W_s$	$W_a$	APE
0.08	0.2499	0.2287	7.242	0.2426	0.216	10.9673
0.09	0.2536	0.2404	5.2086	0.2467	0.2282	7.5179
0.1	0.256	0.2512	1.8833	0.2495	0.2394	4.0181
$\lambda_1$	$W_s$	$W_a$	APE	$W_s$	$W_a$	APE
5	0.256	0.2512	1.8833	0.2495	0.2394	4.0181
6	0.2471	0.234	5.3222	0.2398	0.2215	7.6266
7	0.2364	0.2158	8.7155	0.2283	0.2028	11.1766
$\beta_0$	$W_s$	$W_a$	APE	$W_s$	$W_a$	APE
3	0.256	0.2512	1.8833	0.2495	0.2394	4.0181
3.5	0.2608	0.239	8.3445	0.2534	0.2272	10.359
4	0.2651	0.2305	13.0297	0.2569	0.2185	14.9535

**Table 5.** Effect of  $\alpha, \eta$  and  $\lambda_0$  on  $W_s$  and  $W_a$  for different values of  $m_i, 1 \leq i \leq 3$ .

$\alpha$	Case 1			Case 2		
	$W_s$	$W_a$	APE	$W_s$	$W_a$	APE
0.18	0.2826	0.2724	3.6019	0.2754	0.2586	6.1251
0.24	0.2598	0.2545	2.0616	0.2532	0.2425	4.2402
0.3	0.2371	0.2343	1.2088	0.2309	0.2236	3.1635
$\eta$	$W_s$	$W_a$	APE	$W_s$	$W_a$	APE
1	0.3521	0.3306	6.09	0.3474	0.304	12.5033
1.5	0.3176	0.3021	4.8909	0.3123	0.2848	9.1209
2	0.2857	0.2756	3.5043	0.2797	0.2615	6.4918
$\lambda_0$	$W_s$	$W_a$	APE	$W_s$	$W_a$	APE
0.75	0.3087	0.2739	11.2676	0.3019	0.2615	13.3879
0.8	0.2813	0.263	6.4949	0.2746	0.251	8.6249
0.85	0.256	0.2512	1.8833	0.2495	0.2394	4.0181

From Tables 4 and 5, we can observe that the waiting time is directly proportional to  $\theta$  and  $\beta_0$ . However, it is completely opposite for the rest of the parameters. Also, APE is comparatively less in Case 1 than in Case 2. From the numerical analysis discussed above we can arrive at a conclusion that the results obtained are similar to practical situations that are observed.

## 7. Conclusion and Future Possibility

This investigation involves a bulk retrial G-queue under state dependent, voluntary services, Bernoulli feedback along with delay in repair of the repairman. Performance indices like the long run probabilities and average queue size of the system are discussed which can manage and reduce the queue size. The main motive of our investigation is that we present a relative study corresponding to the exact and approximate expected waiting time for the developed model so that the system planners and decision makers can reduce the waiting time of the customers in the queue which in turn improves the grade of service of their respective systems. This proposed model is applicable in industrial queues, management and production industry, communication networks, supermarkets, etc. Basically, in all these huge sectors it is next to impossible to build a model in which the server never fails or never deactivates. The concept of repairing of the server and the delay in repair is also very relative in today's world of high speed technology. Thus, this

investigation is relative and supportive for those areas wherein a server can be inactive so that resources are better utilized. This model is designed such that it is helpful in avoiding daily overcrowding problems faced by communication and networking systems. The practical example and the numerical simulation discussed in Section 6 depicts that the results are validated by real life congestion situations. This study can be further improved by including the concepts of bulk service, optional vacations, etc. but it will make the model more tricky and complex and will also demand lot of computational efforts.

#### **Conflict of Interest**

The authors confirm that conflicts of interests are none.

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#### **References**

- Artelajo, J.R. (1994). New results in retrial queueing systems with breakdown of servers. *Statistica Neerlandica*, 48(1), 23-36.
- Chauhan, D. (2018). Maximum entropy analysis of unreliable  $M^X/(G_1, G_2)/1$  queue with Bernoulli vacation schedule. *International Journal of Statistics and Applied Mathematics*, 3(6), 110-118.
- El-Affendi, & Kouvatso, M.A. (1983). A maximum entropy analysis of the M/G/1 and G/M/1 queueing systems at equilibrium. *Acta Informatica*, 19(4), 339–355.
- Gelenbe, E. (1991). Product form queueing networks with negative and positive customers. *Journal of Applied Probability*, 28(3), 656-63.
- Jain, M., & Meena, R.K. (2020). Availability analysis and cost optimization of M/G/1 fault-tolerant machining system with imperfect fault coverage. *Arabian Journal for Science and Engineering*, 45(3), 2281-2295.
- Jain, M., & Upadhyaya, S. (2012). Optimal repairable  $M^X/G/1$  queue with Bernoulli feedback and set-up. *International Journal of Mathematics in Operational Research*, 4(6), 679-702.
- Jain, M., Kumar, P., & Meena, R.K. (2020). Fuzzy metrics and cost optimization of a fault tolerant system with vacationing and unreliable server. *Journal of Ambient Intelligence and Humanized Computing*, 1-  
<https://doi.org/10.1007/s12652-020-01951-x>.
- Jain, M., Shekhar, C., & Meena, R.K. (2019). Performance analysis and control F-policy for fault tolerant system with working vacation. *OPSEARCH*, 56(2), 409-431.
- Jaynes, E.T. (1957). Information theory and statistical mechanics. *Physical Reviews*, 106(4), 620-630.
- Pandey, R.V., & Tripathi, P. (2014). Maximum entropy analysis of unreliable server  $M^X/G/1$  queue with essential and multiple optional services. *International Journal of Mathematical Archive*, 5(1), 125-133.
- Prakash, O., & Mukesh (2016). Contribution of maximum entropy principle in the field of queueing theory. *Communication in Statistics- Theory and Methods*, 45(12), 3464-3472.
- Rajadurai, P., Saravanarajan, M.C., & Chandrasekaran, V.M. (2018). A study on M/G/1 retrial queue with subject to server breakdown and repair under multiple vacation policy. *Alexandria Engineering Journal*, 57(2), 947-962.

- Sengupta, S., Basak, S., & Peters II, R.A. (2019). Particle Swarm Optimization: A survey of historical and recent developments with hybridization perspectives. *Machine Learning and Knowledge Extraction*, 1(1), 157-191.
- Sethi, R., Bhagat, A., & Garg, D. (2019). ANFIS based machine repair model with controlled policies and working vacation. *International Journal of Mathematical, Engineering and Management Sciences*, 4(6), 1522-1533.
- Upadhyaya, S. (2014). Performance analysis of a batch arrival retrial queue with Bernoulli feedback. *International Journal of Mathematics in Operations Research*, 6(6), 680-703.
- Upadhyaya, S. (2018). Performance analysis of a discrete-time Geo/G/1 retrial queue under J-vacation policy. *International Journal of Industrial and Systems Engineering*, 29(3), 369-388.
- Upadhyaya, S. (2020a). Cost optimization of a discrete-time retrial queue with Bernoulli feedback and starting failure. *International Journal of Industrial and Systems Engineering*, 36(2), 165-196.
- Upadhyaya, S. (2020b). Investigating a general service retrial queue with damaging and licensed units: an application in local area networks. *OPSEARCH*, 57, 716-745.
- Wang, K.H., Wang, J., Liou, C.D., & Zhang, X. (2019). Particle swarm optimization to the retrial machine repair problem with working breakdowns under the N-policy. *Queueing Models and Service Management*, 2(1), 061-081.
- Yang, S.P., Wu, J.B., & Liu, Z.M. (2013). An M [X]/G/1 retrial G-queue with single vacation subject to the server breakdown and repair. *Acta Mathematicae Applicatae Sinica, English Series*, 29(3), 579-596.
- Zhou, M., Liu, L., Chai, X., & Wang, Z. (2019). Equilibrium strategies in a constant retrial queue with setup time and the N-policy. *Communications in Statistics-Theory and Methods*, 49(7), 1695-1711.

## Appendix A

The remaining symbols of Theorem 2 are given below

$$e'' = -2\lambda_0\lambda_2C_1B_v^{*'}(\eta) + \lambda_0\lambda_2^2C_1^2B_v^{*''}(\eta) - \lambda_0\lambda_2C_2B_v^{*'}(\eta);$$

$$e''' = 3\lambda_0\lambda_2^2C_1^2B_v^{*''}(\eta) - 3\lambda_0\lambda_2C_2B_v^{*'}(\eta) - \lambda_0\lambda_2^3C_1^3B_v^{*'''}(\eta) + 3\lambda_0\lambda_2^2C_1C_2B_v^{*''}(\eta) - \lambda_0\lambda_2C_3B_v^{*'}(\eta);$$

$$V(1) = (1 - B_v^{*'}(\eta)); V'(1) = \lambda_2C_1 \left[ B_v^{*'}(\eta) + \frac{(1 - B_v^{*'}(\eta))}{\eta} \right];$$

$$V''(1) = \lambda_2^2C_1^2B_v^{*''}(\eta) + \lambda_2C_2B_v^{*'}(\eta) + \frac{(1 - B_v^{*'}(\eta))}{\eta^2}(\lambda_2C_2\eta + 2\lambda_2^2C_1^2); P_0 = \frac{\theta^2 pD}{D_1}$$

$$V'''(1) = \{\lambda_2^3C_1^3\eta^2(B_v^{*''}(\eta) - \eta B_v^{*'''}(\eta)) + \eta\lambda_2(1 - B_v^{*'}(\eta))(4\lambda_2C_1C_2 + \eta C_3) + 3\lambda_2^2C_1C_2\eta^3B_v^{*''}(\eta) + 4\lambda_2^2C_1(1 - B_v^{*'}(\eta))(\eta C_2 + 2\lambda_2^2C_1) + \eta\lambda_2B_v^{*'}(\eta) [2\eta\lambda_2C_1C_2 - \eta^2C_3 - \eta\lambda_1C_1C_2 - 6\lambda_2\lambda_1C_1^3 + 8\lambda_2^2C_1^3]\} / \eta^3;$$

$$f' = \lambda_0V(1) + \eta p; f'' = \lambda_0V(1) + \eta p + \lambda_0V'(1); f''' = 2\lambda_0V'(1) + \lambda_0V''(1); f'''' = 3\lambda_0V''(1) + \lambda_0V'''(1);$$

$$\omega = rB_0^*(\theta)(m_o + \sum_{i=1}^k m_i B_i^*(\theta)) + \left( \frac{\lambda_1 C_1}{\theta} - 1 \right) (1 - B_0^*(\theta))(m_o + \sum_{i=1}^k m_i B_i^*(\theta)) \\ + \lambda_3 C_1 (1 - B_0^*(\theta))(h_0^1 + g_0^1) + B_0^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta))(h_i^1 + g_i^1);$$

$$\omega_1 = -2r\lambda_1 C_1 [B_0^{**}(\theta)(m_o + \sum_{i=1}^k m_i B_i^*(\theta)) + B_0^*(\theta)(\sum_{i=1}^k m_i B_i^{**}(\theta))] + \lambda_1^2 C_1^2 \\ [B_0^{***}(\theta) + 2B_0^{**}(\theta)(\sum_{i=1}^k m_i B_i^*(\theta)) + B_0^*(\theta)(\sum_{i=1}^k m_i B_i^{***}(\theta))] - \lambda_1 C_2 (B_0^{**}(\theta) + \\ B_0^*(\theta)(\sum_{i=1}^k m_i B_i^{**}(\theta))) + 2(1 - \lambda_3 C_1 (h_0^1 + g_0^1)) [(1 - B_0^*(\theta)) \left( 1 - \frac{\lambda_1 C_1}{\theta} \right) - B_0^{**}(\theta)] \\ + \theta F_0''(1) + \lambda_3 C_2 (h_0^1 + g_0^1) (1 - B_0^*(\theta)) - \lambda_3^2 C_1^2 (1 - B_0^*(\theta)) [(h_0^1 + g_0^1) - 2h_0^1 g_0^1] \\ - 2(B_0^*(\theta) + \lambda_1 C_1 B_0^{**}(\theta)) \left( \frac{\lambda_1 C_1}{\theta} \sum_{i=1}^k m_i (\theta B_i^{**}(\theta) + 1 - B_i^*(\theta)) + \lambda_3 C_1 \sum_{i=1}^k m_i (1 - B_i^*(\theta))(h_i^1 + g_i^1) \right) \\ \theta B_0^*(\theta) \sum_{i=1}^k m_i F_i''(1) + 2 \frac{\lambda_1 \lambda_3 C_1^2}{\theta} B_0^*(\theta) \sum_{i=1}^k m_i (\theta B_i^{**}(\theta) + 1 - B_i^*(\theta))(h_i^1 + g_i^1) \\ + 2 \sum_{i=1}^k m_i (1 - B_i^*(\theta))(B_0^*(\theta) + B_0^{**}(\theta)) + \lambda_3^2 C_1^2 B_0^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta))(h_i^1 g_i^1 - (h_i^2 + g_i^2)) \\ + \lambda_3 C_2 B_0^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta))(h_i^1 + g_i^1);$$

$$F_i''(1) = \theta^2 \{ \{-\theta \lambda_1^2 C_1^2 B_i^{***}(\theta) + \lambda_1 C_2 \theta B_i^{**}(\theta) + \lambda_1 C_2 (1 - B_i^*(\theta))\} + \\ 2\lambda_1 C_1 \theta \{ \lambda_1 C_1 \theta B_i^{**}(\theta) + \lambda_1 C_1 (1 - B_i^*(\theta)) \} \} / \theta^4, \quad 0 \leq i \leq k;$$

$$F_i'''(1) = \{ 4\lambda_1^2 C_1 C_2 \theta B_i^{**}(\theta) + \theta^2 [\lambda_1^3 C_1^3 \theta B_i^{***}(\theta) - 3\lambda_1^2 C_1 B_i^{***}(\theta) (C_2 \theta + \lambda_1 C_1^2) + \lambda_1 C_3 \\ (\theta B_i^{**}(\theta) + (1 - B_i^*(\theta)))] - 2\lambda_1^2 C_1 \theta B_i^{**}(\theta) (3\lambda_1^2 C_1^2 + C_2 \theta) - 6\lambda_1^2 C_1 (1 - B_i^*(\theta)) \\ (\lambda_1 C_1^2 - C_2 \theta) \} / \theta^4, \quad 0 \leq i \leq k;$$

$$\psi_1 = (m_o + \sum_{i=1}^k m_i B_i^*(\theta)) (3r\lambda_1^2 C_1^2 B_o^{**}(\theta) - 3r\lambda_1 C_2 B_o^{**}(\theta) - \lambda_1 C_3 B_o^{**}(\theta) - \lambda_1^3 C_1^3 B_o^{***}(\theta)) \\ + 3\lambda_1^2 C_1 C_2 B_o^{**}(\theta) + \sum_{i=1}^k m_i B_i^{**}(\theta) (B_o^{**}(\theta) (6r\lambda_1^2 C_1^2 + 3\lambda_1^2 C_1 C_2) - 3\lambda_1^3 C_1^3 B_o^{***}(\theta)) \\ + (3rB_o^{**}(\theta) - 3\lambda_1 C_1 B_o^{**}(\theta)) \sum_{i=1}^k m_i (\lambda_1^2 C_1^2 B_i^{**}(\theta) - \lambda_1 C_2 B_i^{**}(\theta)) + B_o^{**}(\theta) \\ \sum_{i=1}^k m_i (\lambda_1^3 C_1^3 B_i^{***}(\theta) + 3\lambda_1^2 C_1 C_2 B_i^{**}(\theta) - \lambda_1 C_3 B_i^{**}(\theta));$$

$$\begin{aligned} \psi_2 = & -6(1 - B_o^*(\theta))[1 + \lambda_3^2 C_1^2 h_o^1 g_o^1 - \lambda_3 C_1 (h_o^1 + g_o^1)] - 3\theta F_o''(1)(1 - \lambda_3 C_1 (h_o^1 + g_o^1)) \\ & + \theta F_o'''(1) + 3\lambda_3^2 C_1^2 (1 - B_o^*(\theta))(h_o^2 + g_o^2) - 3\lambda_3 C_2 (1 - B_o^*(\theta))(h_o^1 + g_o^1) \\ & - 3 \frac{\lambda_1 \lambda_3^2 C_1^3}{\theta} (\theta B_o^*(\theta) + 1 - B_o^*(\theta))(h_o^2 + g_o^2) + 6 \frac{\lambda_1 C_1}{\theta} (\theta B_o^*(\theta) + 1 - B_o^*(\theta)) \\ & [1 - \lambda_3 C_1 (h_o^1 + g_o^1) + \lambda_3^2 C_1^2 h_o^1 g_o^1] + 3 \frac{\lambda_1 \lambda_3 C_1 C_2}{\theta} (\theta B_o^*(\theta) + 1 - B_o^*(\theta))(h_o^1 + g_o^1) \\ & + 3\lambda_3 C_1 (1 - B_o^*(\theta))(2\lambda_3 C_2 h_o^1 g_o^1 - \lambda_3^2 C_1^2 (h_o^1 g_o^2 + h_o^2 g_o^1)) + (1 - B_o^*(\theta)) \\ & [\lambda_3^3 C_1^3 (h_o^3 + g_o^3) - 2\lambda_3^2 C_1 C_2 (h_o^2 + g_o^2) - \lambda_1^2 C_1 C_2 (h_o^2 + g_o^2) + \lambda_3 C_3 (h_o^1 + g_o^1)]; \end{aligned}$$

$$\begin{aligned} \psi_3 = & 6(B_o^*(\theta) + \lambda_1 C_1 B_o^*(\theta)) \left[ -\sum_{i=1}^k m_i (1 - B_i^*(\theta)) + \frac{\lambda_1 C_1}{\theta} \sum_{i=1}^k m_i (\theta B_i^*(\theta) + 1 - B_i^*(\theta)) \right] \\ & + \lambda_3 C_1 (h_o^1 + g_o^1) \sum_{i=1}^k m_i (1 - B_i^*(\theta)) - \lambda_3^2 C_1^2 \sum_{i=1}^k m_i (1 - B_i^*(\theta)) - 3\theta \sum_{i=1}^k m_i F_i''(1) \\ & [B_o^*(\theta) - \lambda_3 C_1 B_o^*(\theta) + \lambda_1 C_1 B_o^*(\theta)] + \theta B_o^*(\theta) \sum_{i=1}^k m_i F_i'''(1) + \sum_{i=1}^k m_i (1 - B_i^*(\theta)) \\ & [-\lambda_1^3 C_1^3 B_o^{*m}(\theta) + 3\lambda_1^2 C_1 C_2 B_o^{*m}(\theta) + \lambda_1 C_3 B_o^*(\theta)] + 3(\lambda_1^2 C_1^2 B_o^{*m}(\theta) - \lambda_1 C_2 B_o^*(\theta)) \\ & \sum_{i=1}^k m_i (1 - B_i^*(\theta)) \left( 1 + \frac{\lambda_1 C_1}{\theta} + \lambda_3 C_1 (h_i^1 + g_i^1) \right) - 3B_o^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta)) [-\lambda_3^2 C_1^2 \\ & (h_i^2 + g_i^2) + \lambda_3 C_2 (h_i^1 + g_i^1)] - 3\lambda_1 C_1 B_o^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta)) [-\lambda_3^2 C_1^2 (h_i^2 + g_i^2) \\ & + \lambda_3 C_2 (h_i^1 + g_i^1)] + B_o^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta)) [\lambda_3^3 C_1^3 (h_i^3 + g_i^3) - \lambda_3^2 C_1 C_2 (h_i^2 + g_i^2) \\ & + \lambda_3 C_3 (h_i^1 + g_i^1)] + 3\lambda_3 C_1 B_o^*(\theta) \sum_{i=1}^k m_i (1 - B_i^*(\theta)) [-\lambda_3^2 C_1^2 (h_i^1 g_i^2 + h_i^2 g_i^1) + 2\lambda_3 C_2 h_i^1 g_i^1] \\ & + \frac{\lambda_1 C_1}{\theta} \sum_{i=1}^k m_i (1 - B_i^*(\theta) + 1 - B_i^*(\theta)) [-6\lambda_3 C_3 (h_i^1 + g_i^1) (B_o^*(\theta) - \lambda_1 C_1 B_o^*(\theta)) \\ & - 3B_o^*(\theta) (\lambda_3 C_2 (h_i^1 + g_i^1) - \lambda_3^2 C_1^2 (h_i^2 + g_i^2))] + 6\lambda_3^2 C_1^2 B_o^*(\theta) (h_i^1 g_i^1)]; \end{aligned}$$

$$\psi = \psi_1 + \psi_2 + \psi_3 ; \quad j_i = 1 - B_i^*(\theta), 0 \leq i \leq k \quad j_i' = \lambda_1 C_1 B_i^*(\theta), 0 \leq i \leq k ;$$

$$j_i'' = -\lambda_1^2 C_1^2 B_i^{*m}(\theta) + \lambda_1 C_2 B_i^*(\theta), 0 \leq i \leq k ;$$

$$j_i''' = \lambda_1^3 C_1^3 B_i^{*m}(\theta) - 3\lambda_1^2 C_1 C_2 B_i^{*m}(\theta) + \lambda_1 C_3 B_i^*(\theta), 0 \leq i \leq k ;$$



$$T = \lambda_o (B_v^*(\theta) - 1) - \eta p ; T_1 = -\lambda_o \lambda_2 C_1 B_v^{*'}(\theta) ; T_2 = \lambda_o \lambda_2^2 C_1^2 B_v^{*''}(\theta) - \lambda_o \lambda_2 C_2 B_v^{*'}(\theta);$$

$$T_3 = -\lambda_o \lambda_2^3 C_1^3 B_v^{*'''}(\theta) + 3\lambda_o \lambda_2^2 C_1 C_2 B_v^{*''}(\theta) - \lambda_o \lambda_2 C_3 B_v^{*'}(\theta);$$

$$r_1 = 1 - S^*(\lambda_o) ; \rho = T r_1 + f ; \rho_1 = T_1 r_1 + f' ; \rho_2 = T_2 r_1 + f'' ; \rho_3 = T_3 r_1 + f''' ;$$

$$\phi_i' = -\lambda_3 C_1 h_i^1, 0 \leq i \leq k, \quad \phi_i'' = \lambda_3^2 C_1^2 h_i^2 - \lambda_3 C_2 h_i^1, 0 \leq i \leq k,$$

$$\phi_i''' = -\lambda_3^3 C_1^3 h_i^3 + 3\lambda_3^2 C_1 C_2 h_i^2 - \lambda_3 C_3 h_i^1, 0 \leq i \leq k.$$

$$\tau_i' = -\lambda_3 C_1 g_i^1, 0 \leq i \leq k ; \tau_i'' = \lambda_3^2 C_1^2 g_i^2 - \lambda_3 C_2 g_i^1, 0 \leq i \leq k,$$

$$\tau_i''' = -\lambda_3^3 C_1^3 g_i^3 + 3\lambda_3^2 C_1 C_2 g_i^2 - \lambda_3 C_3 g_i^1, 0 \leq i \leq k.$$

